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CHAPTER 8

TRANSMISSION-LINE EQUATIONS

The transmission line may be analyzed by the solution of Maxwell's field equations or by the methods of ordinary circuit analysis. The solution of the field equations involves the determination of the field intensities in three-dimensional space; hence three space variables in addition to the time variable are involved. Although this method may be used to analyze a few systems having relatively simple geometry, in most practical cases the mathematical complications resulting from four independent variables are usually insurmountable. The solution of Maxwell's field equations reveals that the energy propagates through the dielectric medium as an electromagnetic wave, the conductors serving to guide the energy flow.

In the circuit method, the effects of the electric and magnetic fields are taken into consideration by the use of the circuit parameters, *i.e.*, the capacitance, inductance, resistance, and conductance. By this procedure, the mathematical analysis is reduced to a problem involving one space variable in addition to the time variable. The circuit method, however, does not yield the complete solution. In a later chapter we shall find, using the Maxwellian method, that an infinite number of electromagnetic field configurations, known as modes, may be associated with a given transmission line. The *principal mode* corresponds to the field configuration which exists at frequencies for which the spacing between conductors is appreciably less than a quarter wavelength. The *higher modes* appear when the separation distance between conductors is of the order of magnitude of a quarter wavelength or greater, or when there are impedance discontinuities on the line.

In this chapter we shall analyze the transmission line using the circuit method. The Maxwellian method will be considered in later chapters.

8.01. Derivation of the Transmission-line Equations.—Consider the transmission line of Fig. 1, which is assumed to have uniformly distributed parameters. The resistance, inductance, capacitance, and conductance per unit length are represented by R, L, C, and G, respectively.

The equations for the instantaneous voltage Δv across an incremental length of line Δx, and the shunt current through it are

$$\Delta v = i(R\,\Delta x) + (L\,\Delta x)\frac{\partial i}{\partial t} \tag{1}$$

$$\Delta i = v(G\,\Delta x) + (C\,\Delta x)\frac{\partial v}{\partial t} \tag{2}$$

Replacing Δv in Eq. (1) by $(\partial v/\partial x)\,\Delta x$ and dividing by Δx, w[illegible] similar operation for the current equation, we obtain the differential e[illegible]tions of the transmission line,

$$\frac{\partial v}{\partial x} = iR + L\frac{\partial i}{\partial t} \tag{3}$$

$$\frac{\partial i}{\partial x} = vG + C\frac{\partial v}{\partial t} \tag{4}$$

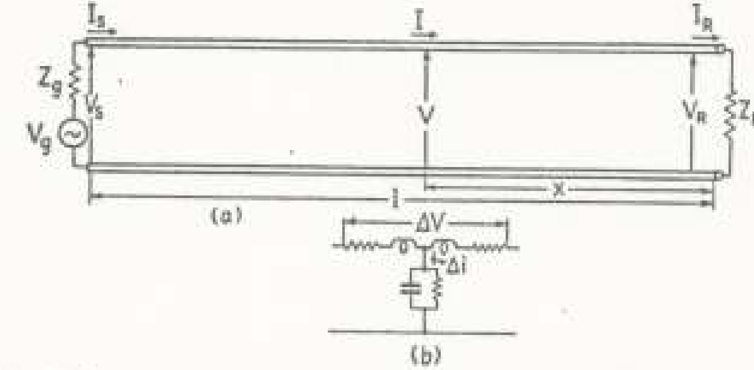


FIG. 1.—(a) Transmission line, and (b) equivalent circuit of a differential element of line.

In order to illustrate the nature of the solution, consider the case of a lossless line in which we have $R = G = 0$. Equations (3) and (4) then reduce to

$$\frac{\partial v}{\partial x} = L\frac{\partial i}{\partial t} \tag{5}$$

$$\frac{\partial i}{\partial x} = C\frac{\partial v}{\partial t} \tag{6}$$

Now differentiate Eq. (5) with respect to x and substitute Eq. (6) for $\partial i/\partial x$ in the resulting equation. Similarly, differentiate Eq. (6) with respect to x and substitute Eq. (5) for $\partial v/\partial x$. This process gives

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} \tag{7}$$

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} \tag{8}$$

These are known as the *wave equations* for the lossless line. Equations of this type frequently occur in the analysis of electrical, mechanical, and acoustical systems. Solutions of Eqs. (7) and (8) are of the form $f_1[t - (x/v)]$ or $f_2[t + (x/v)]$.

termined ... $(x/v)]$ represents a wave of arbitrary waveform (de... ...e particular function chosen) which travels in the $+x$ direction with ... velocity v. Similarly the function $f_2[t + (x/v)]$ represents a wave traveling in the $-x$ direction with a velocity v. Figures 2a and 2b illustrate waves traveling in the $+x$ and $-x$ direction, respectively, for several successive instants of time. Consider the function $f_1[t - (x/v)]$. If we were to ride along with the peak of the wave, it would be necessary for our displacement x to vary with time in such a manner as to hold $t - (x/v)$ constant. Thus, as time t increases, x must increase in a positive direction. We therefore conclude that $f_1[t - (x/v)]$ represents a wave traveling in the $+x$ direction.

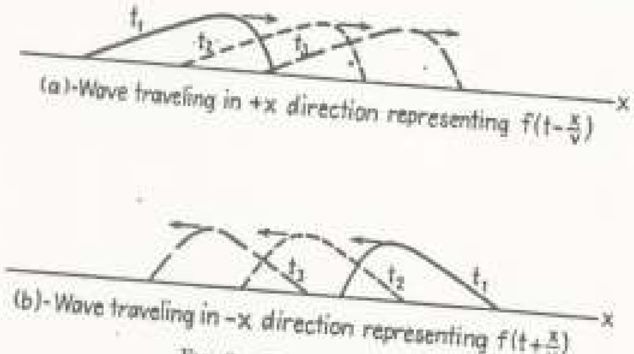


Fig. 2.—Traveling waves.

To find the velocity v, let us substitute $v = f_1[t - (x/v)]$ into Eq. (7). We obtain $\partial^2 v/\partial x^2 = (1/v^2)f_1''[t - (x/v)]$ and $\partial^2 v/\partial t^2 = f_1''[t - (x/v)]$. Inserting these into Eq. (7) and solving for the velocity, we obtain

$$v = \frac{1}{\sqrt{LC}} \tag{9}$$

For a lossless line with a dielectric having a relative permittivity of unity, the velocity v is equal to the velocity of light, *i.e.*, $v = 3 \times 10^8$ meters per second.

8.02. Sinusoidal Impressed Voltage.—In most practical applications we are concerned with voltages and currents having a sinusoidal time variation. For mathematical convenience, such a variation may be represented by the time function $\epsilon^{j\omega t}$. We therefore let[1]

$$v = V\epsilon^{j\omega t} \tag{1}$$

$$i = I\epsilon^{j\omega t} \tag{2}$$

where V and I are complex quantities which are functions of ... not of time t. Inserting Eqs. (1) and (2) into (8.01-3 and 4), the tim... ...ction $\epsilon^{j\omega t}$ cancels out and we obtain equations in which the voltage and current are functions of the space variable alone,

$$\frac{dV}{dx} = Iz \tag{3}$$

$$\frac{dI}{dx} = Vy \tag{4}$$

where

$$z = R + j\omega L \tag{5}$$

$$y = G + j\omega C \tag{6}$$

represent, respectively, the series impedance and shunt admittance per unit length of line.

To obtain an explicit equation for voltage, differentiate Eq. (3) with respect to x and substitute dI/dx from Eq. (4). The current equation is obtained by differentiating Eq. (4) with respect to x and substituting dV/dx from Eq. (3). These operations yield

$$\frac{d^2V}{dx^2} = \gamma^2 V \tag{7}$$

$$\frac{d^2I}{dx^2} = \gamma^2 I \tag{8}$$

where $\gamma = \sqrt{zy}$ is the *propagation constant* of the line. In general, γ is complex and may be separated into real and imaginary parts, hence we let

$$\gamma = \sqrt{zy} = \alpha + j\beta \tag{9}$$

where α is known as the *attenuation constant* and β is the *phase constant*.

The solutions of Eqs. (7) and (8) which also satisfy (3) and (4) are

$$V = A\epsilon^{\gamma x} + B\epsilon^{-\gamma x} \tag{10}$$

$$I = \frac{A}{Z_0}\epsilon^{\gamma x} - \frac{B}{Z_0}\epsilon^{-\gamma x} \tag{11}$$

where

$$Z_0 = \sqrt{\frac{z}{y}} \tag{12}$$

is the *characteristic impedance* of the line.

[1] An instantaneous voltage of the form $v = V_m \cos(\omega t + \theta)$ may be written ... $= \text{Re}\,V_m\epsilon^{j(\omega t+\theta)}$, where Re signifies that we take the real part of the quantity following it. Thus, expanding $\epsilon^{j(\omega t+\theta)} = \cos(\omega t + \theta) + j\sin(\omega t + \theta)$ and discarding the imaginary part, we have $\text{Re}\,\epsilon^{j(\omega t+\theta)} = \cos(\omega t + \theta)$. Now write $v = \text{Re}[(V_m\epsilon^{j\theta})\epsilon^{j\omega t}]$, where θ is the phase angle of the voltage at zero time. Letting $V = V_m\epsilon^{j\theta}$, we have $v = \text{Re}\,V\epsilon^{j\omega t}$. It is customary to drop the designation Re although it is implied and should be reinserted if we wish to obtain actual values of the voltage or current at any instant of time. Thus, we have $v = V\epsilon^{j\omega t}$.

multiplyi ...ous voltage and current equations may be obtained by thus ...s. (10) and (11) by $e^{j\omega t}$ as indicated in Eqs. (1) and (2),

$$v = Ae^{j\omega t+\gamma x} + Be^{j\omega t-\gamma x} \tag{13}$$

$$i = \frac{A}{Z_0} e^{j\omega t+\gamma x} - \frac{B}{Z_0} e^{j\omega t-\gamma x} \tag{14}$$

These equations contain terms of the form $f_1(\omega t + \gamma x)$ and $f_2(\omega t - \gamma x)$, indicating the traveling-wave nature of the solution. In Fig. 1, the distance x is measured from the receiving end of the line. The terms containing $e^{j\omega t+\gamma x}$ represent waves traveling in the $-x$ direction and are therefore the *outgoing* waves of voltage and current. The terms containing $e^{j\omega t-\gamma x}$ represent waves traveling in the $+x$ direction and constitute the reflected waves of voltage and current. The ratio of voltage to current for either the outgoing or reflected wave is equal to the characteristic impedance of the line.

The constants A and B in the transmission-line equations may be evaluated in terms of known boundary conditions. Let us evaluate these in terms of the conditions at the receiving end of the line. At the receiving end we have $x = 0$, $V = V_R$, $I = I_R$ and $Z_R = V_R/I_R$. Equations (10) and (11) then become $V_R = A + B$ and $I_R = (1/Z_0)(A - B)$, from which we obtain

$$A = \frac{V_R}{2}\left(1 + \frac{Z_0}{Z_R}\right) \qquad B = \frac{V_R}{2}\left(1 - \frac{Z_0}{Z_R}\right) \tag{15}$$

Inserting these into Eqs. (10) and (11), and using $V_R/I_R = Z_R$, we obtain the transmission-line equations

$$V = \frac{V_R}{2}\left(1 + \frac{Z_0}{Z_R}\right)e^{\gamma x} + \frac{V_R}{2}\left(1 - \frac{Z_0}{Z_R}\right)e^{-\gamma x} \tag{16}$$

$$I = \frac{I_R}{2}\left(1 + \frac{Z_R}{Z_0}\right)e^{\gamma x} + \frac{I_R}{2}\left(1 - \frac{Z_R}{Z_0}\right)e^{-\gamma x} \tag{17}$$

The terms in Eqs. (16) and (17) may be regrouped to express these equations in hyperbolic function form. The hyperbolic functions are

$$\cosh \gamma x = \frac{e^{\gamma x} + e^{-\gamma x}}{2} \tag{18}$$

$$\sinh \gamma x = \frac{e^{\gamma x} - e^{-\gamma x}}{2} \tag{19}$$

$$\tanh \gamma x = \frac{e^{\gamma x} - e^{-\gamma x}}{e^{\gamma x} + e^{-\gamma x}} \tag{20}$$

In hyperbolic function form, Eqs. (16) and (17) become

$$V = V_R\left(\cosh \gamma x + \frac{Z_0}{Z_R}\sinh \gamma x\right) \tag{21}$$

$$I = I_R\left(\cosh \gamma x + \frac{Z_R}{Z_0}\sinh \gamma x\right) \tag{22}$$

$$Z = \frac{V}{I} = Z_0\left(\frac{Z_R + Z_0 \tanh \gamma x}{Z_0 + Z_R \tanh \gamma x}\right) \tag{23}$$

The impedance Z is the ratio of voltage to current at any point on the line distant x from the receiving end. This is also the impedance which would be obtained if the line were cut at the point x and the impedance were measured looking toward the load.

8.03. Line Terminated in Its Characteristic Impedance.—If a transmission line is terminated in an impedance equal to its characteristic impedance, *i.e.*, if $Z_R = Z_0$, then the reflected-wave terms in Eqs. (8.02-16 and 17) vanish, leaving only the outgoing waves,

$$V = V_R e^{\gamma x} \tag{1}$$

$$I = I_R e^{\gamma x} \tag{2}$$

$$Z = \frac{V}{I} = Z_0 \tag{3}$$

Therefore, if the line is terminated in an impedance equal to its characteristic impedance, the impedance at any point on the line is equal to the characteristic impedance of the line. All of the energy in the outgoing wave is then absorbed in the terminating impedance and there is no reflection.

It is interesting to express Eqs. (1) and (2) in terms of the conditions at the sending end of the line. At the sending end we have $x = l$, $V = V_S$, and $I = I_S$. Equations (1) and (2) then become

$$V_S = V_R e^{\gamma l} \tag{4}$$

$$I_S = I_R e^{\gamma l} \tag{5}$$

Solving these for V_R and I_R and inserting these into Eqs. (1) and (2), with the additional substitution $s = l - x$, where s is the distance from the sending end to the point where V and I are taken, we have

$$V = V_S e^{-\gamma s} = V_S e^{-\alpha s} e^{-j\beta s} \tag{6}$$

$$I = I_S e^{-\gamma s} = I_S e^{-\alpha s} e^{-j\beta s} \tag{7}$$

... in these equations signifies a decrease in magnitude or an atte... of the outgoing wave as it travels toward the receiving end of the line. The attenuation constant α is given in nepers per unit length of line. The factor $e^{-j\beta s}$ denotes a phase shift of βs radians in the distance s, or β radians per unit length of line. Figure 3 shows the variation of the magnitude of the voltage or current as a function of distance for a line terminated in an impedance equal to its characteristic impedance.

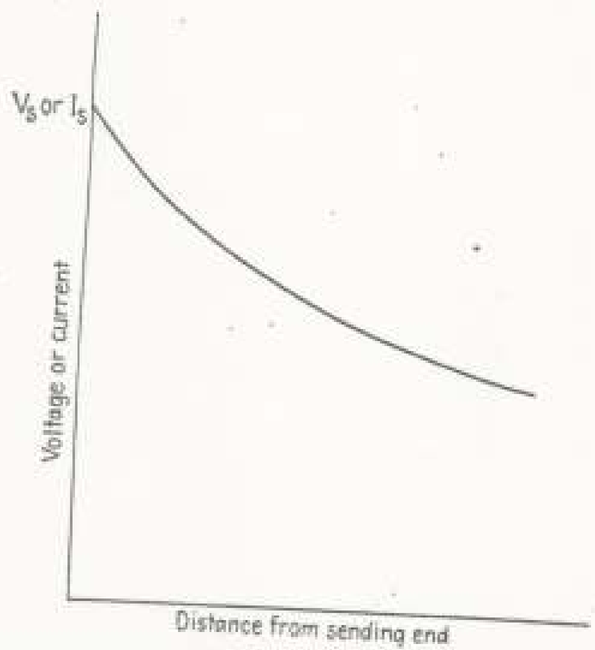


FIG. 3.—Magnitude of voltage and current as a function of distance along a line terminated in its characteristic impedance.

The voltages and currents in the above equations are the amplitudes or peak values of the sinusoidally varying functions. The scalar amplitudes are $|V| = V_S e^{-\alpha s}$ and $|I| = I_S e^{-\alpha s}$. For low-loss lines the voltage and current are in phase; the power flow at any point on a line terminated in its characteristic impedance is:

$$P = \tfrac{1}{2}|V||I|$$

$$= \frac{|V_S|^2}{2Z_0}e^{-2\alpha s} \qquad (8)$$

The power loss per unit length of line is the space rate of decrease of the transmitted power, or $P_L = -(dP/dx)$. Inserting Eq. (8), w... in

$$P_L = 2\alpha\left(\frac{|V_S|^2}{2Z_0}e^{-2\alpha s}\right) = 2\alpha P$$

$$\alpha = \frac{P_L}{2P} \qquad (9)$$

Consequently, the attenuation constant is the ratio of the power loss per unit length of line to twice the transmitted power.

8.04. Propagation Constant and Characteristic Impedance.—The propagation constant γ contains an attenuation constant α and a phase constant β. The phase constant represents the number of radians of phase shift per unit length of line. The *wavelength* λ is the distance required for a phase shift of 2π radians, or

$$\lambda = \frac{2\pi}{\beta} \qquad (1)$$

The *phase velocity* v is the product of the frequency times wavelength, or

$$v = f\lambda = \frac{\omega}{\beta} \qquad (2)$$

The propagation constant γ and characteristic impedance Z_0 are dependent upon the series impedance z and shunt admittance y per unit length of line as expressed by Eqs. (8.02-9 and 12).

For most transmission lines operating at frequencies above 100 kilocycles we find that $\omega L \gg R$ and $\omega C \gg G$, *i.e.*, the reactance and susceptance, are large in comparison with the resistance and conductance. We shall refer to such lines as low-loss lines. Simplified expressions may be derived for γ and Z_0 for this case. Expanding Eqs. (8.02-9 and 12) by the binomial series and retaining the first few terms of the series, we obtain

$$\gamma = \sqrt{zy} = (R + j\omega L)^{1/2}(G + j\omega C)^{1/2}$$

$$\approx \left(\frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}}\right) + j\omega\sqrt{LC} \qquad (3)$$

$$Z_0 = \sqrt{\frac{z}{y}} = (R + j\omega L)^{1/2}(G + j\omega C)^{-1/2}$$

$$\approx \sqrt{\frac{L}{C}}\left[1 - j\left(\frac{R}{2\omega L} - \frac{G}{2\omega C}\right)\right] \qquad (4)$$

we have assumed that $\omega L \gg R$ and $\omega C \gg G$, the characteristic imped as given by Eq. (4), is substantially a pure resistance of value

$$Z_0 = \sqrt{\frac{L}{C}} \tag{5}$$

The real part of Eq. (3) is the attenuation constant, whereas the imaginary part is the phase constant. Inserting Eq. (5) into (3), we obtain

$$\alpha = \frac{R}{2Z_0} + \frac{GZ_0}{2} \tag{6}$$

$$\beta = \omega\sqrt{LC} \tag{7}$$

The wavelength and phase velocity for the low-loss line are

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}} \tag{8}$$

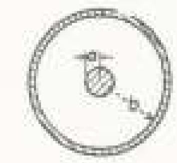
$$v = f\lambda = \frac{1}{\sqrt{LC}} \tag{9}$$

For a lossless line, *i.e.*, $R = G = 0$, the attenuation constant is zero and the outgoing and reflected waves experience no attenuation as they travel along the line. It can be shown that the phase velocity for a lossless line is equal to the velocity of light. The effect of losses in the line is to decrease both the wavelength λ and phase velocity v and to introduce attenuation in the line.

8.05. Transmission-line Parameters.—The R, L, C, and G parameters of several different types of transmission lines are given in Table 1. The resistance given in this table is the skin-effect resistance as computed by the methods of Chap. 15. The conductance is that resulting from dielectric losses in the insulating medium. The attenuation constants α_c and α_d are those resulting from losses in the conductor and dielectric, respectively. The total attenuation constant is the sum of the two terms, or $\alpha = \alpha_c + \alpha_d$. The attenuation constants and characteristic impedance equations are for low-loss lines.

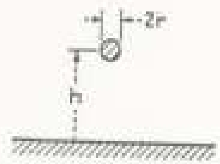
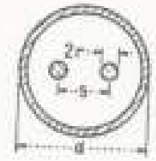
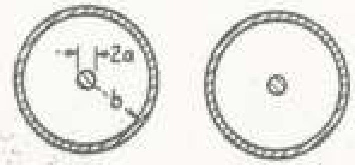
The skin-effect resistance varies inversely as the radius of the conductor. In general, therefore, the attenuation constant decreases as the radius increases. A coaxial line has minimum attenuation for the ratio $b/a = 3.6$, corresponding to a characteristic impedance of approximately 77 ohms.

TABLE 1.—TRANSMISSION-LINE CONSTANTS

		
Inductance, L henrys per meter of line	$\frac{\mu_0}{\pi}\cosh^{-1}\frac{d}{2r}$ or $\frac{\mu_0}{\pi}\ln\frac{d}{r}$ if $d \gg r$	$\frac{\mu_0}{2\pi}\ln\frac{b}{a}$
Capacitance, C farads per meter of line	$\frac{\pi\epsilon}{\cosh^{-1} d/2r}$ or $\frac{\pi\epsilon}{\ln d/r}$ if $d \gg r$	$\frac{2\pi\epsilon}{\ln b/a}$
Resistance, R ohms per meter of line	$\frac{1}{r}\sqrt{\frac{\mu_0 f}{\pi\sigma}}$	$\left(\frac{1}{a}+\frac{1}{b}\right)\sqrt{\frac{\mu_0 f}{4\pi\sigma}}$
Conductance, G mhos per meter of line	ωDC	ωDC
Characteristic impedance in ohms $Z_0 = \sqrt{\frac{L}{C}}$	$120\cosh^{-1}\frac{d}{2r}$ or $276\log_{10}\frac{d}{r}$ if $d \gg r$	$\frac{138}{\sqrt{\epsilon_r}}\log_{10}\frac{b}{a}$
Attenuation constant due to conductor losses $\alpha_c = \frac{R}{2Z_0}$	$\frac{\sqrt{\mu_0 f/\pi\sigma}}{552\, r\log_{10} d/r}$	$\frac{[(1/a)+(1/b)]\sqrt{\mu_0\epsilon_r f/4\pi\sigma}}{276\log_{10} b/a}$
Attenuation constant due to dielectric losses $\alpha_d = \frac{GZ_0}{2}$	$\frac{GZ_0}{2}$	$\frac{GZ_0}{2}$

$\mu_0 = 4\pi \times 10^{-7}$ henry/meter
$\epsilon = \epsilon_0\epsilon_r$
$\epsilon_0 = 8.85 \times 10^{-12}$ farad/meter
ϵ_r = relative permittivity (dielectric constant)
D = dissipation factor of dielectric (for low-loss dielectrics, D = P.F., where P.F. is the power factor of the dielectric. See Appendix II for power factors of typical dielectrics.)
σ = conductivity of conductor
= 5.80×10^7 mhos/meter for copper
= 6.14×10^7 mhos/meter for silver

TABLE 2.

Transmission-line configuration	Characteristic impedance $Z_0 = \sqrt{\frac{L}{C}}$
	$Z_0 = 138 \log_{10} \frac{2h}{r}$
	$Z_0 = 276 \log_{10} \frac{d}{r}$
	$Z_0 = \frac{120}{\sqrt{\epsilon_r}} \left\{ \ln \left[2a \left(\frac{1-b^2}{1+b^2} \right) - \frac{1+4a^2}{16a^4} (1 - 4b^2) \right] \right\}$ where $a = \frac{s}{2r}$ $b = \frac{s}{d}$
	$Z_0 = \frac{138 \log_{10} b/a}{\sqrt{1 + (w/s)(\epsilon_r - 1)}}$ $s \ll \lambda$
	$Z_0 = \frac{276 \log_{10} b/a}{\sqrt{1 + (w/s)(\epsilon_r - 1)}}$ $s \ll \lambda$

If a line is terminated by an impedance equal to its charact[illegible] impedance, the decibel loss in the line is

$$\text{db} = 20 \log_{10} \frac{|V_S|}{|V_R|} = 20 \log_{10} \epsilon^{\alpha x} = 8.686\alpha \qquad (1)$$

Table 2 gives the characteristic impedance of several of the more common types of transmission lines.

8.06. Lossless Line Equations.—In most microwave transmission lines we have $\omega L \gg R$ and $\omega C \gg G$ and, consequently, the transmission lines have characteristics approximating those of a lossless line. Let us therefore consider the transmission-line equations for the theoretical case of a lossless line.

A lossless line would have zero attenuation constant and hence $\gamma = j\beta$. Hyperbolic functions of imaginary angles may be written as trigonometric functions of real angles, that is

$$\sinh j\beta x = j \sin \beta x \qquad \tanh j\beta x = j \tan \beta x$$
$$\cosh j\beta x = \cos \beta x \qquad \coth j\beta x = -j \cot \beta x \qquad (1)$$

For the lossless line, Eqs. (8.02-21, 22, and 23) become

$$V = V_R \left(\cos \beta x + j \frac{Z_0}{Z_R} \sin \beta x \right) \qquad (2)$$

$$I = I_R \left(\cos \beta x + j \frac{Z_R}{Z_0} \sin \beta x \right) \qquad (3)$$

$$Z = Z_0 \left(\frac{Z_R + jZ_0 \tan \beta x}{Z_0 + jZ_R \tan \beta x} \right) \qquad (4)$$

Consider the case of a lossless line which is short-circuited at the distant end. We then have $Z_R = 0$ and $V_R = 0$. Remembering that $I_R = V_R/Z_R$, Eqs. (2), (3), and (4) become

$$V = jI_R Z_0 \sin \beta x \qquad (5)$$
$$I = I_R \cos \beta x \qquad (6)$$
$$Z = jZ_0 \tan \beta x \qquad (7)$$

Equations (5) and (6) represent *standing waves* of voltage and current on the line as shown in Fig. 4. The standing wave is produced by a combination of an outgoing wave and a reflected wave, traveling in opposite directions on the line. Figure 4 may be visualized as representing the amplitudes of voltages and currents which have a sinusoidal time variation. The voltage is zero at the receiving end and has its maximum value at points corresponding to $\beta x = n\pi/2$ or $x = n\lambda/4$, where n is any odd integer. The voltage and current are in space quadrature, as evidenced by Fig. 4, and also in time quadrature as indicated by the j term in Eq. (5).

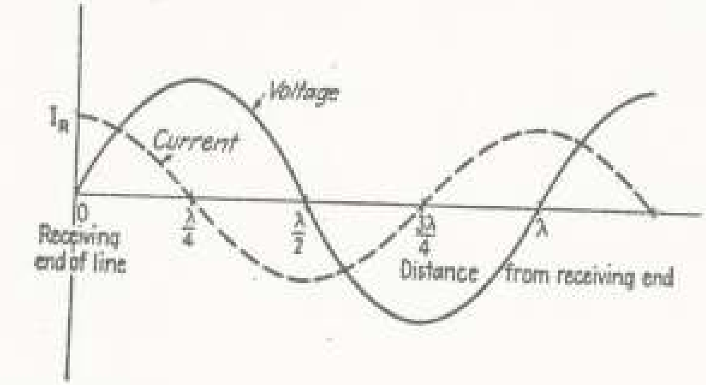


FIG. 4.—Standing waves of voltage and current on a short-circuited lossless line.

The impedance of the short-circuited lossless line, as given by Eq. (7), is plotted in Fig. 5. The input impedance is a pure reactance, alternating between capacitive and inductive reactance as βx increases. Antiresonance occurs when the line is an odd integral number of quarter wavelengths long and resonance occurs when it is an even integral number of quarter wavelengths long. The antiresonant input impedance of a lossless line

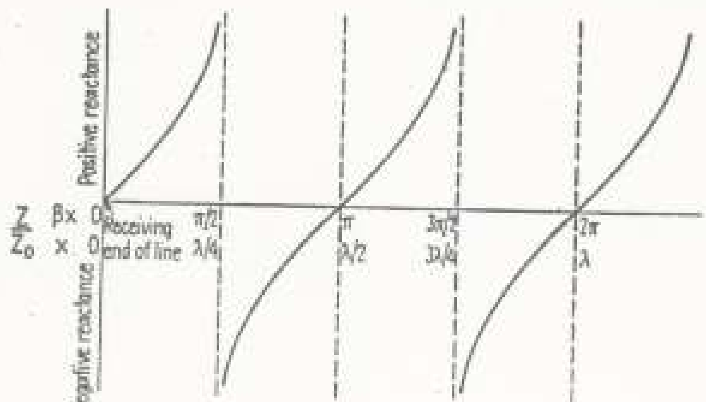


FIG. 5.—Impedance ratio Z/Z_0 for a short-circuited lossless line as a function of x and βx.

would be a pure resistance of theoretically infinite value, whereas the resonant impedance would be zero. In the practical case of a low-loss line the impedance is a very large pure resistance for antiresonance and a very small pure resistance for resonance.

Now consider the lossless line which is open-circuited at the distant end. For this case we have $Z_R = \infty$, $I_R = 0$, and $V_R = I_R Z_R$. Equations (2), (3), and (4) then become

$$V = V_R \cos \beta x \tag{8}$$

$$I = j\frac{V_R}{Z_0} \sin \beta x \tag{9}$$

$$Z = -jZ_0 \cot \beta x \tag{10}$$

The voltage and current standing waves are similar to those shown in Fig. 4, but with voltage and current interchanged. The impedance is a

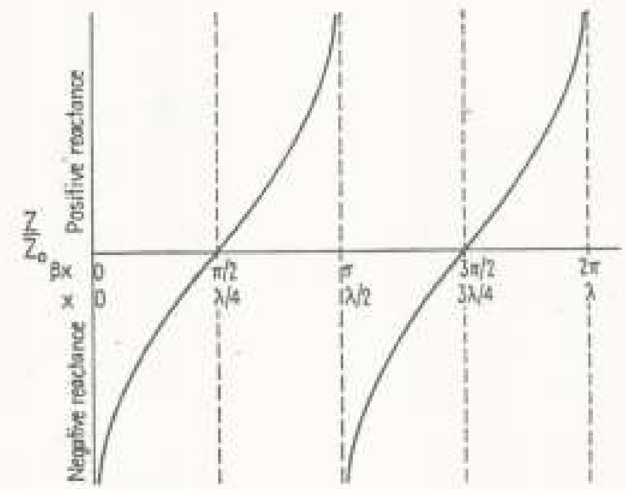


FIG. 6.—Impedance ratio Z/Z_0 for an open-circuited lossless line as a function of x and βx.

pure reactance as shown in Fig. 6. Resonance occurs when the line is an odd integral number of quarter wavelengths long and antiresonance when it is an even integral number of quarter wavelengths long.

We can now conclude that the short-circuited and open-circuited lines are either resonant or antiresonant when the length l is $l = n\lambda/4$ or when $\beta l = n\pi/2$, where n is given by

	Short-circuited line	Open-circuited line
Resonance..........	n = even integer	n = odd integer
Antiresonance..........	n = odd integer	n = even integer

and tl ... with Losses.—If the line loss is short-circuited at the distant end, we again ... $V_R = 0$, and $I_R = V_R/Z_R$. Equations (8.02-21, 22, and 23) then reduce to

$$V = I_R Z_0 \sinh \gamma x \quad (1)$$

$$I = I_R \cosh \gamma x \quad (2)$$

$$Z = Z_0 \tanh \gamma x \quad (3)$$

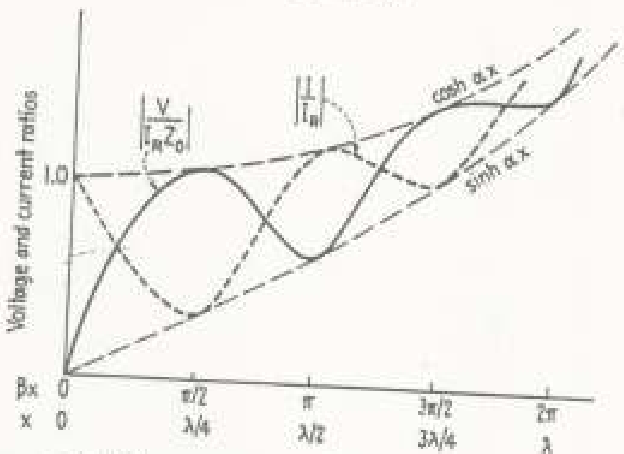


FIG. 7.—Voltage ratio $\left|\frac{V}{I_R Z_0}\right|$ and current ratio $\left|\frac{I}{I_R}\right|$ as a function of x and βx for a short-circuited line having losses.

Inserting $\gamma = \alpha + j\beta$ into these equations and applying the identities for the hyperbolic function of the sum of two angles,[1] we obtain

$$\frac{V}{I_R Z_0} = \sinh \alpha x \cos \beta x + j \cosh \alpha x \sin \beta x \quad (4)$$

$$\frac{I}{I_R} = \cosh \alpha x \cos \beta x + j \sinh \alpha x \sin \beta x \quad (5)$$

$$\frac{Z}{Z_0} = \frac{\tanh \alpha x + j \tan \beta x}{1 + j \tanh \alpha x \tan \beta x} \quad (6)$$

The scalar values of $|V/I_R Z_0|$ and $|I/I_R|$ are plotted against x and βx in Fig. 7. Equation (4) shows that when $\beta x = n\pi/2$, the ratio $|V/I_R Z_0|$ has the value $\cosh \alpha x$ or $\sinh \alpha x$, depending upon whether n is an odd or even integer. Likewise, Eq. (5) shows that when $\beta x = n\pi/2$, the current ratio I/I_R has the values $\cosh \alpha x$ and $\sinh \alpha x$ for even and odd integer values of n, respectively. The curves $\cosh \alpha x$ and $\sinh \alpha x$ therefore represent the envelope of the curves $|V/I_R Z_0|$ and $|I/I_R|$, as shown in Fig. 7. Since I_R and Z_0 are independent of x the ratio $|V/I_R Z_0|$ represents the voltage distribution along the line, and the ratio $|I/I_R|$ represents the current distribution.

Referring to Eq. (6), we find that the scalar impedance ratio $|Z/Z_0|$ varies between the limits $\tanh \alpha x$ and $\coth \alpha x$ as shown in Fig. 8. At the

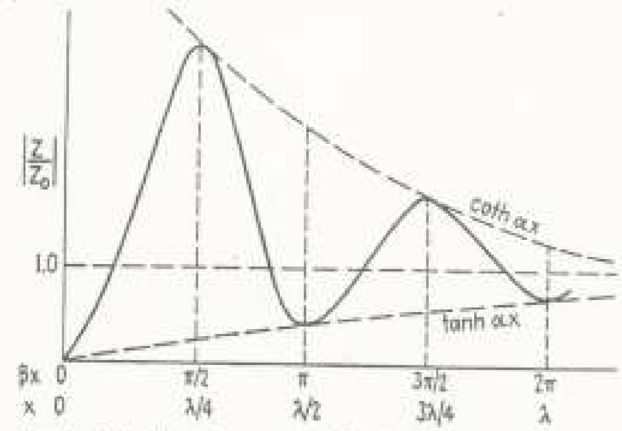


FIG. 8.—Magnitude of the impedance ratio Z/Z_0 for a short-circuited line having losses.

points where the impedance has its maximum and minimum values, the impedances are approximately $Z = Z_0 \coth \alpha x$ and $Z = Z_0 \tanh \alpha x$, respectively.

Figures 7 and 8 represent scalar values, hence all values are plotted as positive quantities. For the lossless line, the curves in Figs. 7 and 8 would degenerate to curves similar to those in Figs. 4 and 5 but with all values plotted as positive quantities.

The variable βx in the above figures may be written $\beta x = \omega x/v$. Consequently, we may consider the above curves as being plotted either against frequency, with line length held constant, or against length of line, with frequency held constant.

8.08. Receiving End Open-circuited.—If the receiving end is open-circuited, we have $Z_R = \infty$, $I_R = 0$; hence Eqs. (8.02-21, 22, 23) yield

$$V = V_R \cosh \gamma x \quad (1)$$

$$I = \frac{V_R}{Z_0} \sinh \gamma x \quad (2)$$

$$Z = Z_0 \coth \gamma x \quad (3)$$

Comparison of these equations with Eqs. (8.07-1, 2, and 3) shows that the ratio $|V/V_R|$ for the open-circuited line has a variation with γx similar

[1] See, for example, B. O. PEIRCE, "A Short Table of Integrals," Ginn and Company, Boston, 1929.

... ratio of the short-circuited line, whereas the ratio $|IZ_0/V_R|$ of the ... -circuited line is similar to the voltage ratio of the short-circuited line. The impedance ratio $|Z/Z_0|$ for the open-circuited line is

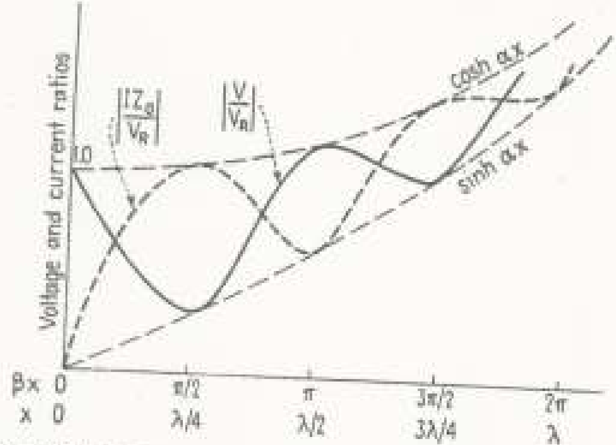


FIG. 9.—Voltage and current ratios for an open-circuited line having losses.

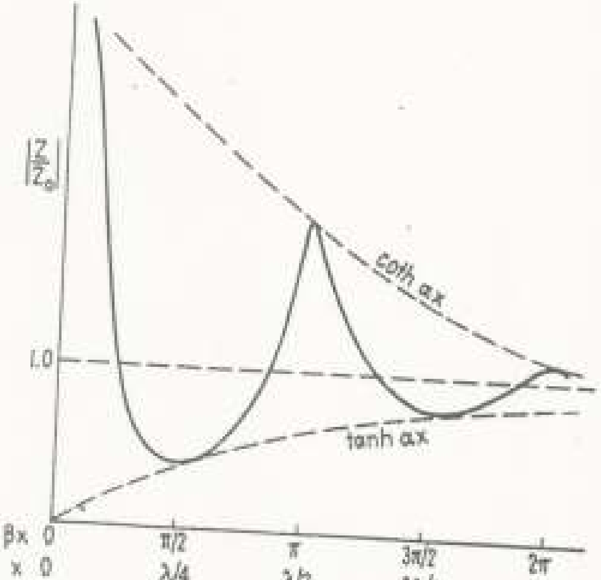


FIG. 10.—Magnitude of the impedance ratio for an open-circuited line having losses.

the reciprocal of that of the short-circuited line. These ratios are shown in Figs. 9 and 10.

8.09. Sending-end Equations.—Thus far we have dealt largely with the transmission-line equations expressed in terms of the voltage, current, and impedance at the receiving end of the line. If the sending-end ...e V_s and current I_s are known, we may readily obtain V_R and I_R ... riting Eqs. (8.02-16 and 17) or (8.02-21 and 22) for the full length of the line (letting $x = l$) and substituting the known values of V_s and I_s for V and I. These equations may then be solved for V_R and I_R.

If only the generated voltage V_g of Fig. 1 is known, it is then necessary to first compute the input impedance of the line by Eq. (8.02-23). The impedance as seen by the generator voltage V_g is the sum of the input impedance to the line and the generator impedance Z_g. The sending-end voltage and current may therefore be obtained from

$$I_s = \frac{V_g}{Z + Z_g} \tag{1}$$

$$V_s = V_g - I_s Z_g \tag{2}$$

where Z is the input impedance of the line.

PROBLEMS

1. Show that the binomial expansion of the terms given in Eqs. (8.04-3 and 4) yields the approximations indicated in these equations.
2. A coaxial line has dimensions $a = 0.75$ cm and $b = 3$ cm. The dielectric is polystyrene with a dielectric constant of 2.5 and power factor 0.0004. Compute the following values at a frequency of 500 megacycles:
   (a) Inductance and capacitance per meter of line.
   (b) Conductance and skin-effect resistance per meter of line.
   (c) Attenuation constant and phase constant.
   (d) Wavelength and phase velocity.
   (e) Input impedance of a quarter-wavelength section of line if (1) short-circuited and (2) open-circuited.
3. A lossless line is one-eighth of a wavelength long and is terminated by a pure resistance which is approximately equal to the characteristic impedance of the line. Show that if the value of the terminating resistance is varied by a small amount either side of the value $R_L = Z_0$, the input impedance of the line will contain a reactive component, the magnitude of which varies directly with R, whereas the resistive component remains substantially constant.
4. A high-frequency voltmeter is constructed of a quarter-wavelength lossless line which is terminated in the heater junction of a thermocouple having a resistance of R ohms. The thermocouple leads are connected to a microammeter.
   (a) Derive an expression for the voltage V_s at the sending end in terms of the current I_R through the thermocouple heater.
   (b) Derive an equation for the input impedance to the line. How does this vary with the value of the thermocouple resistance?
   (c) Compute the input voltage and input impedance if $R = 5$ ohms, $Z_0 = 75$ ohms and $I_R = 15$ ma.

CHAPTER 9

GRAPHICAL SOLUTION OF TRANSMISSION-LINE PROBLEMS

A number of ingenious circle diagrams have been devised to facilitate the graphical solution of transmission-line problems. Basically, all of them spring from the same fundamental relationships which are expressed in the transmission-line equations. In this chapter, we shall consider two types of impedance diagrams, these being referred to as the *rectangular impedance diagram* and the *polar impedance diagram*. First, however, let us derive the transmission-line equations in reflection-coefficient form, since they will be useful in the construction of the impedance diagrams.

9.01. Reflection-coefficient Equations.—The reflection coefficient r_R is defined as the ratio of the reflected voltage to the outgoing voltage at the receiving end of the line. In Eq. (8.02-15) the terms A and B represent the outgoing and reflected voltages at the load, respectively. The reflection coefficient is therefore

$$r_R = \frac{Z_R - Z_0}{Z_R + Z_0} \tag{1}$$

Let us now express the transmission-line equations in reflection-coefficient form. In Eqs. (8.02-16 and 17), let $V'_R = V_R/2[1 + (Z_0/Z_R)]$, where V'_R is the outgoing-voltage wave at the load. Now factor out the term $V'_R e^{\gamma x}$ and, with the substitution of r_R from Eq. (1), we obtain

$$V = V'_R e^{\gamma x}(1 + r_R e^{-2\gamma x}) \tag{2}$$

$$I = \frac{V'_R}{Z_0} e^{\gamma x}(1 - r_R e^{-2\gamma x}) \tag{3}$$

$$Z = \frac{V}{I} = Z_0 \left(\frac{1 + r_R e^{-2\gamma x}}{1 - r_R e^{-2\gamma x}}\right) \tag{4}$$

Equations (2), (3), and (4) are the transmission-line equations expressed in terms of the reflection coefficient. They may be used to evaluate the voltage, current, and impedance at any point on the line. The terms $V'_R e^{\gamma x}$ and $(V'_R/Z_0)e^{\gamma x}$ in these equations represent the outgoing waves of voltage and current, respectively, whereas the terms $r_R V'_R e^{-\gamma x}$ and $-r_R(V'_R/Z_0)e^{-\gamma x}$ represent the reflected waves.

If the line is terminated in an impedance equal to its characteristic impedance, Eq. (1) shows that the reflection coefficient is zero and consequently the reflected-wave terms in Eqs. (2) and (3) are zero, leaving only the outgoing waves.

If the line is short-circuited at the distant end, we have $Z_R = 0$ and $r_R = -1$; for an open-circuited line, we have $Z_R = \infty$ and $r_R = 1$. In the more general case, r_R is complex and may be readily evaluated using Eq. (1).

9.02. The Rectangular Impedance Diagram.[1]—In the construction of impedance diagrams, it is convenient to express the reflection coefficient as an exponential quantity. We therefore let

$$r_R = e^{-2(t_0 + ju_0)} \tag{1}$$

where e^{-2t_0} is the magnitude and $-2u_0$ is the angle of the reflection coefficient. Now let

$$t = t_0 + \alpha x \tag{2}$$

$$u = u_0 + \beta x \tag{3}$$

In Eq. (9.01-4) the term $r_R e^{-2\gamma x}$ then becomes $r_R e^{-2\gamma x} = e^{-2(t+ju)}$ and the impedance ratio may be written:

$$\frac{Z}{Z_0} = \frac{1 + e^{-2(t+ju)}}{1 - e^{-2(t+ju)}} \tag{4}$$

At the receiving end of the line, we have $Z = Z_R$, $\alpha x = 0$, and $\beta x = 0$; therefore Eq. (4) becomes

$$\frac{Z_R}{Z_0} = \frac{1 + e^{-2(t_0+ju_0)}}{1 - e^{-2(t_0+ju_0)}} \tag{5}$$

Equation (4) is the basic relationship used in the construction of impedance diagrams. The impedance diagram is essentially a plot of Eq. (4) which enables us to obtain values of Z/Z_0 if the values of t and u are known, or conversely, to obtain values of t and u if Z/Z_0 is known. When impedances are expressed as a ratio, such as Z/Z_0, they are known as *normalized impedances*.

For convenience, let us separate the real and imaginary parts of Z/Z_0 in Eq. (4), letting[2]

$$\frac{Z}{Z_0} = r + jx = \frac{1 + e^{-2(t+ju)}}{1 - e^{-2(t+ju)}} \tag{6}$$

[1] The theory of the impedance diagram presented here is similar to that given by W. Jackson and L. G. Huxley in The Solution of Transmission-line Problems by the Use of the Circle Diagram of Impedance, *J.I.E.E.* (*London*), vol. 91, part 3, pp. 105–127; September, 1944.

[2] For low-loss lines it may be assumed that Z_0 is real. Writing $Z = R + jX$, we obtain $Z/Z_0 = (R/Z_0) + j(X/Z_0)$. Here $r = R/Z_0$ corresponds to the normalized resistance and $x = X/Z_0$ is the normalized reactance.

...ngular impedance diagram is a plot of Eq. (6) in the form of consta... ...nd constant-u loci in the $r + jx$ plane as shown in Fig. 1. If t is held constant in Eq. (6) and u is allowed to vary, the locus of r and x for various values of u may be shown to be a circle with its center on the r axis, distant coth $2t$ from the origin, and with a radius cosech $2t$. On the

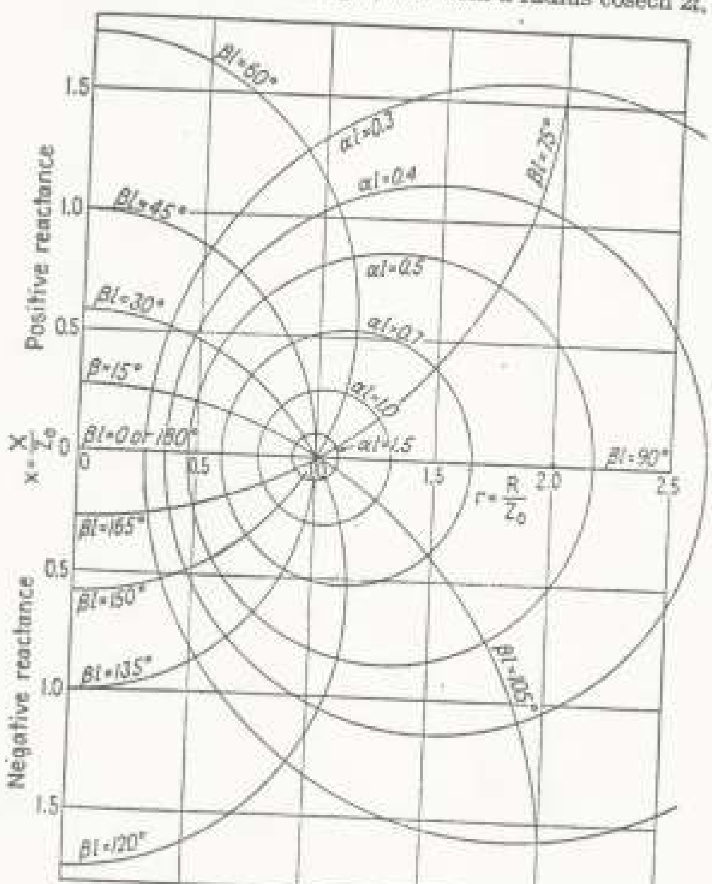


FIG. 1.—Rectangular impedance diagram.

other hand, if u is held constant and t is allowed to vary, the locus of r and x is a circle centered on the x axis, distant $-$ cot $2t$ from the origin and with a radius cosec $2u$. Since t is related to αl by Eq. (2) and u related to βl by Eq. (3), it is customary to designate the constant-t and constant-u circles as αl and βl circles, respectively, as shown in Fig. 1. These constitute two families of orthogonal bipolar circles.[1] Any point in the impedance diagram represents a value of $r + jx$ and a value of $t + ju$ (read on the αl and βl circles), these two quantities being related by Eq. (6).

Let us now trace the steps which are necessary to evaluate the input impedance of a transmission line terminated in a known impedance Z_R.

It is assumed that Z_R, Z_0, α, β, and the length of line l are known. The procedure is as follows:

1. Compute the values of Z_R/Z_0, αl, and βl.
2. Enter the chart at the known value of Z_R/Z_0 and observe the corresponding values of t_0 and u_0 (on the αl and βl circles).
3. Compute the values of $t = t_0 + \alpha l$ and $u = u_0 + \beta l$. These are the values of t and u corresponding to the sending end of the line.
4. Reenter the chart at the new values t and u (on the αl and βl circles) and read the corresponding impedance Z/Z_0. This is the normalized input impedance.

As an example, assume that $Z_R/Z_0 = 2 + j0$, $\alpha l = 0.2$ nepers, and $\beta l = 0.6$ radians. Entering the impedance diagram of Fig. 1 at $Z_R/Z_0 = 2 + j0$, we obtain the values of $t_0 = 0.534$ and $u_0 = 90°$ or 1.57 radians. Step 3 above yields $t = 0.734$ and $u = 124°$ or 2.17 radians. Reentering the impedance diagram at these values of t and u, the normalized input impedance is found to be $Z/Z_0 = 1.08 - j0.5$.

9.03. Polar Impedance Diagram.—In the rectangular impedance diagram, the circle $\alpha l = 0$ has an infinitely large radius. Therefore an infinitely large diagram would be required to solve all possible problems. When dealing with low-loss lines which are open-circuited, short-circuited, or terminated in a pure reactance, the solution is sometimes found to be beyond the limits of any practical diagram. In the polar impedance diagram, introduced by P. H. Smith,[1,2] the entire impedance diagram is contained within a circle of any desired radius.

The general plan of construction of the polar impedance diagram will be given here, followed by several illustrative problems. A more detailed description of the construction of the diagram is given in Sec. 9.07.

In the polar impedance diagram, the impedance $Z/Z_0 = r + jx$ and the quantity $t + ju$ in Eq. (9.02-6) are both related to a new variable $p + jq$. Let

$$p + jq = \epsilon^{-2(t+ju)} \tag{1}$$

and Eq. (9.02-6) then becomes

$$\frac{Z}{Z_0} = r + jx = \frac{1 + (p + jq)}{1 - (p + jq)} \tag{2}$$

[1] Bipolar circles of the type shown in Fig. 1 are encountered in a number of engineering applications. For example, the αl and βl circles correspond to the electric and magnetic field lines of a two-conductor transmission line.

[1] SMITH, P. H., Transmission-Line Calculator, *Electronics*, vol. 12, pp. 29–31; January, 1939.

[2] SMITH, P. H., An Improved Transmission-Line Calculator, *Electronics*, vol. 17, p. 130; January, 1944.

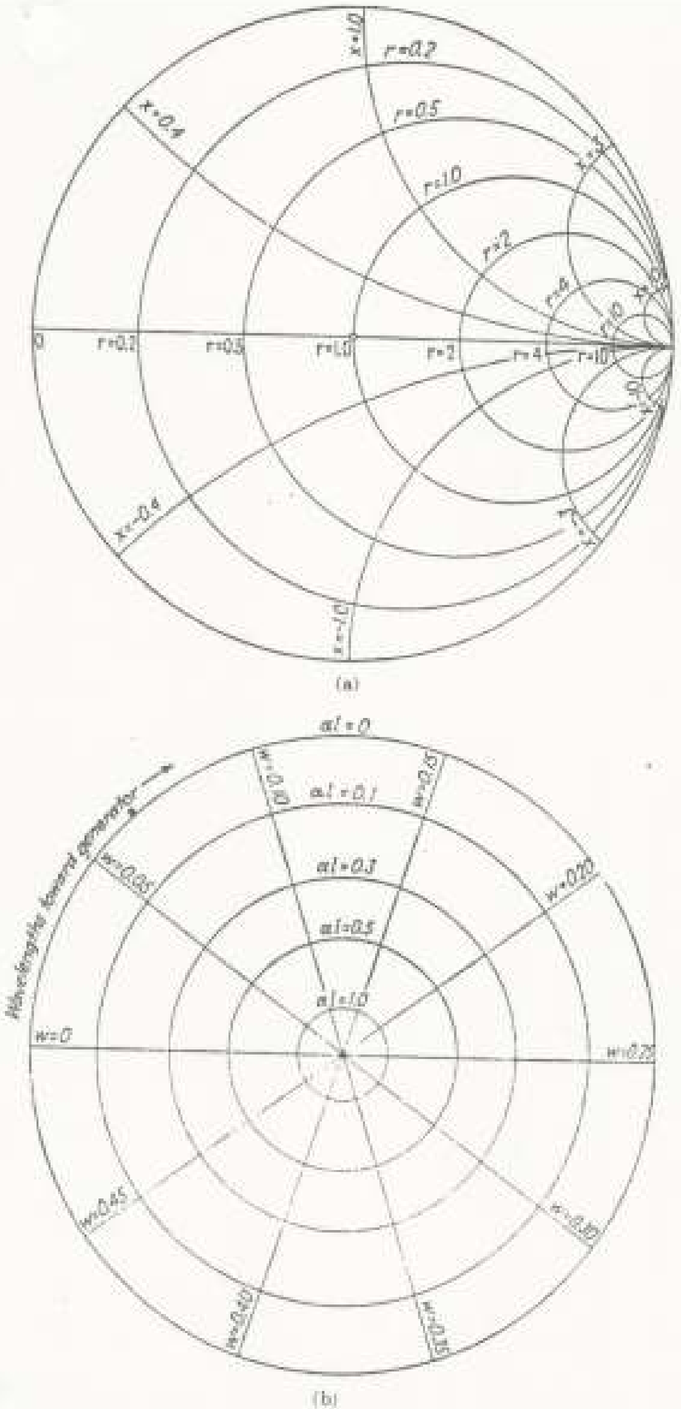


FIG. 2.—Polar impedance diagram. (a) Constant-r and constant-x loci; (b) constant-t and constant-w loci.

Let p and q represent the rectangular coordinate axes. Equ. (2) may be used to obtain families of constant-r and constant-x loci in the $p + jq$ plane. Similarly, Eq. (1) may be used to plot families of constant-t and constant-u loci in the $p + jq$ plane. Any point in the polar impedance diagram then defines three quantities, (1) a value of $r + jx$, (2) a value of $t + ju$, and (3) a value of $p + jq$. These three quantities are interrelated by Eqs. (1) and (2). In the solution of transmission-line problems, we are interested in obtaining values of $r + jx$ corresponding to known values of $t + ju$, or vice versa. Once the diagram has been constructed, the p and q coordinate axes have no further use and, therefore, are omitted in the final impedance diagram.

The constant-r and constant-x loci form two families of orthogonal circles in the $p + jq$ plane as shown in Fig. 2a. The constant-r circles have centers on the p axis, distant $r/(1 + r)$ from the origin, and have a radius of $1/(r + 1)$. The entire impedance diagram is contained in a circle of unit radius, with center at the origin. The constant-x circles have centers at $p = 1$, $q = 1/x$ and have radii $1/x$. The upper half of the diagram of Fig. 2a represents positive reactance, whereas the lower half represents negative reactance. The constant-r and constant-x circles all pass through the point (1, 0).

The constant-t loci consist of a family of circles having centers at the origin and radii e^{-2t}. These are designated [1] αl in Fig. 2b. The constant-u loci are radial lines passing through the origin. However, it is more convenient to replace u by a new variable $w = u/2\pi$. Substituting u from Eq. (9.02-3), with $\beta = 2\pi/\lambda$ and $w_0 = u_0/2\pi$, we obtain

$$w = w_0 + \frac{x}{\lambda} \tag{3}$$

Therefore, w is a measure of the length of line in wavelengths. The constant-w lines are also radial lines passing through the origin, with a slope $-\tan 4\pi w$ as shown in Fig. 2b.

The polar impedance diagram of Fig. 3 is obtained by superimposing Figs. 2a and 2b. To simplify the final diagram, the constant-w lines have been omitted, although the values are given on the scales marked "wave-lengths toward the generator" and "wavelengths toward the load" along the outer rim of the diagram.

[1] The impedance diagram of Fig. 3 contains constant-αl circles rather than the decibel or standing-wave-ratio circles which are usually included on the polar diagram. The αl circles facilitate the solution of problems where the line has losses. The method of solving problems including the effect of line losses is practically the same for either the rectangular or the polar diagram. The standing-wave ratio may be readily obtained from Fig. 5 after the value of t_0 has been determined from either impedance diagram.

It is interesting to observe that both the rectangular and polar diagrams may be used to solve problems in terms of admittances as well as in terms of impedances. The normalized admittance at any point on the line is the reciprocal of the normalized impedance, thus Eq. (9.02-4) may be written

$$\frac{Y}{Y_0} = \frac{1 - e^{-2(t+jw)}}{1 + e^{-2(t+jw)}}$$

where $Y_0 = 1/Z_0$ is the characteristic admittance of the line. Equation (4) gives the same families of curves in the rectangular and polar diagrams as Eq. (9.02-4). The constant-r and constant-x circles become constant-conductance and constant-susceptance circles, respectively, when dealing with admittances. Otherwise, the use of the diagram is exactly the same for either impedances or admittances.

9.04. Use of the Polar Impedance Diagram.—The polar impedance diagram is used in much the same manner as the rectangular impedance diagram. To determine the input impedance of a line terminated in a known impedance, the procedure is as follows:

1. Compute Z_R/Z_0, αl, and l/λ.
2. Enter the impedance diagram at the known value of Z_R/Z_0 and read the corresponding values of t_0 on the αl circles and w_0 on the "wavelengths toward the generator" scale.
3. Compute the values of $t = t_0 + \alpha l$ and $w = w_0 + (l/\lambda)$.
4. Reenter the diagram at the new values of t and w and read the corresponding normalized sending-end impedance.

Referring to Fig. 4, assume that point P corresponds to the terminal impedance Z_R/Z_0 and that the line is lossless. As we move toward the generator on the transmission line, the impedance point moves in the clockwise direction on the constant-αl circle along the path PQ_1. However, if the line has losses, then the impedance point spirals inward as indicated by the path PQ_2. As the length of the line increases, the impedance point continues to spiral inward, eventually winding up on the point $Z/Z_0 = 1$.

If we move toward the generator on the transmission line, the impedance point moves in the clockwise direction in the impedance diagram and the values of w are read on the "wavelengths toward the generator" scale. If we move toward the load, the impedance point in the diagram moves in the counterclockwise direction and the values of w are read on the "wavelengths toward the load" scale. One complete revolution on the impedance diagram corresponds to a half wavelength of line.

9.05. Standing-wave Ratio.—The standing-wave ratio for a lossless line is defined as $\rho = \frac{|V_{max}|}{|V_{min}|}$, where $|V_{max}|$ and $|V_{min}|$ are the magnitudes of the maximum and minimum voltages. The standing-wave ratio is

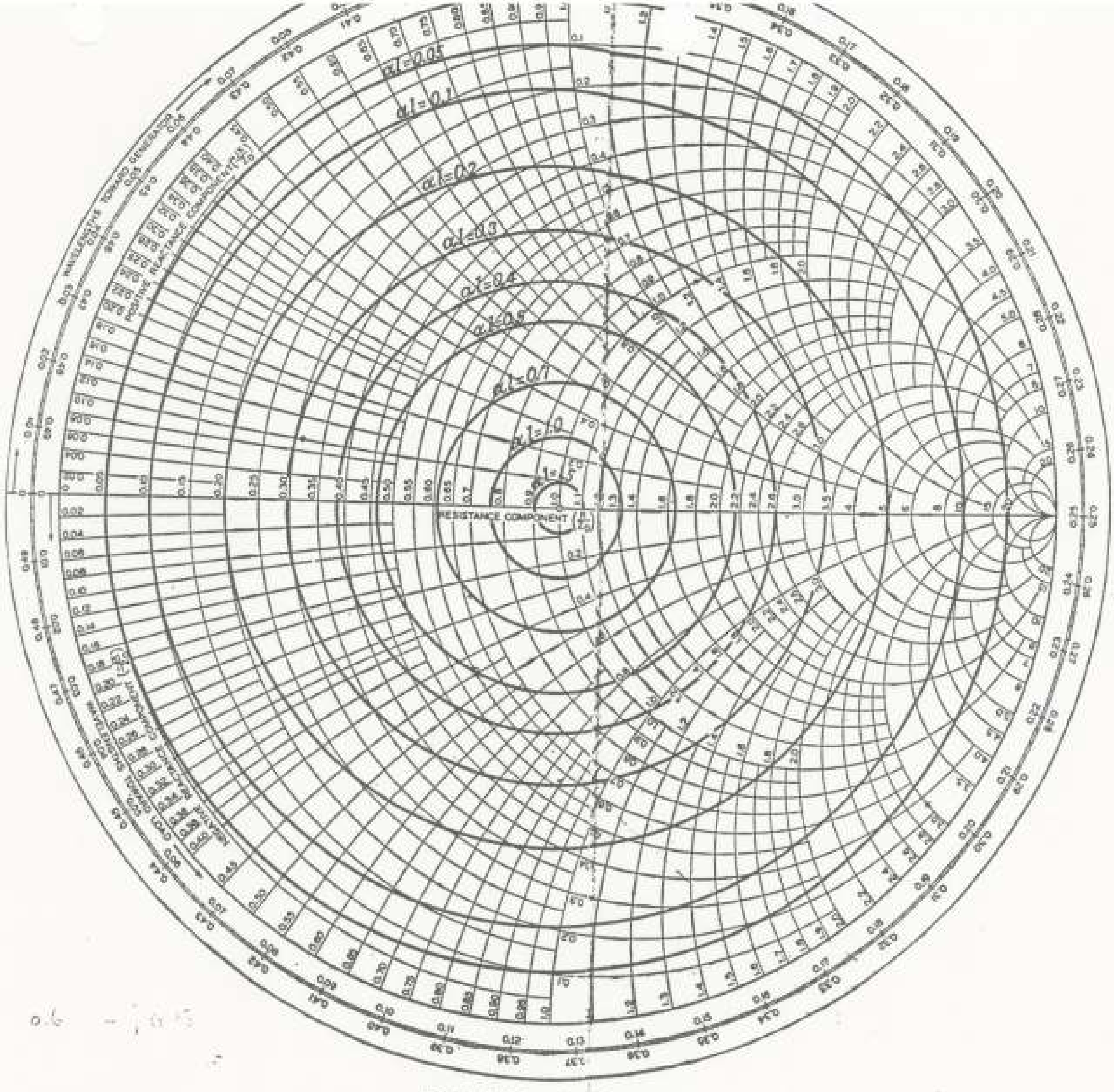


FIG. 3.—Polar impedance diagram.

onsiderable importance, since it is a quantity which can readi com-
ted from laboratory measurements.

To obtain expressions for the maximum and minimum voltages, we re-
rn to Eq. (9.01-2) and substitute Eq. (9.02-1) for r_R. For a lossless line,
= 0 and we have

$$V = V'_R e^{j\beta x}[1 + e^{-2u_0} e^{-2j(v_0 + \beta x)}] \tag{1}$$

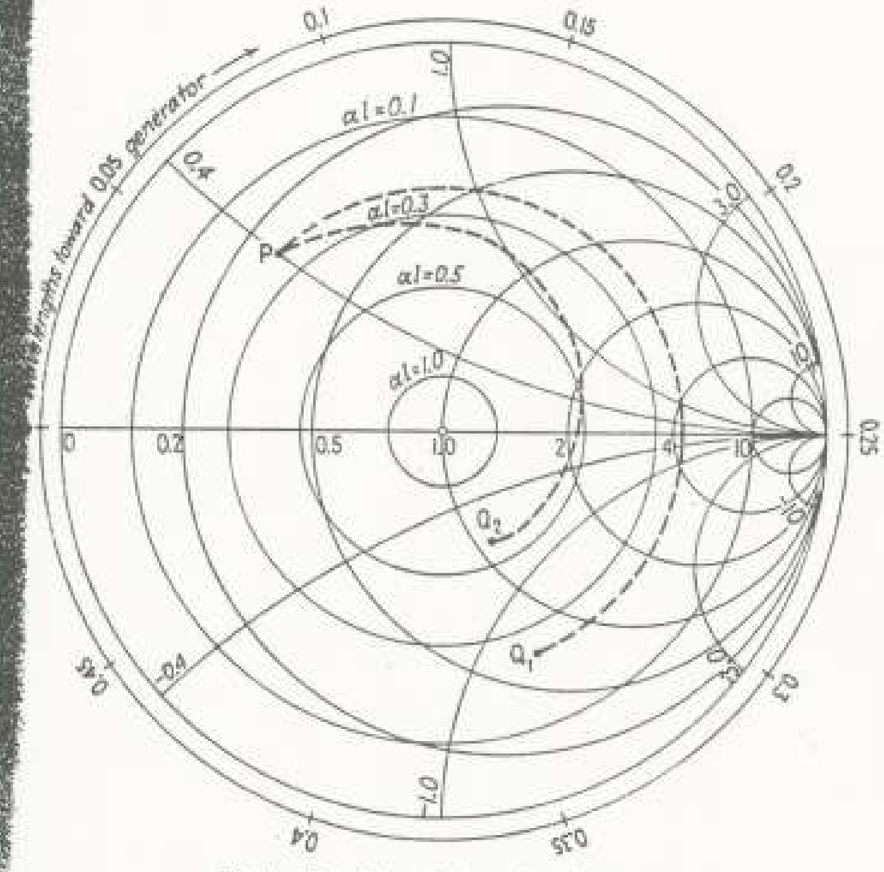


FIG. 4.—Use of the polar impedance diagram.

Maximum voltage occurs at that point on the line which makes the second
term inside the bracket in phase with the first term. This occurs at that
point on the line where $e^{-2j(v_0 + \beta x_{max})} = 1$, yielding

$$|V_{max}| = V'_R(1 + e^{-2u_0}) \tag{2}$$

Minimum voltage occurs when $e^{-2j(v_0 + \beta x_{min})} = -1$, yielding

$$|V_{min}| = V'_R(1 - e^{-2u_0}) \tag{3}$$

...ing-wave ratio is therefore

$$\rho = \left|\frac{V_{max}}{V_{min}}\right| = \frac{1 + e^{-2t_0}}{1 - e^{-2t_0}} = \frac{1 + |r_R|}{1 - |r_R|} \qquad (4)$$

where $|r_R| = e^{-2t_0}$ is the scalar magnitude of the reflection coefficient.

Figure 5 shows a graph of the standing-wave ratio as a function of t_0. If the normalized impedance Z_R/Z_0 is known, the value of t_0 may be obtained from Fig. 3, and Fig. 5 may then be used to determine the standing-wave ratio.

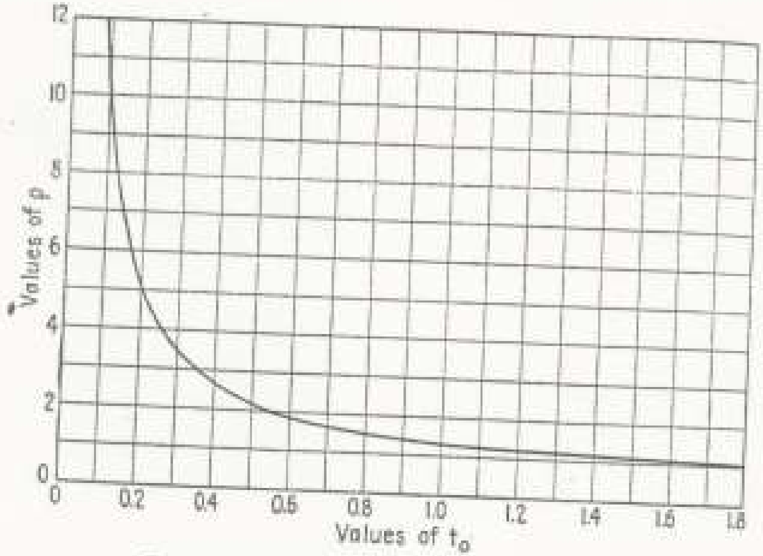


FIG. 5.—Standing-wave ratio ρ as a function of t_0.

The maximum and minimum voltages, respectively, occur at points on the line where

$$2(u_0 + \beta x_{max}) = n\pi \qquad n \text{ is even} \qquad (5)$$

$$2(u_0 + \beta x_{min}) = n\pi \qquad n \text{ is odd} \qquad (6)$$

These may be expressed as

$$\frac{x_{max}}{\lambda} = \frac{n}{4} - \frac{u_0}{2\pi} = \frac{n}{4} - w_0 \qquad n \text{ is even} \qquad (7)$$

$$\frac{x_{min}}{\lambda} = \frac{n}{4} - \frac{u_0}{2\pi} = \frac{n}{4} - w_0 \qquad n \text{ is odd} \qquad (8)$$

where $-2u_0$ is the phase angle of the reflection coefficient as given by Eq. (9.02-1).

If a line has attenuation, the successive values of $|V_{max}|$ and $|V_{min}|$ vary along the line and consequently there is no fixed value for the standing wave ratio.

9.06. Illustrative Examples.—The use of the polar impedance diagram will now be illustrated by several examples.

Example 1. Determine the input impedance of a line having the following characteristics:

$$Z_0 = 75 \text{ ohms} \qquad \frac{l}{\lambda} = 0.2$$

$$Z_R = 150 + j100 \text{ ohms} \qquad \alpha l = 0.15 \text{ neper}$$

We first compute $Z_R/Z_0 = 2 + j1.333$. Entering the polar impedance diagram at this value of normalized impedance, we obtain $t_0 = 0.35$ and $w_0 = 0.210$ (on the "wavelengths toward the generator" scale). Now compute $t = t_0 + \alpha l = 0.50$ and $w = w_0 + (l/\lambda) = 0.410$. These are the values of t and w at the sending end of the line. Reentering the impedance diagram at these values of t and w, we obtain the normalized input impedance $Z/Z_0 = 0.60 - j0.46$, or $Z = 45 - j34.5$ ohms. If the line were lossless the standing-wave ratio from Fig. 5, corresponding to $t_0 = 0.350$ would be $\rho = 2.97$.

Example 2. A lossless line is terminated in a capacitance such that $Z_R/Z_0 = 0 - j0.5$. Determine the lengths of line required for (a) resonance ($Z/Z_0 = 0$) and (b) antiresonance ($Z/Z_0 = \infty$).

The outer circle of the impedance diagram corresponds to zero resistance. Entering the diagram at $Z_R/Z_0 = 0 - j0.5$, we obtain $t_0 = 0$ and $w_0 = 0.425$. The lengths of line required for the impedance point to move to (a) $Z/Z_0 = 0$ and (b) $Z/Z_0 = \infty$ are

(a) $l = (0.5 - 0.425)\lambda = 0.075\lambda$.

(b) $l = (0.75 - 0.425)\lambda = 0.325\lambda$.

The antiresonant line is a quarter wavelength longer than the resonant line.

Example 3. A line which is 0.4λ long is short-circuited at one end and has a lumped impedance (normalized) of $Z_1/Z_0 = 0.5 + j0.2$ shunted across the input terminals. The value of αl is 0.2 nepers. Find the normalized input impedance.

In dealing with parallel circuits, it is advisable to use admittances. The impedance diagram may be used to convert impedances into admittances and vice versa. To determine the normalized admittance Y_1/Y_0, enter the impedance diagram at the point corresponding to the normalized impedance $Z_1/Z_0 = 0.5 + j0.2$. The normalized admittance is halfway around the diagram on the same constant-αl circle (or on a straight line through the center of the impedance diagram to the same constant-αl circle). This gives $Y_1/Y_0 = 1.72 - j0.72$.

Now consider the input admittance to the line only (with Y_1 disconnected). The admittance of the short circuit is $Y_R = \infty$ or $Y_R/Y_0 = \infty$. Entering the diagram at this value of admittance, we read $t_0 = 0$ and $w_0 = 0.25$. At the sending end of the line, we have $t = t_0 + \alpha l = 0.2$ and $w = w_0 + (l/\lambda) = 0.65$. The w scale returns to zero at $w = 0.5$; hence the point corresponding to $w = 0.65$ is 0.15 beyond the zero point (on the "wavelengths toward the generator" scale). The normalized input admittance of the line is, therefore, $Y_L/Y_0 = 0.54 + j1.23$. The normalized input admittance with Y_1 connected is then $Y/Y_0 = (Y_L/Y_0) + (Y_1/Y_0) = 2.26 + j0.51$. The normalized impedance is obtained by entering the diagram at $Y/Y_0 = 2.26 + j0.51$ and proceeding halfway around the diagram. Thus the input impedance is $Z/Z_0 = 0.43 - j0.09$.

9.07. Construction of the Polar Impedance Diagram.—The polar diagram consists of families of constant r, x, t, and w loci plotted in the $p + jq$ plane. In order to construct the diagram, it is first necessary to obtain

... relating each of the quantities r, x, t, and w, to p and q. The equat... determine the respective loci.

Returning to Eq. (9.03-2), we add 1.0 to both sides of the equation yielding

$$(r+1) + jx = \frac{2}{1-(p+jq)} \qquad (1)$$

To separate the real and imaginary parts, multiply the numerator and denominator of the right-hand side by the conjugate of the denominator yielding

$$(r+1) + jx = \frac{2(1-p)}{(1-p)^2+q^2} + \frac{j2q}{(1-p)^2+q^2} \qquad (2)$$

Equating the real and imaginary terms on both sides of Eq. (2), we obtain

$$r+1 = \frac{2(1-p)}{(1-p)^2+q^2} \qquad (3)$$

$$x = \frac{2q}{(1-p)^2+q^2} \qquad (4)$$

Equation (3) may be written as

$$p^2 - \frac{2pr}{r+1} + q^2 = -\frac{r-1}{r+1} \qquad (5)$$

Adding $r^2/(r+1)^2$ to both sides of the equation to complete the square gives

$$\left(p - \frac{r}{r+1}\right)^2 + q^2 = \frac{1}{(r+1)^2} \qquad (6)$$

This is the equation of the constant-r circles in Figs. 2a and 3. The centers of the circles are on the p axis at $r/(r+1)$ and the radii are $1/(r+1)$. The circle corresponding to $r = 0$ has unit radius and is centered at the origin in the $p + jq$ plane. The circle $r = \infty$ has zero radius and is centered at $p = 1$, $q = 0$.

To obtain the equation expressing the loci of the constant-x circles, (4) is rearranged to give

$$(p-1)^2 + q^2 - \frac{2q}{x} = 0$$

Adding $1/x^2$ to both sides of the equation completes the square and gives

$$(p-1)^2 + \left(q - \frac{1}{x}\right)^2 = \frac{1}{x^2} \qquad (7)$$

This is the equation of the constant-x circles, with centers at $p = 1$, $q = 1/x$, and radii $1/x$. The circle $x = \infty$ has zero radius.

To obtain the constant-t and constant-w circles, we return to Eq (...-1) and write this in the form

$$p + jq = \epsilon^{-2t}(\cos 2u - j \sin 2u) \qquad (9)$$

Separating real and imaginary parts, we obtain

$$p = \epsilon^{-2t} \cos 2u \qquad (10)$$

$$q = -\epsilon^{-2t} \sin 2u \qquad (11)$$

Squaring these two equations and adding gives the equation of the constant-t circles.

$$p^2 + q^2 = \epsilon^{-4t} \qquad (12)$$

These circles are centered at the origin and have radii ϵ^{-2t}.

Dividing Eqs. (10) and (11) and inverting, we obtain

$$\frac{q}{p} = -\tan 2u \qquad (13)$$

Substitution of $u = 2\pi w$ into Eq. (13) gives

$$\frac{q}{p} = -\tan 4\pi w \qquad (14)$$

Hence the constant-w loci are straight lines passing through the origin and having a slope of $-\tan 4\pi w$.

PROBLEMS

1. Derive the equations for the constant-t and constant-u circles in the $r + jx$ plane of the rectangular impedance diagram. Show that these are circles and give the location of the centers and values of the radii.
2. A lossless coaxial line, having a characteristic impedance of 50 ohms, is $\frac{3}{8}$ of a wavelength long and is terminated in an impedance $Z_R = 75 + j60$ ohms. A condenser, having a capacitance of 4×10^{-12} farads is connected in series with the center conductor one-eighth of a wavelength from the receiving end. Using the impedance diagram, find the input impedance of the line at a frequency of 1,000 megacycles.
3. A line having a characteristic impedance of 50 ohms is used in the grid-plate circuit of an oscillator. The grid-plate capacitance is 1.8×10^{-12} farads and the line is tuned by means of a small air condenser at the far end. If the line is 15 centimeters long what value of capacitance would be required if the oscillator is to have a frequency of 600 megacycles?
4. A generator is connected to a load impedance by means of two coaxial lines in cascade. The first line is 0.6λ long and has $Z_{01} = 50$ ohms and $\alpha = 0.4$ neper per meter. The second line is 0.4λ long and has $Z_{02} = 75$ ohms and $\alpha = 0.3$ neper per meter. The load impedance is $Z_R = 45 - j75$ and the wavelength is $\lambda = 1$ meter. What is the input impedance?
5. A generator having an internal impedance Z_g is connected to a lossless transmission line having a characteristic impedance Z_0. Show that if $Z_g = Z_0$, the voltage at the receiving end of the open-circuited line will be equal to one-half the internal voltage of the generator, regardless of the length of line.

CHAPTER 10

TRANSMISSION-LINE NETWORKS

Transmission lines are often used as network elements in microwave systems. They may be used as resonant or antiresonant circuits, as reactive circuit elements in filter networks, as impedance transformers, as attenuators, or as circuit elements in various types of measuring systems.

Lumped-parameter networks are usually unsatisfactory at microwave frequencies, since the values of inductance and capacitance used in such networks are extremely small and slight variations due to mechanical vibration, temperature effects, etc., seriously alter the characteristics of the network. Transmission-line networks offer the advantages of greater stability, ease of adjustment, and much higher Q's than are possible with lumped-parameter circuits. The basic principles of transmission-line networks will be considered in this chapter.

10.01. Resonant and Antiresonant Lines.—The properties of lossless lines were considered in Sec. 8.06. Let us now see what effect losses have upon the input impedances of open-circuited and short-circuited lines.

The input impedance of short-circuited and open-circuited lines of length l are obtained by substituting l for x in Eqs. (8.07-3) and (8.08-3), respectively, yielding

$$Z = Z_0 \tanh \gamma l \quad \text{short-circuited line} \quad (8.07\text{-}3)$$

$$Z = Z_0 \coth \gamma l \quad \text{open-circuited line} \quad (8.08\text{-}3)$$

Now substitute $\gamma = \alpha + j\beta$ into these equations and use the identities $\tanh(\alpha + j\beta)l = \dfrac{\tanh \alpha l + j \tan \beta l}{1 + j \tanh \alpha l \tan \beta l}$ and $\coth \gamma l = \dfrac{1}{\tanh \gamma l}$ to obtain

$$Z = Z_0 \left(\frac{\tanh \alpha l + j \tan \beta l}{1 + j \tanh \alpha l \tan \beta l} \right) \quad \text{short-circuited line}$$

$$Z = Z_0 \left(\frac{1 + j \tanh \alpha l \tan \beta l}{\tanh \alpha l + j \tan \beta l} \right) \quad \text{open-circuited line}$$

For resonance or antiresonance, we have $\beta l = n\pi/2$, where n is an odd or even integer as specified at the end of Sec. 8.06. Consider the short-circuited line. Resonance occurs when n is even and antiresonance when n is odd. Inserting the corresponding values of βl into Eq. (1), together with the approximation $\tanh \alpha l \approx \alpha l$ for small values of αl, we [obtain] the resonant and antiresonant impedances

$$Z = Z_0 \tanh \alpha l \approx Z_0 \alpha l \qquad \text{resonance} \qquad (3)$$

$$Z = Z_0 \coth \alpha l \approx \frac{Z_0}{\alpha l} \qquad \text{antiresonance} \qquad (4)$$

The resonant and antiresonant impedances of open-circuited lines are also given by Eqs. (3) and (4), respectively.

Let us now investigate the variation of impedance resulting from small frequency deviations either side of the resonant or antiresonant frequency. Let $\beta l = (n\pi/2) + \delta$, where δ is a small angular departure from the resonant or antiresonant value of βl. We then have

$$\tan \beta l = \frac{\tan (n\pi/2) + \tan \delta}{1 - \tan (n\pi/2) \tan \delta}$$

If n is even, this becomes $\tan \beta l = \tan \delta \approx \delta$, and if n is odd, $\tan \beta l = -(1/\tan \delta) \approx -(1/\delta)$. For the short-circuited line in the vicinity of resonance, n is even. Insertion of $\tan \beta l = \delta$ and $\tanh \alpha l = \alpha l$ into Eq. (1), gives

$$Z = Z_0 \left(\frac{\alpha l + j\delta}{1 + j\delta \alpha l} \right) \qquad \text{resonance} \qquad (5)$$

Similarly, for the short-circuited line in the vicinity of antiresonance, we have n odd, $\tan \beta l = -1/\delta$, and $\tanh \alpha l = \alpha l$, yielding

$$Z = Z_0 \left(\frac{1 + j\delta \alpha l}{\alpha l + j\delta} \right) \qquad \text{antiresonance} \qquad (6)$$

Equations (5) and (6) apply equally well for the open-circuited line in the vicinity of resonance and antiresonance. For small values of αl and δ, we have $\delta \alpha l \ll 1$, and the scalar values of the input impedance become approximately

$$Z = Z_0 \sqrt{(\alpha l)^2 + \delta^2} \qquad \text{resonance} \qquad (7)$$

$$Z = \frac{Z_0}{\sqrt{(\alpha l)^2 + \delta^2}} \qquad \text{antiresonance} \qquad (8)$$

It now remains to relate δ to the frequency. Let ω be the impressed angular frequency and ω_0 be the resonant or antiresonant angular frequency. From our previous assumption, we have $\beta l = \omega l/v = (n\pi/2) + \delta$. At the resonant or antiresonant frequency, we have $\omega_0 l/v = n\pi/2$. Combining these two expressions, we obtain

$$\delta = (\omega - \omega_0) \frac{l}{v} \qquad (9)$$

... the qu... $\omega - \omega_0$ represents the difference between the impressed angular ...ency and the resonant or antiresonant angular frequency.

10.02. The Q of Resonant and Antiresonant Lines.—One of the principal advantages of transmission-line networks is the high values of Q which are attainable by this means. Whereas a Q of 300 represents a relatively high value for lumped L-C circuits, Q's of the order of several thousand are attainable with lines. A high Q implies a high degree of frequency selectivity and therefore a sharply tuned circuit.

The Q of lumped L-C circuits is usually defined by the ratio of reactance to resistance, *i.e.*,

$$Q = \frac{\omega L}{R} \qquad (1)$$

where ωL is the inductive reactance and R is the circuit resistance.

A more general definition of Q, which is applicable to any case of electrical or mechanical resonance, is

$$Q = 2\pi \frac{\text{peak energy storage}}{\text{energy dissipated per cycle}} \qquad (2)$$

Multiplying the numerator and denominator of Eq. (2) by the frequency and remembering that the product of energy dissipation per cycle times frequency is the power loss, we have

$$Q = \omega \frac{\text{peak energy storage}}{\text{average power loss}} = \frac{\omega W_S}{P_L} \qquad (3)$$

where W_S is the peak energy storage and P_L is the time-average power loss.

In order to show that Eqs. (1) and (3) are equivalent, multiply the numerator and denominator of Eq. (1) by $\frac{1}{2}I^2$, where I is the amplitude of the current flowing in the inductance. This gives

$$Q = \omega \frac{\frac{1}{2}LI^2}{\frac{1}{2}I^2R} = \omega \frac{W_S}{P_L} \qquad (4)$$

where $W_S = \frac{1}{2}LI^2$ is the peak energy storage in the inductance and $P_L = \frac{1}{2}I^2R$ is the time-average power loss.

Let us now apply the definition given by Eq. (3) to derive an expression for the Q of resonant and antiresonant lines. The final expression for Q is the same regardless of whether we choose an open-circuited or a short-circuited line operating at resonance or antiresonance.

Consider an antiresonant short-circuited line. The peak energy storage may be evaluated without serious error by assuming that the line is lossless. The voltage and current at any point on the line are then in time quadrature; therefore the energy storage in the electric field has its maximum value when the energy storage in the magnetic field is zero, and vice versa. The peak values of the energy storage in the electric and magnetic fields are equal and we may choose either for the purpose of evaluating the Q.

Consider the peak energy storage in the electric field. The capacitance of a differential length of line dx is $C\,dx$ and the peak energy storage is $dW_S = \frac{1}{2}(C\,dx)V^2$, where V is the voltage amplitude. Substituting the voltage from Eq. (8.06-5) into this expression and integrating between the limits $\beta x = 0$ and $\beta x = n\pi/2$ (where n is odd for the antiresonant line), the peak energy storage becomes

$$W_S = \frac{C(I_R Z_0)^2}{2\beta}\int_0^{n\pi/2} \sin^2 \beta x \, d(\beta x) = \frac{n\pi C}{8\beta}(I_R Z_0)^2 \qquad (5)$$

where I_R is the amplitude of the current at the receiving end.

The time-average power loss is obtained from $P_L = \frac{1}{2}VI\cos\theta$, where V and I are the amplitudes of the voltage and current at the input terminals, and θ is the phase angle between V and I. The voltage and current at the input terminals are found by inserting $\beta l = n\pi/2$, $\cosh \alpha l = 1$, and $\sinh \alpha l = \alpha l$ into Eqs. (8.07-4) and (8.07-5), yielding the scalar values $V = I_R Z_0$ and $I = I_R \alpha l$, and $\theta = 0$. The time-average power loss is therefore

$$P_L = \frac{I_R^2 Z_0 \alpha l}{2} \qquad (6)$$

Inserting Eqs. (5) and (6) into (3), together with $\beta l = n\pi/2$, $Z_0 = \sqrt{L/C}$, and $\beta = \omega\sqrt{LC}$, we obtain an expression for the Q

$$Q = \frac{\beta}{2\alpha} \qquad (7)$$

Thus, the Q is the phase constant divided by twice the attenuation constant. It is also interesting to note that if we neglect the conductance in Eq. (8.04-6) so that $\alpha = R/2Z_0$, and insert this, together with the above expressions for β and Z_0, into Eq. (7), the expression for the Q reduces to the familiar form $Q = \omega L/R$ where L and R are the series inductance and resistance per unit length of line.

The Q is a measure of the frequency selectivity of a resonant or antiresonant circuit. To show this relationship, we return to Eqs. (10.01-7 and 8). Let ω represent the angular frequency for which $\delta = \alpha l$. At this frequency, the input impedance is $\sqrt{2}$ times the resonant impedance, or $1/\sqrt{2}$ times the antiresonant impedance, as the case may be. Substitution of δ from Eq. (10.01-9) into the above expression yields $(\omega - \omega_0)/v_c = \alpha$, where ω_0 is the resonant angular frequency. From Eq. (7) we obtain

$Q = \beta/2$ $\omega_0/2\alpha v_c$; hence

$$Q = \frac{\omega_0}{2(\omega - \omega_0)} = \frac{f_0}{2\Delta f} \tag{8}$$

where f_0 is the resonant frequency and $\Delta f = (\omega - \omega_0)/2\pi$.

The resonant and antiresonant impedances may be expressed in terms of the Q, by inserting $\beta l = n\pi/2$ and α from Eq. (7) into (10.01-3) and (10.01-4), yielding

$$Z = \frac{Z_0 n\pi}{4Q} \qquad \text{resonance} \tag{9}$$

$$Z = \frac{4Z_0Q}{n\pi} \qquad \text{antiresonance} \tag{10}$$

In general, the Q of a line is increased by increasing either the size of the conductors or the spacing between conductors. Increasing the size of the conductors decreases the skin-effect resistance, whereas increasing the spacing between conductors increases the inductance per unit length of line. However, in open-wire lines, the radiation losses increase as the separation distance increases.

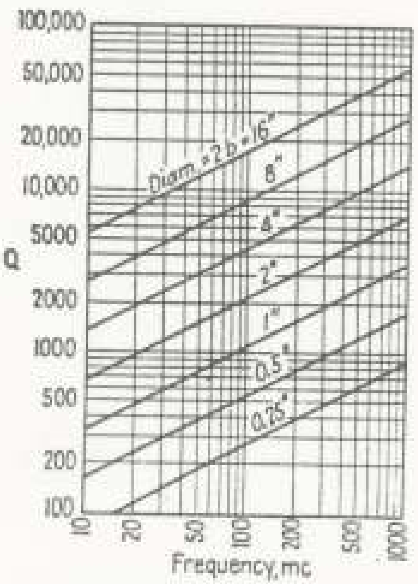


FIG. 1.—Q of copper coaxial lines having the optimum ratio $b/a = 3.6$.

For the coaxial line, the attenuation constant, as given in Table 1, becomes a minimum when the ratio b/a is 3.6. This corresponds to a characteristic impedance of approximately 77 ohms. Since β is independent of b and a, it follows from Eq. (7) that maximum Q likewise occurs when $b/a = 3.6$. Figure 1 is a plot of the Q of copper coaxial lines as a function of frequency for various sizes of lines, all having the optimum value $b/a = 3.6$.

10.03. Lines with Reactance Termination.—Lines are often used as tuned elements in vacuum-tube circuits where they are shunted by the interelectrode capacitances and conductances of the tube. A line may be tuned by means of a small variable condenser shunted across either end of the line. Let us observe what effect lumped reactances have upon the resonant and antiresonant frequencies.

In high-Q circuits the resistance has very little effect upo resonant frequency; hence we shall confine our attention to lossless li. which are terminated by pure reactances. Consider the lines shown in Fig. 2, with lumped reactances X_1 at the sending end and X_R at the receiving end. In Fig. 2a the reactance X_1 is assumed to be connected in series with a zero-impedance generator. This is equivalent to the series L-C circuit of Fig. 2b, the input impedance having zero value at the resonant frequency. In Fig. 2c, the reactance X_1 is assumed to be shunted across an infinite-impedance generator, this being analogous to the parallel L-C circuit of Fig. 2d.

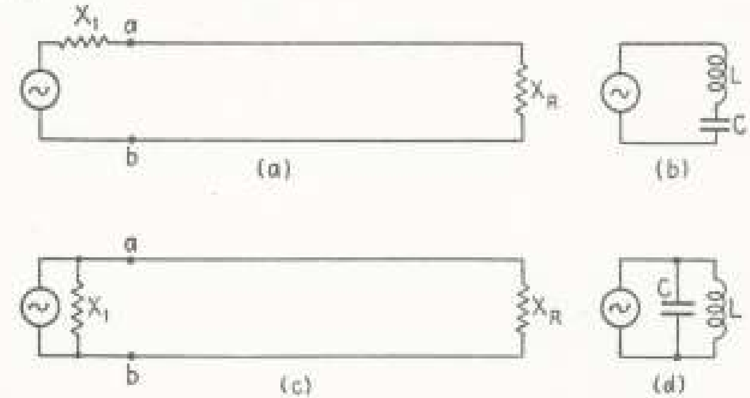


FIG. 2.—Lines with reactance terminations and their lumped-circuit equivalents.

A condition of resonance exists when the reactances looking both ways at any pair of terminals, such as at ab in Fig. 2a or 2c, are equal in magnitude and opposite in sign. In Fig. 2a this results in a resonant input impedance, whereas in Fig. 2b, the input impedance is antiresonant.

The input reactance of the line alone at terminals ab is obtained from Eq. (8.06-4),

$$Z = jZ_0\left(\frac{X_R + Z_0 \tan \beta l}{Z_0 - X_R \tan \beta l}\right) \tag{1}$$

Applying the above criterion, resonance or antiresonance occurs when

$$X_1 = -Z_0\left(\frac{X_R + Z_0 \tan \beta l}{Z_0 - X_R \tan \beta l}\right) \tag{2}$$

Solving for $\tan \beta l$ and substituting $\beta = \omega/v_c$, we have

$$\tan \frac{\omega l}{v_c} = Z_0\left(\frac{X_1 + X_R}{X_1X_R - Z_0^2}\right) \tag{3}$$

In these equations X_1 and X_R take positive values for inductive reactance and negative values for capacitive reactance.

If the [illegible] is short-circuited at the receiving end, we have $X_R = 0$, and Eq. (3) [illegible]es to

$$\tan \frac{\omega l}{v_c} = -\frac{X_1}{Z_0} \tag{4}$$

and if open-circuited, $X_R = \infty$, and

$$\tan \frac{\omega l}{v_c} = \frac{Z_0}{X_1} \tag{5}$$

If there is no reactance at the sending end, we set $X_1 = 0$ for the series connection and $X_1 = \infty$ for the parallel connection.

Resonant lines have a multiplicity of resonant and antiresonant frequencies which may be either harmonically or inharmonically related. If the line contains no lumped reactance, the resonant and antiresonant frequencies are harmonically related. For example, the short-circuited line without reactance termination at either end is resonant when $\tan \omega l/v_c = 0$ and antiresonant when $\tan \omega l/v_c = \pm\infty$. The corresponding resonant and antiresonant frequencies are given by

$$f = \frac{nv_c}{4l} \quad \text{or} \quad l = \frac{n\lambda}{4} \tag{6}$$

where n is an even integer for resonance and odd for antiresonance. The resonant and antiresonant frequencies are therefore harmonically related.

In general, if the line is terminated by lumped reactances the resonant and antiresonant frequencies are not harmonically related. This is apparent since the frequency appears on both sides of Eq. (3). The question sometimes arises: how can we design a line having reactance terminations so as to be simultaneously resonant or antiresonant at any two or more specified frequencies? Such a problem might arise if we were to design an oscillator or amplifier to deliver a relatively large output at a harmonic of the fundamental frequency. The interelectrode capacitance of the tube constitutes a lumped reactance shunting the input end of the line, as shown in Fig. 2. Therefore the antiresonant frequencies will, in general, be inharmonically related unless we specifically design the circuit to have harmonic antiresonant frequencies.

Equation (10.03-3) shows that there are four variables which we are at liberty to adjust. These are the two terminating reactances, the characteristic impedance of the line, and the length of line. By proper adjustment of the four variables, it should be possible to make the circuit either resonant or antiresonant at four specified frequencies which may be either harmonically or inharmonically related.

As a specific example, consider a short-circuited line with a capacitance C shunted across its input terminals. We wish to find the length of line and value of C required to make the circuit antiresonant at th[illegible] angular frequencies ω_1 and ω_2. For the two specified frequencies Eq. [illegible] becomes

$$\tan \frac{\omega_1 l}{v_c} = \frac{1}{\omega_1 C Z_0} \qquad \tan \frac{\omega_2 l}{v_c} = \frac{1}{\omega_2 C Z_0} \tag{7}$$

Dividing these equations gives

$$\frac{\tan \omega_1 l/v_c}{\tan \omega_2 l/v_c} = \frac{\omega_2}{\omega_1} \tag{8}$$

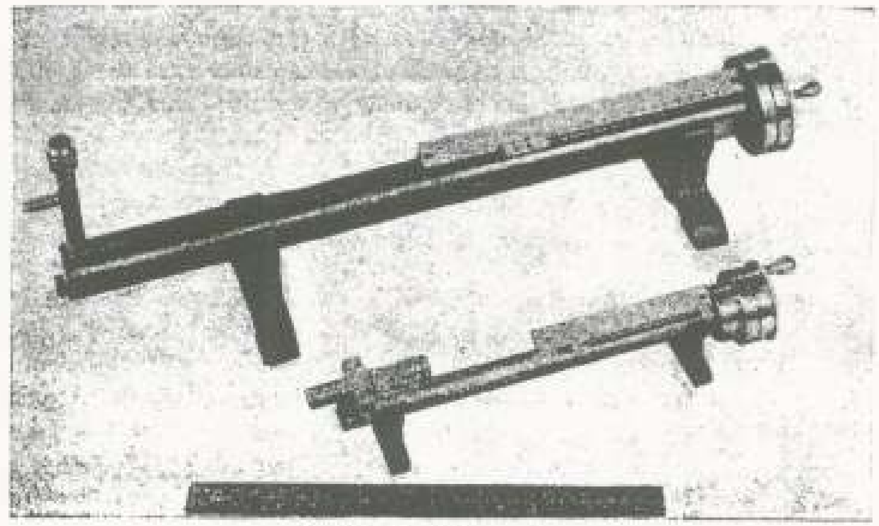


FIG. 3.—Coaxial wavemeters.

If the two frequencies are harmonically related, we have $\omega_2 = n\omega_1$, and Eq. (8) reduces to

$$\tan \frac{\omega_1 l}{v_c} = n \tan n \frac{\omega_1 l}{v_c} \tag{9}$$

This is a transcendental equation of the form $\tan \theta = n \tan n\theta$. Solutions for $\omega_1 l/v_c$ may be obtained either by Newton's method [1] or by graphical methods. The length of line may then be computed from the known value of $\omega l_1/v_c$.

10.04. Measurement of Wavelength.—The coaxial wavemeter shown in Fig. 3 consists of a coaxial line which is short-circuited at both ends. One end contains a sliding piston which is adjustable by means of a worm gear. The microwave source is coupled to the wavemeter by means of a coaxial line which terminates in a small coupling loop inside the coaxial

[1] DOHERTY, R. E., and E. G. KELLER, "Mathematics in Modern Engineering," chap. 4, John Wiley & Sons, Inc., New York, 1936.

microammete[illegible] a resonance indicator. Resonance occurs when the coaxial line is an integral number of half wavelengths long. The microammeter reading has its maximum value at resonance and decreases abruptly as the coaxial wavemeter is tuned away from resonance. A centimeter scale and vernier on the dial indicate the wavelength.

10.05. Measurement of Impedances at Microwave Frequencies.—An unknown impedance can be measured by connecting it to a low-loss transmission line having a known value of characteristic impedance and measuring: (1) the standing wave ratio $\rho = |V_{max}|/|V_{min}|$ and (2) the distance from the terminating impedance to the first voltage maximum or minimum.

The standing wave ratio is a measure of the impedance mismatch of a line. If the line is terminated in its characteristic impedance, the standing wave ratio is unity. The standing wave ratio increases as the terminal impedance becomes increasingly greater than or less than the characteristic impedance, approaching infinite value for the open-circuited or short-circuited line.

For a lossless line, the voltage maximum and current minimum occu[r] at the same point on the line. In Sec. 9.05 it was shown that the voltage maximum may be represented by $|V_{max}| = V_R'(1 + |r_R|)$, where $|r_R|$ is the magnitude of the reflection coefficient. By a similar method, the current minimum can be shown to be $|I_{min}| = (1 - |r_R|)(V_R'/Z_0)$. At the point where the voltage is a maximum and the current a minimum, the impedance is

$$Z = \frac{|V_{max}|}{|I_{min}|} = Z_0 \frac{1 + |r_R|}{1 - |r_R|} = \rho Z_0 \tag{1}$$

where ρ is the standing-wave ratio given by Eq. (9.05-4). In a similar manner, it may be shown that the impedance at the voltage minimum (current maximum) is

$$Z = \frac{Z_0}{\rho} \tag{2}$$

Now write Eq. (8.06-4) for the impedance at the point of maximum voltage on the line, letting $x = x_{max}$ and $Z = \rho Z_0$.

$$\rho Z_0 = Z_0 \left(\frac{Z_R + jZ_0 \tan \beta x_{max}}{Z_0 + jZ_R \tan \beta x_{max}} \right) \tag{3}$$

Solving this for Z_R and substituting $\beta = 2\pi/\lambda$, we have

$$Z_R = Z_0 \left(\frac{\rho - j \tan \frac{2\pi x_{max}}{\lambda}}{1 - j\rho \tan \frac{2\pi x_{max}}{\lambda}} \right) \tag{4}$$

If ρ, Z_0, x_{max}, and λ are known, the terminal impedance may [be rea]dily computed from Eq. (4). A similar relationship may be derive[d for] the terminating impedance in terms of the distance from x_{min} to the voltage minimum.

The impedance diagram can be used to evaluate an unknown load impedance. As an example, assume that the values $\rho = 2$ and $x_{max}/\lambda = 0.15$

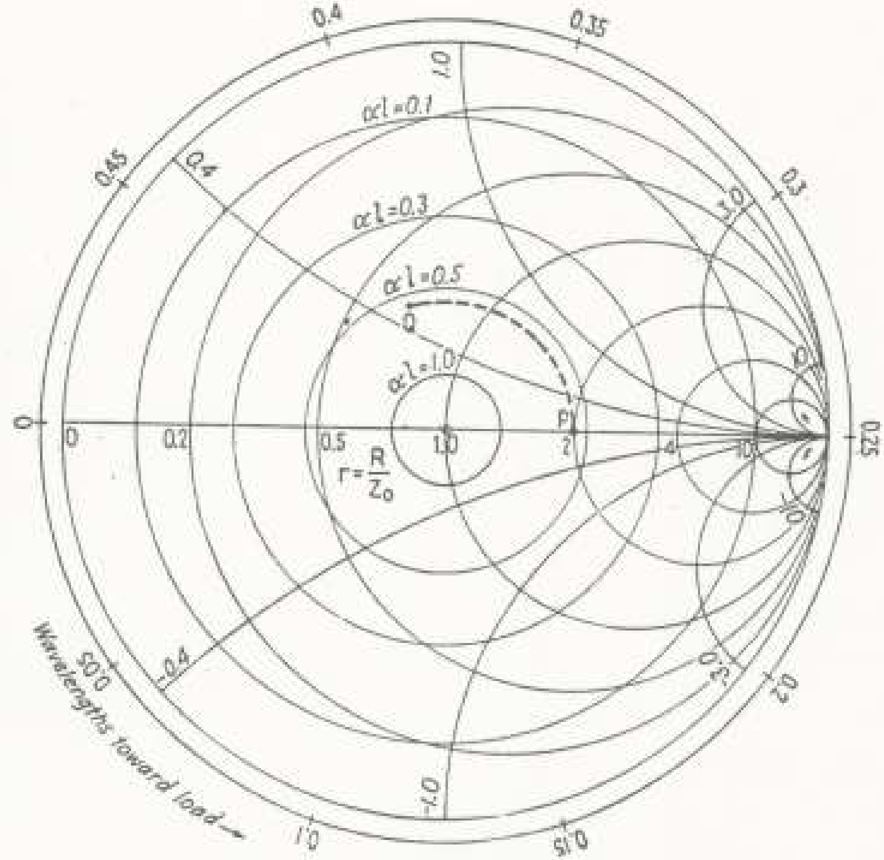
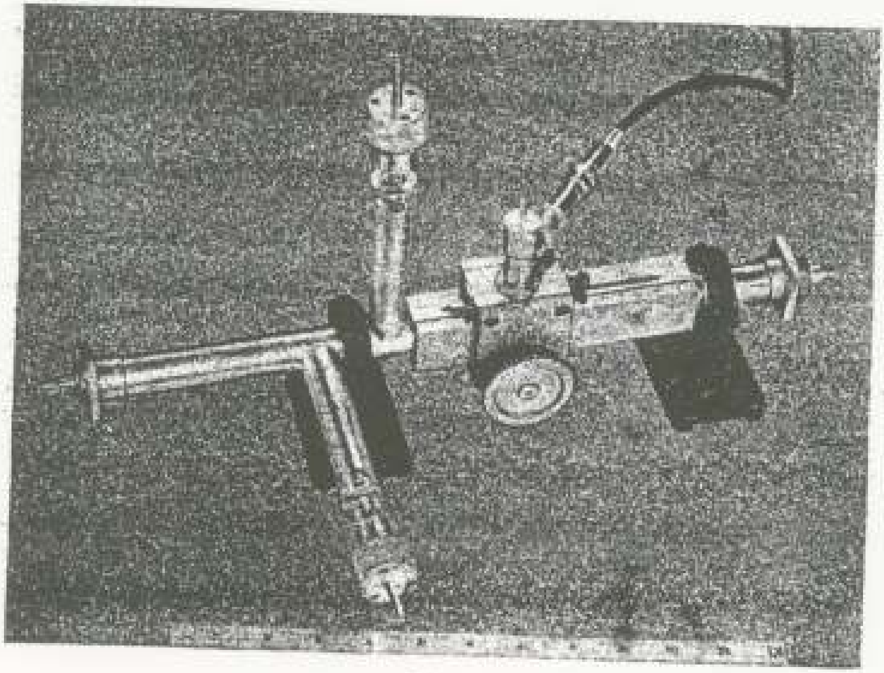


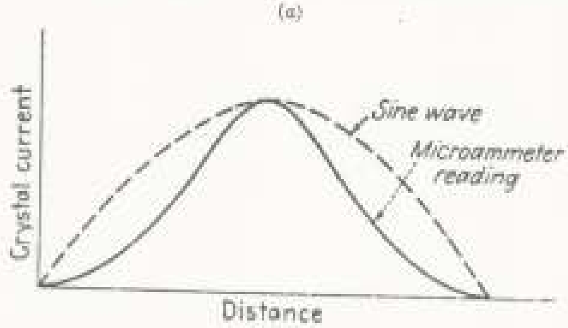
FIG. 4.—Use of the impedance diagram for determining an unknown impedance.

have been experimentally determined. The normalized impedance at the voltage-maximum point on the line, from Eq. (1), is $Z/Z_0 = \rho = 2.0 + j0$. Enter the impedance diagram at this value of impedance (point P in Fig. 4) and observe the corresponding values of $\alpha l = 0.54$ and $w_0 = 0.25$. Now proceed in a counterclockwise direction on the constant-αl circle to point Q where $w = w_0 + (x_{max}/\lambda) = 0.40$ (on the "wavelengths toward the load" scale). This corresponds to moving from the voltage-maximum point to the recei end of the line. The normalized impedance at point Q is $Z_R/Z_0 = \quad + j0.48$. This is the normalized load impedance.

Figure 5a shows a standing-wave indicator which is used to measure the standing-wave ratio at wavelengths of from approximately 5 to 15 centi-



(a)



(b)

FIG. 5.—Standing-wave indicator and method of crystal calibration.

meters. This consists of a sliding probe which extends a short distance into a slotted section of coaxial line. The probe is connected to a crystal detector and microammeter. This device actually measures the electric intensity in the coaxial line. However, the electric intensity is proportional to the voltage between conductors, hence the standing-wave indicator may be assumed to be a voltage-measuring device. le on the standing-wave indicator is used to measure the distance ween the terminal impedance and the voltage maximum or minimum.

To calibrate the standing-wave indicator, the coaxial line is short-circuited at its output terminals and data are taken for a curve of microammeter reading plotted against probe position, as shown in Fig. 5b. The sine-wave curve, also shown in Fig. 5b, is the curve which would be obtained if the crystal had a linear volt-ampere characteristic. The relative calibration curve is then obtained by plotting a curve representing the sine wave values as ordinates and the corresponding microammeter readings as abscissas. Care must be taken to couple the probe loosely to the coaxial line in order to avoid disturbing the standing wave on the coaxial line.

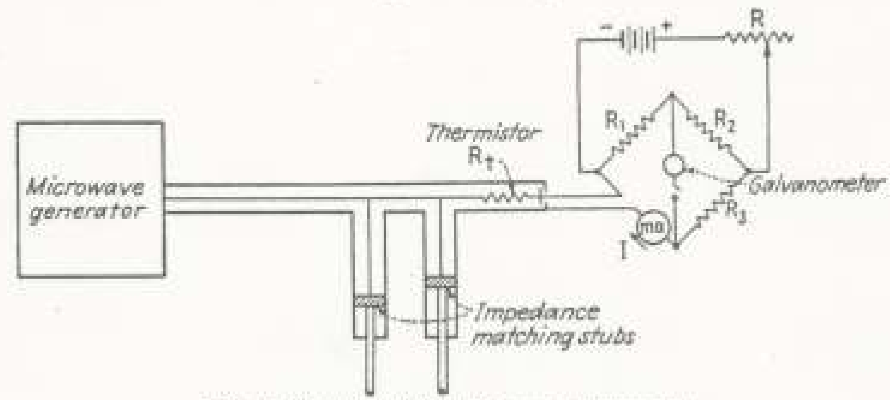


FIG. 6.—Thermistor bridge for power measurement.

10.06. Power Measurement at Microwave Frequencies.—It is often necessary to measure the power output of oscillators or amplifiers operating at microwave frequencies. The thermistor bridge or bolometer bridge, shown in Fig. 6, provides a convenient and reasonably accurate method of measuring power. One arm of the bridge, designated as R_t in Fig. 6, contains a resistance element which has a relatively high temperature coefficient of resistance. This element is connected in such a manner as to absorb the power from the microwave source whose power output is being measured.

With the microwave source disconnected, the bridge is balanced and the milliammeter reading in the R_t branch is observed. The microwave source is then connected and the stubs are adjusted for maximum power transfer to R_t, as indicated by a maximum unbalance of the bridge. The unbalance results from the fact that the value of R_t changes as the power dissipation in it increases. Balance is restored by decreasing the d-c power loss in R_t by an amount exactly equal to the microwave power dissipation in this resistance. This is accomplished by means of the rheostat in the battery cur ... If I_1 is the direct current through R_t for the initial balance and ... the direct current when the bridge is balanced with the microwave source connected, the microwave power is $P_{ac} = (I_1^2 - I_2^2)R_t$. The milliammeter may be calibrated to read the power in watts.

If the element R_t is a semiconductor having a negative temperature coefficient of resistance, it is known as a thermistor. Negative temperature-coefficient materials suitable for thermistors include uranium oxide, a mixture of nickel oxide and manganese oxide, and silver sulphide. Conducting wires are sometimes used which have a positive temperature coefficient of resistance. These are known as bolometers or barretters. One commercial form consists of a straight platinum wire, 0.06 mil in diameter,

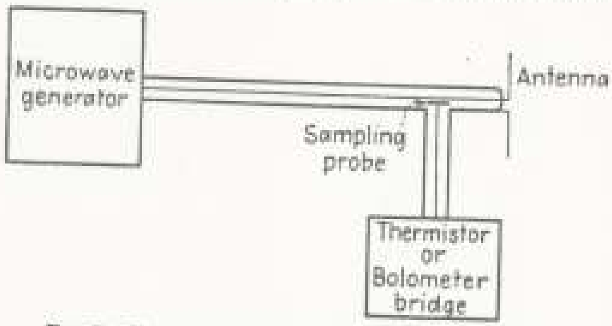


FIG. 7.—Power measurement by the sampling method.

which is mounted in a cylinder for use in standard coaxial line fittings. This particular element has a maximum power rating of 32.5 milliwatts and may be used to measure values of power as low as 10 microwatts.

In order to minimize skin-effect errors, thermistors and barretters usually have very small diameters; consequently their power rating is quite small. Higher power levels may be measured by inserting a calibrated attenuator between the source and R_t in Fig. 6. Sections of coaxial line having a high-loss dielectric are sometimes used as attenuators. However, the attenuation characteristics of this type of attenuator are likely to vary with temperature and humidity. Another type of attenuator consists of a glass rod upon which has been deposited a thin layer of platinum. The platinized glass rod is used as the center conductor in a coaxial line which is inserted between the power source and the thermistor or bolometer bridge.

The power consumed in a load impedance may be measured by a sampling method shown in Fig. 7. In this method the power-measuring circuit is loosely coupled to the coaxial line which connects the microwave source to the load. A small fraction of the total power enters the power-measuring bridge, which may be of the thermistor or bolometer type. In this method, it is necessary to calibrate the power-measuring circuit with the given load impedance in order to determine what fraction of the total power is being drawn off by the power-measuring circuit. A more accura... ...mpling method consists of a coaxial line containing a directional coup... ...imilar to that shown for wave guides in Chap. 18.

10.07. Effect of Impedance Mismatch upon Power Transfer.—A well-known power-transfer theorem states that if a variable load impedance is connected to a constant-voltage generator, maximum power transfer occurs when the load impedance is equal to the complex conjugate of the generator impedance. When this condition prevails the impedances are said to be matched. It is sometimes helpful to be able to determine the power sacrificed as a result of not having matched impedances.

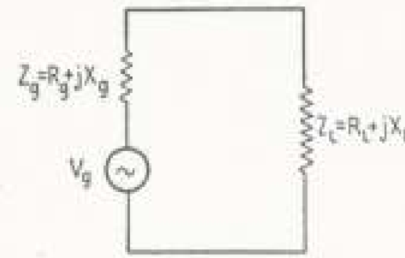


FIG. 8.—Generator and load.

Referring to Fig. 8, let $Z_g = R_g + jX_g$ be the generator impedance and $Z_L = R_L + jX_L$ represent the load impedance. The scalar value of current is

$$|I| = \frac{V_g}{\sqrt{(R_g + R_L)^2 + (X_g + X_L)^2}} \tag{1}$$

and the power consumed in the load is

$$P = |I|^2 R_L = \frac{V_g^2 R_L}{(R_g + R_L)^2 + (X_g + X_L)^2} \tag{2}$$

If the load impedance is the only variable, the power is a maximum when $R_L = R_g$ and $X_L = -X_g$, that is, when the generator and load impedance are conjugates. The power is then

$$P_{max} = \frac{V_g^2}{4R_g} \tag{3}$$

The ratio of the power for any load impedance to the maximum power is found by dividing Eq. (2) by (3),

$$\frac{P}{P_{max}} = \frac{4R_g R_L}{(R_g + R_L)^2 + (X_g + X_L)^2} \tag{4}$$

Now divide the numerator and denominator by R_g^2 and let $R' = R_L/R_g$ and $X' = (X_g + X_L)/R_g$, giving

$$\frac{P}{P_{max}} = \frac{4R'}{(1 + R')^2 + (X')^2} \tag{5}$$

11 we let ... nd X' be the coordinate axes in a rectangular coordinate system, the ... epresenting constant values of P/P_{max} are circles as shown in Fig. 9. This graph makes it possible to evaluate the ratio of actual power to maximum power for any condition of impedance mismatch. The procedure is to compute R' and X' for the particular problem. The value

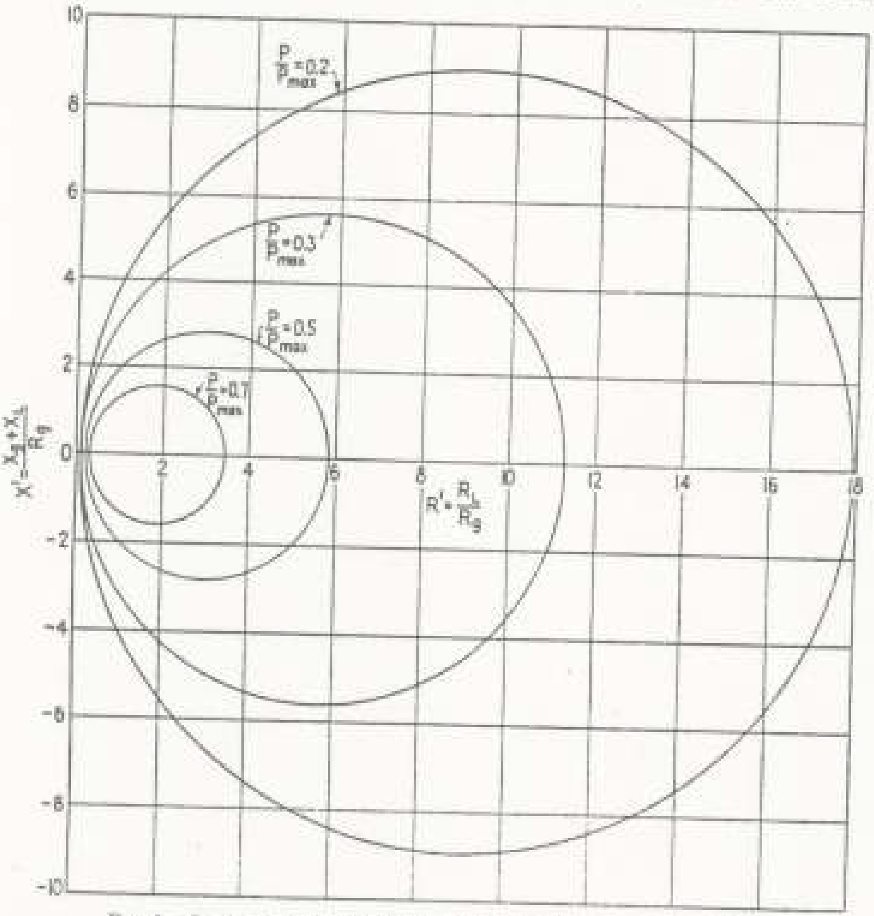


FIG. 9.—Power transfer ratio P/P_{max} for mismatched impedances.

of P/P_{max} is then obtained from Fig. 9. If the load consists of a transmission line with a load impedance at the distant end, the impedance Z_l is the input impedance of the line.

10.08. Power-transfer Theorem.—There is an interesting and useful power-transfer theorem which states that if a generator is connected through one or more pure reactance networks to a load, as shown in Fig. 10, and the conditions are such that there is a conjugate impedance match at one pair of terminals, then there will be a conjugate impedance match at every pair of terminals and maximum power will be tran... from the generator to the load.

This may be readily verified by the use of Thévenin's theorem. If we break into the network at any junction, such as at ab in Fig. 10a, Thévenin's theorem permits us to replace the network to the left of ab by a generator as shown in Fig. 10b. The generator impedance Z'_g in the equivalent network is equal to the impedance looking to the left at terminals ab, and the generator voltage V'_g is the open-circuit voltage at ab. The network to the right of ab in Fig. 10a is replaced by its equivalent impedance Z_2 in Fig. 10b. Now assume that there is a conjugate match of impedances

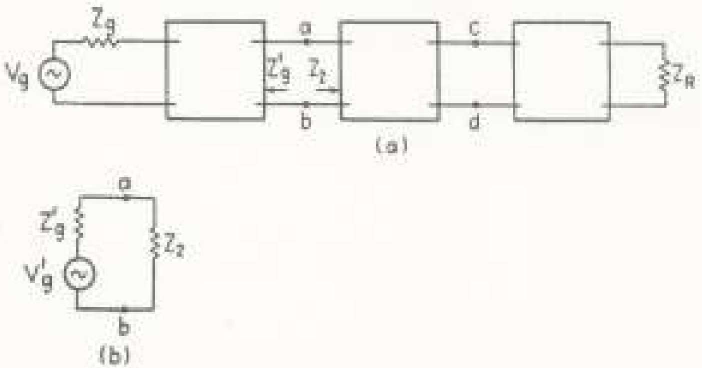


FIG. 10.—(a) Network consisting of a generator connected to a load impedance through pure reactance networks, and (b) equivalent circuit from Thévenin's theorem.

at ab in Figs. 10a and 10b. A conjugate impedance match in the circuit of Fig. 10b signifies that there is a maximum power transfer past the junction ab. If there is a maximum power transfer in the equivalent circuit, there must likewise be maximum power transfer in the original circuit. Since we have maximum power flow past the junction ab in Fig. 10a, and there is no power lost in the reactive networks, it follows that there is a maximum power flow at every junction and likewise maximum power transfer to the load. Consequently, there must be a conjugate impedance match at every junction, since, if there were not a conjugate impedance match at any junction, there could not be maximum power flow past this junction.

For most transmission lines operating at microwave frequencies, we have $\omega L \gg R$ and $\omega C \gg G$; hence the lines may be treated as pure reactance networks. The foregoing power-transfer theorem makes it possible to match impedances at any point on the line between the generator and load and be assured of a conjugate impedance match throughout the entire system, resulting in maximum power transfer to the load.

10.09. Quarter-wavelength and Half-wavelength Lines.—Transmission lines which are either a quarter wavelength or a half wavelength long have special ir ance-transforming properties. Consider the input impedance to a qua vavelength lossless line which is terminated in an impedance Z_R. Inserting $\beta l = \pi/2$ into Eq. (8.06-4), we obtain

$$Z = \frac{Z_0^2}{Z_R} \qquad (1)$$

The input impedance varies inversely as Z_R; therefore the quarter-wavelength line is effectively an impedance inverter. If Z_R is inductive, the input impedance is capacitive, and vice versa. If the load impedance is constant, it is possible to control the magnitude of the input impedance,

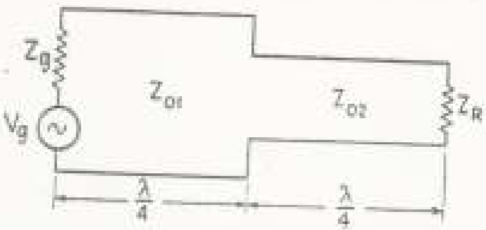


FIG. 11.—Impedance transformer consisting of two quarter-wavelength lines.

but not its phase angle, by a proper choice of Z_0. If a generator having an impedance Z_g is connected to the input end of the line and both Z_g and Z_R are pure resistances, maximum power transfer occurs when

$$Z_g = \frac{Z_0^2}{Z_R} \text{ or } Z_0 = \sqrt{Z_g Z_R} \qquad (2)$$

From a practical point of view, the range of characteristic impedances of coaxial lines is from 5 to 250 ohms, whereas that for open-wire lines is from 90 to 1,000 ohms. These provide the practical limits of impedance transformation using the quarter-wavelength line.

Let us now see what effect variations in frequency have upon the power transfer. Assume that a lossless line, terminated by an impedance Z_R, has matched impedances at the generator at the frequency for which the line is a quarter wavelength long. If Z_g and Z_R are approximately equal, the variation in power transfer with frequency will be relatively small. However, if Z_g and Z_R differ greatly the power transfer decreases rapidly as the frequency departs from the frequency at which the line is a quarter wavelength long, and therefore the impedance transformer is highly frequency selective.

Two or more quarter-wavelength sections of line having different characteristic impedances, such as shown in Fig. 11, may be used to obtain an impedance transformer which is less frequency selective than that of a single quarter-wavelength section.

Now consider the input impedance of a half-wavelength which is terminated in an impedance Z_R. Inserting $\beta l = \pi$ into Eq. 3-4), we obtain

$$Z = Z_R \qquad (3)$$

Therefore, the input impedance of the half-wavelength line is equal to the terminating impedance, or the half-wavelength line is effectively a one-to-one ratio transformer.

10.10. Single-stub Impedance Matching.—From a practical point of view, the use of the quarter-wavelength line as an impedance transformer is restricted largely to the matching of resistive impedances where the frequency and impedances are constant. Stub impedance-matching systems

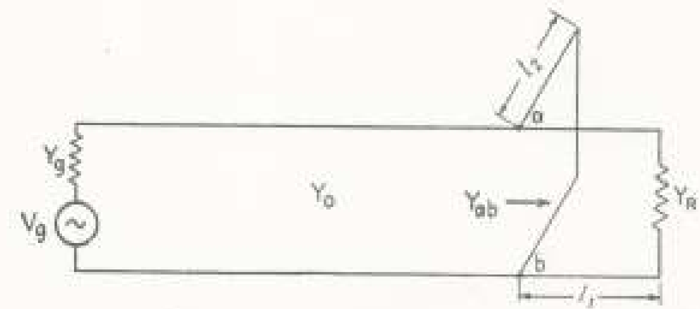


FIG. 12.—Single-stub impedance matching.

are more versatile in that they may be used to match complex impedances and are more readily adjustable. A stub consists of an open-circuited or short-circuited line which is shunted across the transmission line between the generator and the load. One or more such stubs may be used for impedance matching.

If a single stub is used, as shown in Fig. 12, it is necessary to have both the length of the stub and the distance from the stub to the load adjustable in order to match all possible load impedances. Let us analyze this case. Since we are dealing with parallel circuits, it is convenient to use admittances. The characteristic admittance of the line $Y_0 = 1/Z_0$ may be assumed to be a pure conductance for a low-loss line.

Maximum power transfer requires a conjugate admittance (or impedance) match at the generator and, likewise, a conjugate admittance match at points ab where the stub is connected to the transmission line. In the following discussion, it is assumed that the line is lossless and therefore that the characteristic admittance Y_0 is a pure conductance. If the generator admittance happens to be equal to the characteristic admittance, that is, if $Y_g = Y_0$, then the admittance looking to the left at ab is Y_0 and the admittance looking to the right at ab (including the stub) must likewise will b… standing waves on the line between the generator and the stub, althou… standing waves will exist on the section of line from the stub to the load and on the stub itself. …Under these conditions, there

If the generator admittance is not equal to Y_0, then the admittance looking to the left at ab is not equal to Y_0, and therefore the admittance looking to the right at ab must likewise have some value other than Y_0 for a conjugate admittance match. The section of line between the generator and stub then serves as an impedance transformer and has standing waves of voltage and current along its length. It should be noted that the absence of standing waves on a line is not always an indication of maximum power transfer.

In the following analysis, we shall assume a lossless line with $Y_g = Y_0$. Maximum power transfer then requires that $Y_{ab} = Y_0$, where Y_{ab} is the admittance looking to the right at ab including the stub. The admittance of a lossless stub is a pure susceptance. The stub is located at that point on the line where the real part of the admittance, looking toward the load, is Y_0. The stub length is then adjusted so that its susceptance is equal and opposite to the susceptance of the line at this point, thereby canceling the susceptance and leaving $Y_{ab} = Y_0$.

The reflection coefficient equations will be used in order to gain facility in the use of these equations. The admittance at any point on the line looking toward the load (with the stub disconnected) is obtained by inverting Eq. (9.01-4)

$$\frac{Y}{Y_0} = \frac{1 - r_R e^{-2\gamma x}}{1 + r_R e^{-2\gamma x}} \tag{1}$$

where the reflection coefficient is given by $r_R = (Y_0 - Y_R)/(Y_0 + Y_R)$. Now let

$$r_R = \frac{Y_0 - Y_R}{Y_0 + Y_R} = |r_R| e^{-j2u_0} \tag{2}$$

where $|r_R|$ is the real part of the reflection coefficient and $-2u_0$ is the angle. Substitution of Eq. (2) into (1), together with $\gamma = j\beta$ (for a lossless line), gives

$$\frac{Y}{Y_0} = \frac{1 - |r_R| e^{-2j(u_0+\beta x)}}{1 + |r_R| e^{-2j(u_0+\beta x)}} = \frac{1 - |r_R| e^{-j\psi}}{1 + |r_R| e^{-j\psi}} \tag{3}$$

where $\psi = 2(u_0 + \beta x)$

Using the trigonometric identity $e^{-j\psi} = \cos\psi - j\sin\psi$ and separating the real and imaginary terms, we obtain from Eq. (3),

$$\frac{Y}{Y_0} = \frac{1 - |r_R|^2}{1 + 2|r_R|\cos\psi + |r_R|^2} + \frac{j2|r_R|\sin\psi}{1 + 2|r_R|\cos\psi + |r_R|^2} \tag{4}$$

The stub is located at that point on the line where the re… t of the admittance is equal to Y_0, or where the real part of Eq. (4) ha… t value. This requires that

$$\cos\psi = -|r_R| \tag{5}$$

Inserting $\psi = 2(u_0 + \beta l_1)$, where l_1 is the distance between the receiving end of the line and the stub, we have

$$\cos 2(u_0 + \beta l_1) = -|r_R|$$

$$l_1 = \frac{\lambda}{2\pi}\left(-u_0 + \frac{1}{2}\cos^{-1} - |r_R|\right) \tag{6}$$

The values of $|r_R|$ and u_0 may be evaluated in terms of known values of Y_0 and Y_R by using Eq. (2). These are then substituted into Eq. (6) to obtain the distance l_1 from the load to the stub.

From Eq. (5), we obtain $\sin\psi = \sqrt{1 - |r_R|^2}$. Substitution of this, together with Eq. (5), into (4), yields

$$\frac{Y}{Y_0} = 1 + \frac{j2|r_R|}{\pm\sqrt{1 - |r_R|^2}} \tag{7}$$

For a conjugate admittance match, under the assumed conditions, the susceptance portion of Eq. (7) must be canceled by an equal and opposite stub susceptance. Assume that the characteristic admittance of the stub and line are equal. The normalized stub admittance must then be

$$\frac{Y_{stub}}{Y_0} = -\frac{j2|r_R|}{\pm\sqrt{1 - |r_R|^2}} \tag{8}$$

The normalized admittances of the short-circuited and open-circuited stubs are obtained from Eqs. (8.06-7 and 10),

$$\frac{Y_{stub}}{Y_0} = -j\cot\frac{2\pi l_2}{\lambda} \qquad \text{short-circuited stub} \tag{9}$$

$$\frac{Y_{stub}}{Y_0} = j\tan\frac{2\pi l_2}{\lambda} \qquad \text{open-circuited stub} \tag{10}$$

The length of stub is found by equating either Eq. (9) or (10) to Eq. (8) and solving for l_2.

The position of the stub and its length may also be determined from standing-wave measurements on the line. With the stub disconnected, the maximum and minimum voltages are observed and the distance from the load to the first voltage maximum is obtained. Equation (9.05-4) is then used to obtain the value of $|r_R|$ and Eq. (9.05-7) yields u_0. These values may then be substituted into Eqs. (6) and (8) and the values of l_1 and l_2 are then determined as outlined above.

SWR $\rho = \left|\frac{V_{max}}{V_{min}}\right| = \frac{1 + |r_R|}{1 - |r_R|}$ || $\frac{x_{max}}{\lambda} = \frac{n}{4} - \frac{u_0}{2\pi}$

$r_R \to Y_{stub} \to l_2$ (9) || $u_0 \to l_1$ (6)

...enerator having an internal impedance $Z_g = 75$ ohms is connected to a load impe... $Z_R = 250$ ohms by means of a transmission line having a characteristic impeda... of 75 ohms. The wavelength of the impressed signal is $\lambda = 20$ cm. Find the position and length of a short-circuited stub which will yield maximum power transfer to the load.

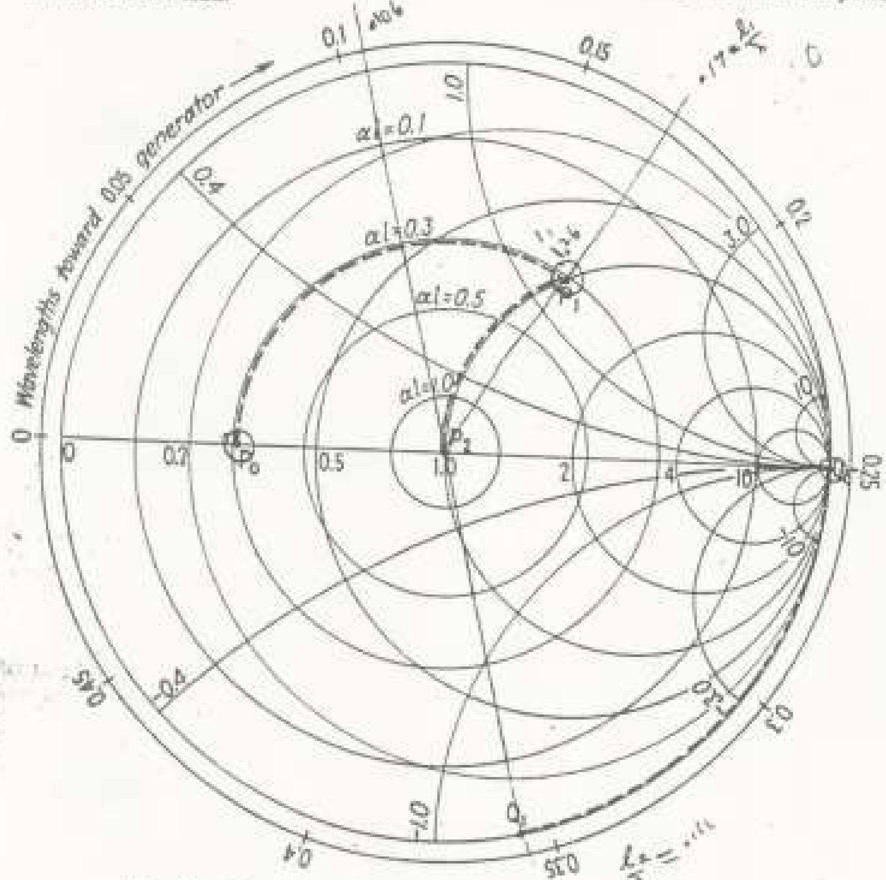


FIG. 13.—Graphical solution of single-stub impedance matching.

Consider first the analytical solution. The admittances are $Y_0 = 1/Z_0 = 0.0133$ mhos and $Y_R = 1/Z_R = 0.004$ mhos. Substitution of these values into Eq. (2) yields

$$r_R = |r_R| \epsilon^{-2jw_R} = 0.537$$

or $|r_R| = 0.537$ and $w_R = 0$. Inserting these values into Eq. (6), we obtain

$$\frac{l_1}{\lambda} = \frac{1}{4\pi}\cos^{-1}(-0.537)$$

There are a multiplicity of values of l_1 which satisfy this equation. These values differ by a half wavelength and any one of the values may be used. The shortest distance corresponds to the second-quadrant angle of 121.5° or 2.12 radians. This

$$\frac{l_1}{\lambda} = 0.169 \qquad l_1 = 3.38 \text{ cm.}$$

The length of the stub is found by equating (8) and (9) and substituting $|r_R| = -0.537$, yielding

$$\cot\frac{2\pi l_2}{\lambda} = 1.264$$

Again we find a multiplicity of solutions differing by a half wavelength. The shortest stub is

$$\frac{l_2}{\lambda} = 0.1065 \qquad l_2 = 2.13 \text{ cm.}$$

Now consider the graphical solution. We enter the polar diagram at the normalized load admittance $Y_R/Y_0 = 0.3 + j0$ (point P_0 in Fig. 13) and observe that $t_0 = 0.30$ (on the αl circles) and $w_0 = 0$ (on the "wavelengths toward the generator" scale). As we move back toward the generator, the admittance point moves in a clockwise direction on the constant-αl circle. At P_1 we intersect the unity conductance circle and at the corresponding point on the transmission line we place the stub. At this point the admittance is $Y_1/Y_0 = 1 + j1.26$ and we have $w_1 = 0.17$. The distance l_1 from the load to the stub is determined by $l_1/\lambda = w_1 - w_0 = 0.17$.

The stub must provide a normalized susceptance of -1.26 mhos. Let us now determine the length of stub. At the short-circuited end of the stub we have $Y/Y_0 = \infty$. Entering the diagram at this point (Q_0 in Fig. 13), we observe that $w_0' = 0.25$. Moving in the clockwise direction around the constant-αl circle we stop at the susceptance $Y_S/Y_0 = -j1.26$. The corresponding value of w' is $w' = 0.356$. Thus, the stub length is obtained from $l_2/\lambda = w' - w_0' = 0.106$. The values of l_1/λ and l_2/λ obtained by the graphical method are in agreement with the analytical solutions.

10.11. Double-stub Impedance Matching.—In order to match variable load impedances using the single stub, it is necessary to vary the length of the stub as well as its position with respect to the load impedance. The stub position may be varied by inserting a telescoping "line stretcher" between the load and the stub. A more convenient impedance transforming system consists of two or more adjustable stubs spaced approximately a quarter wavelength apart as shown in Fig. 14.

Let us assume that $Y_g = Y_0$. To obtain maximum power transfer, it is then necessary to adjust the lengths of the stubs so that the admittance looking to the right at ab is equal to Y_0. Stub 1 serves to make the conductance part of the admittance at ab equal to Y_0, and stub 2 is then adjusted to cancel the susceptance portion of the admittance at ab.

Consider the graphical solution of the double-stub impedance transformer. Assume that the admittance Y_{cd} at points cd in Fig. 14 (with both stubs disconnected) corresponds to point P_0 in Fig. 15. Connecting stub 1 adds a pure susceptance, causing the admittance point to move on the constant-conductance circle to a new position P_1 which is determined by the length of the stub. If the stubs are a quarter wavelength apart,

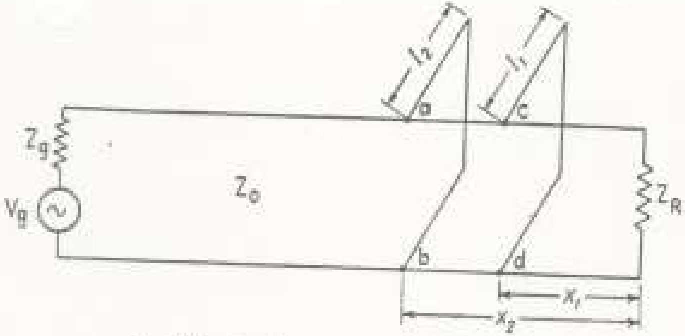


Fig. 14.—Double-stub impedance matching.

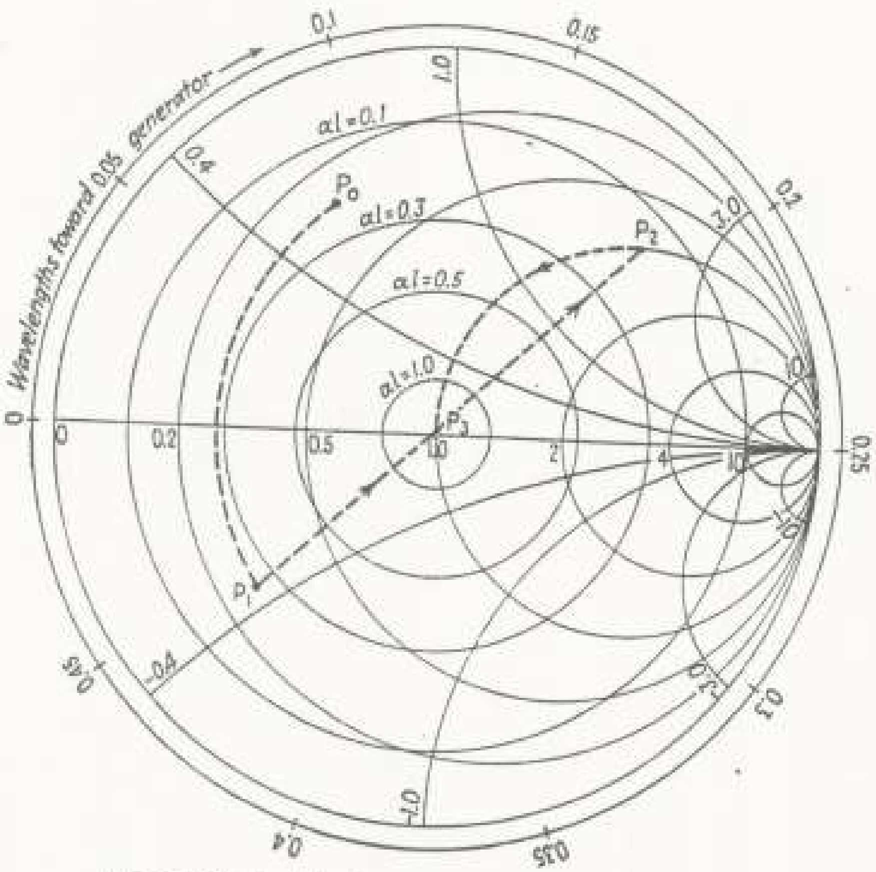


Fig. 15.—Graphical solution of double-stub impedance matching.

[illegible] admittance Y_{ab} (with stub 2 disconnected) is at point P_2 in [illegible] which is halfway around the diagram from P_1 on the same αl cir[illegible] In order to obtain the conditions necessary for matched admittanc[illegible] [illegible]ne length of stub 1 should be such that point P_2 falls on the unity conductance circle. With stub 2 connected and adjusted to cancel the susceptance at P_2, the admittance point moves from P_2 to P_3. The latter point corresponds to $Y/Y_0 = 1 + j0$, which is the requirement for matched admittances.

The double-stub transformer will not match all possible load admittances. Thus, if the load admittance and position of the stub are such as to place

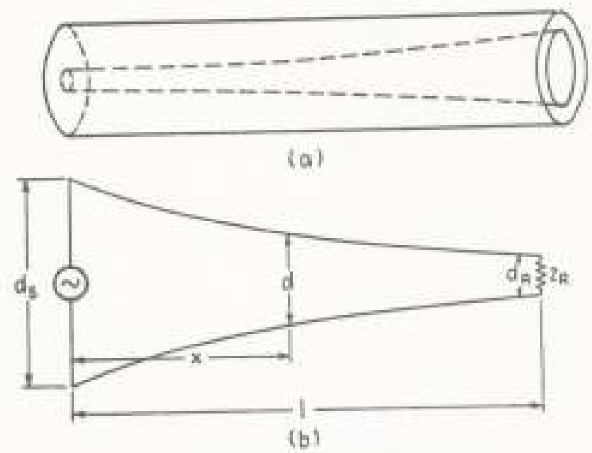


Fig. 16.—Coaxial and open-wire tapered lines.

P_0 on any conductance circle greater than unity (so that P_0 falls inside the unity conductance circle), it is then impossible to obtain matched admittances using two stubs which are spaced a quarter wavelength apart. The range of admittances which can be matched by this method is increased somewhat by spacing the stubs a little less than a quarter wavelength apart. Triple-stub impedance transformers are sometimes used where accurate impedance matching is required.

10.12. The Exponential Line.[1]—Tapered lines, which are several wavelengths long, such as those shown in Fig. 16, provide a gradual impedance transformation over the length of the line. This type of impedance transformer is less frequency selective than the quarter-wavelength line or the stub transformers. An exponential line is a tapered line in which the characteristic impedance varies exponentially along the line.

As a point of departure, let us write the differential equations of the transmission line similar to Eqs. (8.02-3 and 4), but with the distance x measured

[1] Wheeler, H. A., Transmission Lines with Exponential Taper, *Proc. I.R.E.*, vol. 27, pp. 65–71; January, 1939.

... sending end. For a lossless line these are

$$\frac{dV}{dx} = -j\omega LI \tag{1}$$

$$\frac{dI}{dx} = -j\omega CV \tag{2}$$

where the inductance L and capacitance C per unit length of line are now functions of the distance x.

We now assume that the tapered line is terminated in such a manner as to prevent reflections at the receiving end. For this condition, there will be outgoing waves of voltage and current but no reflected waves. Assume that the outgoing wave of voltage has a variation with x given by

$$V = V_S e^{-(\delta+j\beta)x} \tag{3}$$

where V_S is the amplitude of the sending-end voltage and $e^{-\delta x}$ represents an exponential voltage transformation resulting from the line taper. The constant δ will be referred to as the *voltage transformation constant.*

If the voltage is transformed by an amount $e^{-\delta x}$, we would expect that the current would be transformed by the inverse amount, or by an amount $e^{\delta x}$. Hence the current is represented by

$$I = I_S e^{(\delta-j\beta)x} \tag{4}$$

where I_S is the sending-end current amplitude.

The voltage and current given by Eqs. (3) and (4) are in time phase at any point on the line and the power flow is

$$P = \tfrac{1}{2}|V||I| = \tfrac{1}{2}V_S I_S \tag{5}$$

The power is independent of distance x.

Let us now determine what conditions are required to obtain the assumed voltage and current distribution. The impedance at any point along the line is the ratio of voltage to current. Dividing Eqs. (3) and (4), we have

$$Z = \frac{V}{I} = Z_S e^{-2\delta x} \tag{6}$$

where $Z_S = V_S/I_S$. Equation (6) shows that the impedance is transformed by the factor $e^{-2\delta x}$ which is the square of the voltage transformation.

Substitution of the voltage from Eq. (3) and the current from (4) into (1) and (2) gives

$$(\delta + j\beta)Z_S e^{-2\delta x} = j\omega L \tag{7}$$

$$-(\delta - j\beta)\frac{e^{+2\delta x}}{Z_S} = j\omega C \tag{8}$$

The product of these two equations gives

$$\delta^2 + \beta^2 = \omega^2 LC \tag{9}$$

and solving for β

$$\beta = \sqrt{\omega^2 LC - \delta^2} = \omega\sqrt{LC}\sqrt{1 - \frac{\delta^2}{\omega^2 LC}} \tag{10}$$

The phase constant β may be either real or imaginary, a condition similar to that encountered in filter networks. For large values of ω, the phase constant, given by Eq. (10), is real and the voltage and current waves propagate without attenuation (although they are transformed owing to the taper of the line). This corresponds to the pass band in conventional filter theory. For values of ω below a certain critical value, β is imaginary and the voltage and current, as given by Eqs. (3) and (4), are *both* attenuated with distance along the line. This corresponds to the attenuation band in filter theory. Hence the exponential line has properties similar to those of an impedance-transforming high-pass filter.

Cutoff occurs when β is zero, or from Eq. (10), when

$$\omega_c = \frac{\delta}{\sqrt{LC}} = \delta v_c \tag{11}$$

where ω_c is the cutoff angular frequency and $v_c = 3 \times 10^8$ meters per second is the velocity of light. Equation (11) shows that a high transformation constant δ results in a correspondingly high cutoff frequency.

The inductance and capacitance variation along the line are determined by the form of voltage and current distribution. Returning to Eq. (7) and writing this for the sending end ($x = 0$), we obtain $(\delta + j\beta)Z_S = j\omega L_S$ where L_S is the inductance per unit length at the sending end of the line. Dividing this into Eq. (7), gives

$$\frac{L}{L_S} = e^{-2\delta x} \tag{12}$$

A similar procedure applied to Eq. (8) yields

$$\frac{C}{C_S} = e^{2\delta x} \tag{13}$$

where C_S is the capacitance per unit length at the sending end.

If we define the characteristic impedance at any point on the line by the relationship $Z_0 = \sqrt{L/C}$, Eqs. (12) and (13) give

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{L_S}{C_S}}e^{-2\delta x} \tag{14}$$

Fin[...] let us evaluate the terminal impedance Z_R which is necessary to pre[...] reflections. By dividing Eqs. (7) and (8), and substituting $Z_S = Z\epsilon^{2\delta x}$ from Eq. (6), we obtain

$$Z = \sqrt{\frac{L}{C}}\sqrt{\frac{-(\delta - j\beta)}{\delta + j\beta}} \tag{15}$$

Rationalizing Eq. (15) and separating real and imaginary terms, we obtain

$$Z = \sqrt{\frac{L}{C}}\left(\frac{\beta}{\sqrt{\delta^2 + \beta^2}} + \frac{j\delta}{\sqrt{\delta^2 + \beta^2}}\right) \tag{16}$$

Substitution of

$$\delta^2 + \beta^2 = \omega^2 LC = \left(\frac{\omega}{\omega_c}\right)^2 \delta^2 \qquad \text{and} \qquad \beta^2 = \omega^2 LC - \delta^2 = \delta^2\left[\left(\frac{\omega}{\omega_c}\right)^2 - 1\right]$$

from Eqs. (9) and (11) into Eq. (16) gives

$$Z = \sqrt{\frac{L}{C}}\left[\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} + j\frac{\omega_c}{\omega}\right] \tag{17}$$

Equation (17) is an expression for the impedance (ratio of voltage to current) at any point on the exponential line. The impedance at either the transmitting or receiving end of the line is obtained by inserting the values of L and C at the corresponding end of the line into Eq. (17). This gives the impedances which are required to terminate the line so as to prevent reflections.

If the impressed frequency is much greater than the cutoff frequency, then Eq. (17) reduces to $Z = \sqrt{L/C}$, *i.e.*, the impedance at any point on the line is equal to the characteristic impedance at that point. For maximum power transfer, the generator and receiver impedances should then be approximately equal to the characteristic impedances at their respective ends of the line. In general, the condition $\omega \gg \omega_c$ is satisfied if the exponential line is several wavelengths long.

As the impressed frequency approaches cutoff the imaginary term increases. At cutoff the impedance is $Z = j\sqrt{L/C}$. The j signifies that the impedance is a pure reactance; hence there is no power flow.

Figure 17 is a plot of the voltage-transformation term $\epsilon^{-\delta x}$ and the impedance transformation term $\epsilon^{-2\delta x}$ against δx.

Example. An exponential line is desired to transform between resistive impedances $Z_t = 450$ ohms and $Z_R = 75$ ohms at a wavelength of $\lambda = 10$ cm. Find the length of coaxial line and the equation for the taper.

Inserting the values of Z_t and Z_R into Eq. (6), we obtain

$$\frac{Z_R}{Z_t} = \epsilon^{-2\delta l} = 0.167$$

Referring to Fig. 17, we obtain $\delta l = 0.9$. It is desirable to choose a len[...] line such that $\omega \gg \omega_c$. Inserting ω_c from Eq. (11), the inequality becomes ω [...], which may also be written $\delta \ll 2\pi/\lambda$. A value of δ equal to 10 per cent of $2\pi/\lambda$ would yield an impedance Z, which differs from Z_0 by less than 1 per cent. In order to obtain a reasonable length of tapered line, assume a value of $\delta = 0.06$. From the value of δl given above, we obtain $l = 15$ cm.

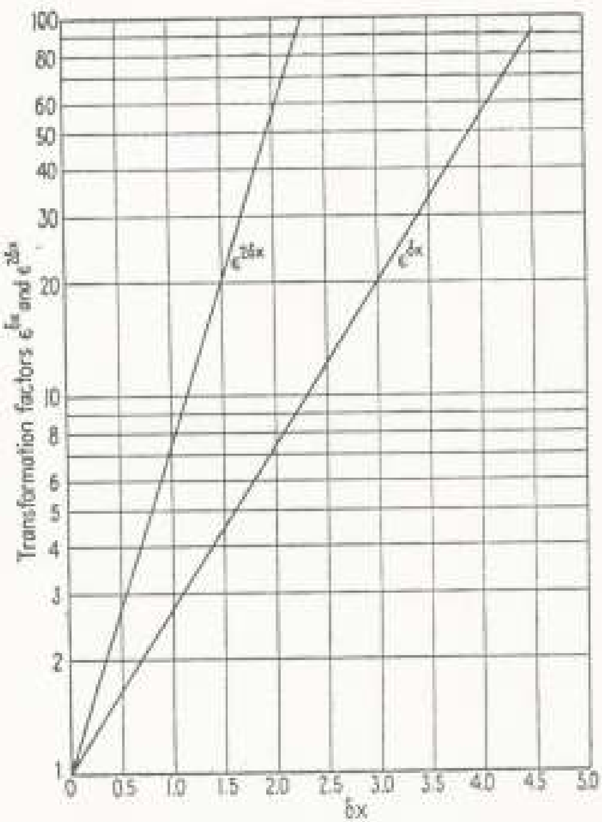


FIG. 17.—Plot of voltage and impedance transformation factors against δx.

The characteristic impedance of the line at the sending end should be approximately 450 ohms and that at the receiving end, approximately 75 ohms. To obtain the equation for the exponential taper, write Eq. (6) in the form $Z/Z_S = \epsilon^{-2\delta x}$. Inserting the characteristic impedance of a coaxial line from Table 1, Chapter 8, for Z, and $Z_S = 450$ ohms, we obtain

$$\frac{138}{450}\log_{10}\frac{b}{a} = \epsilon^{-2\delta x}$$

or

$$\log_{10}\frac{b}{a} = 3.26\epsilon^{-0.2x}$$

where a and b are the radii of the inner and outer conductors, respectively.

Networks Using Transmission-line Elements.[1]—Various types of f[illegible] may be constructed using transmission-line elements. In general, filter networks of this type are band-pass filters with multiple-pass bands due to the multiple-resonance properties of the component lines.

If the filter contains no lumped reactances, the resonant and antiresonant frequencies of the line elements are harmonically related and therefore the multiple-pass bands occur at harmonic intervals on the frequency scale. If lumped reactances are used, the multiple-pass bands, in general, will be inharmonically related.

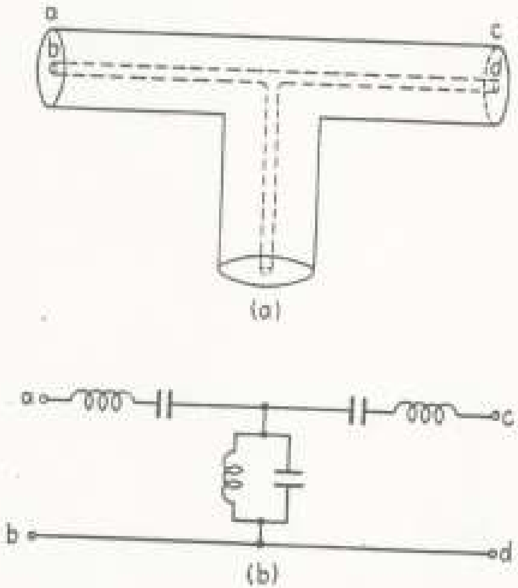


Fig. 18.—Band-pass filter and equivalent circuit

Consider the simple T filter shown in Fig. 18a. In the pass band, this may be represented by the equivalent circuit of Fig. 18b. The lengths l_1 and l_2 may be adjusted so that the series arms are resonant and the shunt arm is antiresonant at the mid-frequency in the first pass band. Let f_m represent this mid-frequency. At even harmonics of f_m the series arms are antiresonant and the shunt arm is resonant. This condition corresponds to maximum attenuation of the impressed signal. At odd harmonics of f_m the same conditions prevail as at the frequency f_m. Consequently, the pass bands are centered at odd harmonics of f_m, whereas the attenuation bands are centered at the even harmonics.

[1] Mason, W. P., and R. A. Sykes, The Use of Coaxial and Balanced Transmission Lines in Filters and Wide-Band Transformers for High Radio Frequencies, *Bell System Tech. J.*, vol. 16, pp. 275–302; July, 1937.

[illegible]re 19a shows another type of band-pass filter using [illegible] [illegible] are joined by a transmission line which is effectively a quarter [illegible] length long at the mid-frequency of the first pass band. The resona[illegible] are tuned by adjustable plungers so as to be resonant at the mid-frequency of the pass band. In general, the multiple pass bands for this type of filter are not harmonically related since the capacitance between the end of the plunger and the bottom wall of the resonator is effectively a lumped reactance. Figure 19b shows the equivalent circuit for this type of filter. The width of the pass band is determined, in part, by the effective Q of the loaded resonators.

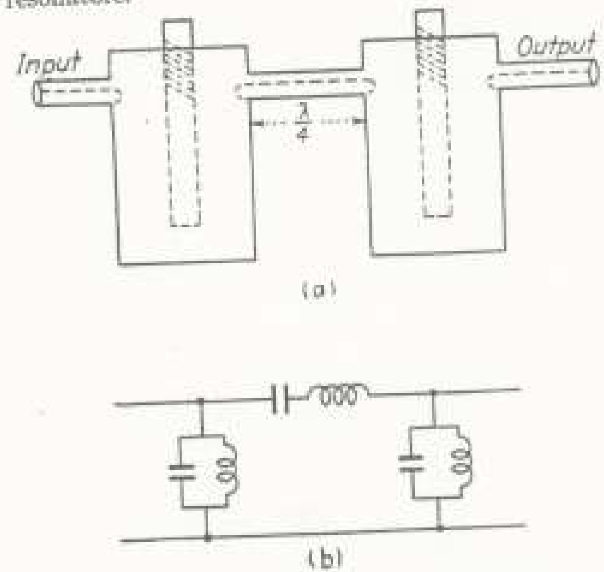


Fig. 19.—Band-pass filter and equivalent circuit.

Low-pass and high-pass filters may be constructed as shown in Fig. 20. The inductive reactances consist of short-circuited lines which are less than a quarter wavelength long.

A rigorous analysis of filters of the type shown in Figs. 18 to 20 is quite laborious. However, several useful relationships may be obtained by analogy with equivalent lumped-parameter networks. For example, the cutoff frequencies of low-pass and high-pass filters with lumped elements are $f_c = 1/\pi\sqrt{LC}$ and $f_c = 1/4\pi\sqrt{LC}$, respectively, where L and C are as shown in Figs. 20b and 20d.[1] The image impedance of the low-pass filter at zero frequency or the high-pass filter at infinite frequency is $Z_k = \sqrt{Z_1 Z_2}$, where Z_1 is the total series impedance and Z_2 is the total

[1] Everitt, W. L., "*Communication Engineering*," 2d ed., chap. 6, McGraw-Hill Book Company, Inc., New York, 1937.

... impedance of the lumped-parameter filter. The attenuation constant ... the lumped-parameter filter, in the attenuation band, is given by cosh ... $|1 + Z_1/2Z_2|$. These relationships may be used as first approximations in the design of filters such as those described above. It is helpful to construct experimental filters with elements which can be adjusted to give the desired characteristics. The dimensions obtained from the experimental filter are then used in the design of the actual filter.

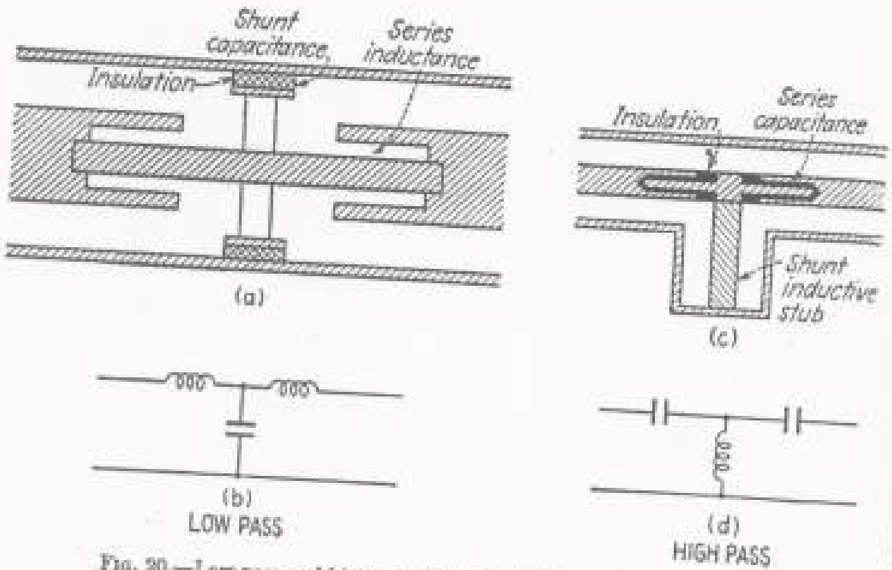


FIG. 20.—Low-pass and high-pass filters and their equivalent circuits.

In order to illustrate the methods of analysis of distributed-parameter filter networks, let us consider the band-pass filter shown in Fig. 18a. It is assumed that the transmission elements are lossless. The image impedance Z_k for the network as a whole is given by

$$Z_k = \sqrt{Z_{oc}Z_{sc}} \tag{1}$$

where Z_{oc} is the impedance looking into terminals ab with terminals cd open-circuited, and Z_{sc} is the impedance at ab with terminals cd short-circuited. The image impedance is the impedance which should be used to terminate the network in order to prevent reflection. It corresponds to the characteristic impedance in transmission-line theory. If the two series branches in Fig. 18a are identical, the network is symmetrical and the image impedance at ab is equal to that at cd. If the series branches are unequal, the network is unsymmetrical and the image impedances at ab and cd are unequal and the network is then an impedance-transforming filter.

The propagation constant $\Gamma = \alpha' + j\beta'$ for the entire filter net... contains an attenuation constant α' and a phase constant β'. The p... tion constant is related to the open-circuited and short-circuited impedances by

$$\cosh \Gamma = \sqrt{\frac{Z_{oc}}{Z_{oc} - Z_{sc}}} \tag{2}$$

We may replace Γ by $\alpha' + j\beta'$ and expand the hyperbolic cosine to obtain

$$\cosh(\alpha' + j\beta') = \cosh \alpha' \cos \beta' + j \sinh \alpha' \sin \beta' \tag{3}$$

For a dissipationless filter, α' is zero in the pass band and β' is either zero or π radians in the attenuation band. Equation (3) then reduces to

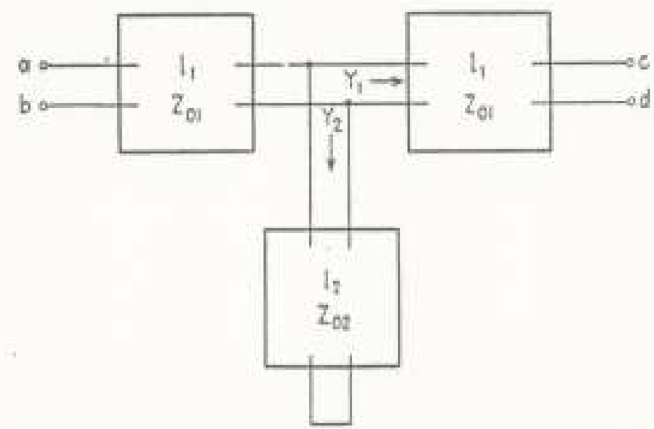


FIG. 21.—Block diagram representing the band-pass filter of Fig. 18a.

$\cosh \Gamma = \cos \beta'$ in the pass band and $\cosh \Gamma = \pm \cosh \alpha'$ in the attenuation band. In the band-pass filter, the pass band lies in the region defined by $-1 < \cosh \Gamma < +1$.

Let us now evaluate the image impedance and propagation constant for the filter shown in Fig. 18. A block diagram of this filter is shown in Fig. 21. Admittances will be used since we are dealing with parallel circuits. Consider first the admittance at terminals ab with cd short-circuited. The admittances Y_1 and Y_2 in Fig. 21 are

$$Y_1 = -jY_{01} \cot \beta l_1$$
$$Y_2 = -jY_{02} \cot \beta l_2 \tag{4}$$

The admittance $Y_1 + Y_2$ terminates the input branch of the filter. The input admittance of a lossless line terminated in an admittance Y_R may be obtained by writing Eq. (8.06-4) in terms of admittances, yielding

$$Y = Y_0\left(\frac{Y_R + jY_0 \tan \beta l}{Y_0 + jY_R \tan \beta l}\right) \tag{5}$$

obtain ... admittance at ab with cd short-circuited,

$$Y_{sc} = jY_{01}\left[\frac{Y_{01}(\tan\beta l_1 - \cot\beta l_1) - Y_{02}\cot\beta l_2}{2Y_{01} + Y_{02}\tan\beta l_1 \cot\beta l_2}\right] \qquad (6)$$

Let us now find the input admittance at ab with cd open-circuited. With cd open-circuited, the admittances Y_1 and Y_2 are

$$Y_1 = jY_{01}\tan\beta l_1$$
$$Y_2 = -jY_{02}\cot\beta l_2 \qquad (7)$$

By substituting $Y_R = Y_1 + Y_2$ into Eq. (5), with the equations for Y_1 and Y_2 as given by Eq. (7), we obtain the open-circuited admittance

$$Y_{0c} = jY_{01}\left(\frac{2Y_{01}\tan\beta l_1 - Y_{02}\cot\beta l_2}{Y_{01} - Y_{01}\tan^2\beta l_1 + Y_{02}\cot\beta l_2 \tan\beta l_1}\right) \qquad (8)$$

To obtain Z_k and $\cosh\Gamma$, substitute $Z_{sc} = 1/Y_{sc}$ and $Z_{0c} = 1/Y_{0c}$, where Y_{sc} and Y_{0c} are given by Eqs. (6) and (8), into Eqs. (1) and (2). Expressing the final result in terms of impedances, we obtain, after considerable manipulation,

$$Z_k = Z_{01}\sqrt{\frac{1 + (Z_{01}\tan\beta l_1/2Z_{02}\tan\beta l_2)}{1 - (Z_{01}\cot\beta l_1/2Z_{02}\tan\beta l_2)}} \qquad (9)$$

$$\cosh\Gamma = \cos 2\beta l_1 + \frac{Z_{01}}{2Z_{02}}\frac{\sin 2\beta l_1}{\tan\beta l_2} \qquad (10)$$

Now consider the special case in which $l_1 = l_2$, hence $\beta l_1 = \beta l_2$. Equation (10) may be written

$$\cosh\Gamma = \left(1 + \frac{Z_{01}}{2Z_{02}}\right)\cos 2\beta l + \frac{Z_{01}}{2Z_{02}} \qquad (11)$$

The mid-frequency of the pass bands occurs when the series branch is resonant and the shunt branch is antiresonant. In Sec. 8.06 it was shown that this occurs when $\beta l = n\pi/2$ where n is an odd integer. Inserting $\beta = \omega/v_c$ into this expression, the mid-frequencies are found to be

$$\frac{\omega_m l}{v_c} = \frac{n\pi}{2} \quad \text{or} \quad f_m = \frac{nv_c}{4l} \qquad n \text{ is odd}$$

where $v_c = 3 \times 10^8$ meters per second is the velocity of light. The condition that $\beta l = n\pi/2$, where n is odd, also corresponds to $\cosh\Gamma = -1$, as is evident by substitution of $\beta l = n\pi/2$ into Eq. (11).

Cutoff occurs when $\cosh\Gamma = 1$. Equation (11) then beco...

$$\cos 2\beta l = \frac{1 - (Z_{01}/2Z_{02})}{1 + (Z_{01}/2Z_{02})} \qquad (12)$$

Comparing this with the identity $\cos 2\beta l = (1 - \tan^2\beta l)/(1 + \tan^2\beta l)$, we obtain

$$\tan\frac{\omega_c l}{v_c} = \pm\sqrt{\frac{Z_{01}}{2Z_{02}}}$$
$$\omega_c = \frac{v_c}{l}\tan^{-1} \pm \sqrt{\frac{Z_{01}}{2Z_{02}}} \qquad (13)$$

The two cutoff frequencies for each band correspond to the positive and negative signs in Eq. (13). The width of the pass band decreases as the ratio $Z_{01}/2Z_{02}$ increases, approaching zero band width as $Z_{01}/2Z_{02} \to \infty$.

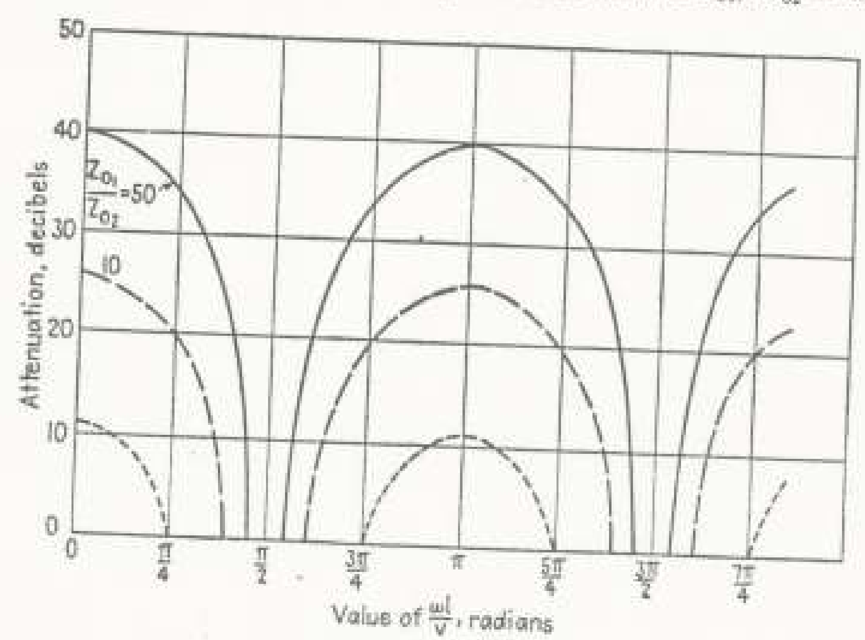


FIG. 22.—Attenuation characteristic of the filter shown in Fig. 18a.

From a practical point of view, the largest attainable ratio is of the order of $Z_{01}/2Z_{02} = 100$, corresponding to a minimum pass-band width of 20 per cent. In the attenuation band we have $\cosh\Gamma = \cos\alpha'$. A plot of α' as a function of βl for various values of $Z_{01}/2Z_{02}$ is shown in Fig. 22.

The [illegible] impedance at the mid-frequency is obtained by inserting $\beta l = n$[illegible] [w]here n is odd) into Eq. (9), yielding

$$Z_k = \sqrt{1 + \frac{Z_{01}}{2Z_{02}}} \tag{14}$$

Now consider another special case in which it is assumed that $Z_{01} = 2Z_{02}$. Equation (10) then becomes

$$\cosh \Gamma = \frac{\sin \beta(2l_1 + l_2)}{\sin \beta l_2} \tag{15}$$

The mid-frequency for this filter occurs when $\cosh \Gamma = 0$, or when $\beta(2l_1 + l_2) = n\pi/2$, where n is even. Inserting $\beta = \omega/v_c$ into this expression, we obtain

$$f_m = \frac{nv_c}{4(2l_1 + l_2)} \qquad n \text{ is even} \tag{16}$$

Cutoff frequencies occur when $\cosh \Gamma = \pm 1$, corresponding to

$$f_{c1} = \frac{v_c}{4(l_1 + l_2)} \qquad f_{c2} = \frac{v_c}{4l_1} \tag{17}$$

The image impedance at the mid-frequency is

$$Z_k = Z_{01} \sqrt{- \tan\left(\frac{\pi/2}{1 + l_2/2l_1}\right) \tan\left[\frac{\pi}{2}\left(\frac{1 + l_2/l_1}{1 + l_2/2l_1}\right)\right]} \tag{18}$$

For narrow bands the image impedance is approximately $Z_k = (4l_1/\pi l_2)Z_{01}$.

The band width of this type of filter decreases as l_2 is made smaller. By making l_2 very small, it is possible to obtain a band-pass filter with a very narrow pass band.

PROBLEMS

1. A lossless line is terminated in a pure resistance which is not equal to the characteristic impedance of the line. Prove that the standing wave has its maximum and minimum values at the receiving end and at integral multiples of quarter-wavelength distances from the receiving end. Derive an expression for the standing-wave ratio for this case. Will the voltage be a maximum or a minimum at the receiving end?
2. A tuned circuit for an oscillator consists of an open-circuited line containing two different sizes of coaxial line as shown in Fig. 23. Derive an expression for the input impedance at terminals ab assuming that the lines are lossless. What are the conditions for antiresonance at ab? Show that the antiresonant frequency can be varied by varying the lengths l_1 and l_2, but keeping $l_1 + l_2$ constant. (*Note:* This is the principle used in tuning the lighthouse-tube oscillator shown in Fig. 6b, Chap. 5.)
3. A silver-plated coaxial line has dimensions $a = 0.5$ cm and $b = 2.0$ [illegible] is to be used at a frequency of 700 megacycles. Assume that the dielectric [illegible].
   (*a*) Compute the attenuation constant and Q of the line.
   (*b*) Evaluate the input impedance of a quarter-wavelength short-circuited section of line and a quarter-wavelength open-circuited section of line.
   (*c*) Compute the data and plot a curve of the scalar value of input impedance as a function of frequence in the vicinity of the antiresonant impedance for the short-circuited line.
4. A line having a characteristic impedance of 75 ohms is terminated in an unknown impedance which is to be measured. The maximum and minimum voltages on the line are found to be 120 volts and 25 volts, respectively, with the maximum voltage point 30 cm from the terminal impedance. The frequency is 300 megacycles.
   (*a*) Compute the value of the terminal impedance and check this value using the impedance diagram.
   (*b*) Find the length and position of a single stub which will match the impedance to the line.

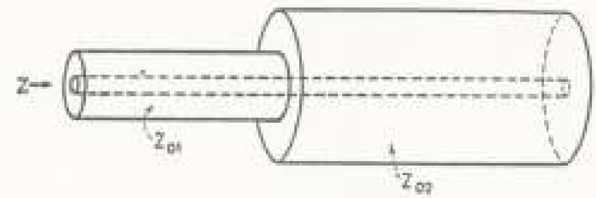


Fig. 23.

5. It is desired to construct an oscillator which will operate at 500 megacycles and deliver an appreciable amount of third harmonic in its output. A short-circuited coaxial line, having a characteristic impedance of 75 ohms, is used for the tuned circuit. This is shunted at its input end by the grid-plate capacitance of the tube which has a value of 1.5×10^{-12} farad. Determine the length of line and additional shunt capacitance which should be added at the input end of the line in order to make it simultaneously antiresonant at 500 and 1,500 megacycles.
6. The input circuit of a microwave receiver consists of a half-wavelength dipole antenna coupled to a coaxial line which is $2\frac{1}{2}$ wavelengths long. The line is terminated by a crystal detector. A short-circuited stub is placed near the receiving end in order to assure maximum power transfer to the detector and to increase the selectivity of the input circuit. The antenna impedance and the characteristic impedance of the line are both 72 ohms. The frequency selectivity of the circuit for crystal resistances of 150 ohms and 1,000 ohms are to be compared. The input circuit is tuned to a wavelength of 20 cm. The impedance diagram is to be used in the following calculations.
   (*a*) Find the length of stub and its position for matched impedances at $\lambda = 20$ cm.
   (*b*) Tabulate the values of complex impedance of the line at the antenna terminals (with the antenna disconnected) at wavelengths of $\lambda = 20 \pm 0.5k$ cm, where k is an integer from 1 to 10. Plot a curve of the scalar value of input impedance against wavelength.
   (*c*) Assume that the antenna can be replaced by a generator having an internal impedance of 72 ohms. Using the impedances of part (*b*) and the power diagram of Fig. 9, plot curves of P/P_{max} against wavelength for the two cases considered.
   (*d*) What conclusions can be drawn regarding the relative selectivity of the two systems?

... a load impedance is given by

$$P = \frac{|V_{max}||V_{min}|}{Z_0}$$

where $|V_{max}|$ and $|V_{min}|$ are the maximum and minimum voltages on the transmission line supplying the load. This relationship is valid regardless of the magnitude or phase angle of the load impedance.

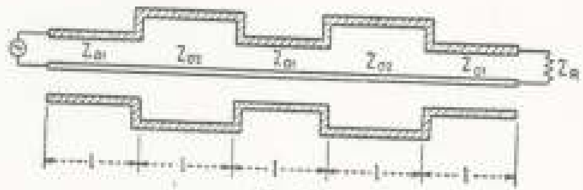


FIG. 24.

8. The system shown in Fig. 24 is a band-pass filter. The lengths l are all a half wavelength long in the middle of the pass band. Assume that $Z_R = 50$ ohms, $Z_{01} = 50$ ohms, and $Z_{02} = 200$ ohms. The wavelength at the middle of the pass band is 10 cm. Using the impedance diagram,
(a) Find the input impedance of the system at wavelengths of $\lambda = 10 \pm 0.3k$ cm where k is an integer from 1 to 10.
(b) Using the input impedances obtained in part (a) and the power diagram of Fig. 9, obtain the corresponding values of P/P_{max}. Plot a curve of P/P_{max} against wavelength.
9. Design a symmetrical coaxial-line filter of the type shown in Fig. 18a to have a mid-frequency of 1,500 megacycles and a bandwidth of 5 megacycles. The image impedance is $Z_R = 150$ ohms at the mid-frequency. Plot a curve of α against βl for the filter.

CHAPTER 11

TRANSMITTING AND RECEIVING SYSTEMS

The fundamental processes involved in the transmission and reception of signals at microwave frequencies are essentially the same as those at ordinary radio frequencies. At the transmitter, the carrier may be generated by any one of the various types of microwave oscillators previously described, or it may be derived from a crystal oscillator followed by a chain of frequency multipliers. The carrier is modulated and the signal is then either impressed directly upon the transmitting antenna or it may be amplified by one or more successive stages of amplification before being radiated. Either amplitude, phase, or frequency modulation can be used. A new type of modulation, known as pulse-time modulation, also offers interesting possibilities at microwave frequencies.

Superheterodyne receivers are commonly employed in microwave systems. The input usually contains a frequency-selective circuit followed by a mixer. In the mixer, the incoming signal is heterodyned against a signal generated by a local oscillator to obtain the difference frequency. This difference frequency is amplified in one or more tuned stages of intermediate-frequency amplification after which it is detected and amplified in an audio- or video-frequency amplifier.

At microwave frequencies, difficulty is often encountered in maintaining a high degree of frequency stability of the carrier oscillator at the transmitter and of the local oscillator at the receiver. Various automatic frequency-control systems have been devised for the purpose of stabilizing these oscillators. Another difficulty arises owing to the fact that the bandwidths of the microwave circuits and the tuned circuits in the intermediate-frequency amplifier are usually relatively large. Consequently these circuits admit a large amount of noise, resulting in a low signal-to-noise ratio. Certain types of microwave oscillators, particularly magnetrons, generate relatively high-noise voltages. These considerations favor the types of modulation in which the signal may be more readily separated from the noise, such as frequency modulation or pulse modulation.

A typical low-power microwave transmitting and receiving system, employing a frequency-modulated klystron oscillator and a superheterodyne receiver, is shown in the block diagram of Fig. 1. At the transmitter the klystron is frequency modulated by impressing the modulating voltage on its anode. Parabolic reflectors are used for directional transmission