some intermediate frequency range such as 100 to 200 Mc. The microwave signals can be detected by using the harmonics of the oscillator which are produced in the internal detector of the meter. Harmonics up to the twentieth can be used conveniently and reliably.\footnote{One commercial model of a heterodyne frequency meter is described by E. Karplus, A Heterodyne Frequency Meter for 10 to 3,000 Megacycles, General Radio Experimenter, vol. 20, nos. 2 and 3, July and August, 1945.}

The elements of a representative heterodyne frequency meter, intended for use between 10 to 3,000 Mc, are shown in Fig. 8.6. The variable-frequency oscillator, tuning from 100 to 200 Mc, is coupled to a crystal rectifier detector. The beat frequency is introduced into the audio amplifier, the output of which is available at either a panel meter or at terminals for use with head phones. The meter is used when the frequency is not sufficiently stable to produce steady, audible beat notes or the beat note is below an audible rate. The quartz-crystal oscillator can be connected to the input terminals for calibration. A 1-Mc quartz oscillator may be used to control a chain of multivibrators to generate low-frequency sequences. The 1-Mc crystal oscillator can be checked against WWV with the aid of a radio receiver.\footnote{Details of operation of a heterodyne frequency meter can be found in an instruction book for the General Radio type 720-A heterodyne frequency meter. Several useful techniques of wide applicability are described.}

\footnotetext[1]{One commercial model of a heterodyne frequency meter is described by E. Karplus, A Heterodyne Frequency Meter for 10 to 3,000 Megacycles, General Radio Experimenter, vol. 20, nos. 2 and 3, July and August, 1945.}

\footnotetext[2]{Details of operation of a heterodyne frequency meter can be found in an instruction book for the General Radio type 720-A heterodyne frequency meter. Several useful techniques of wide applicability are described.}

\begin{equation}
\begin{align*}
\omega_0^2 &= \frac{1}{LC} \\
Q_a &= \frac{\omega_0 L}{R_a} \\
R_a &= \frac{\omega_0 L Q_a}{R_a} = \left(\frac{\omega_0 L}{R_a}\right)^2
\end{align*}
\end{equation}

The three quantities defined by these relations can be measured experimentally to provide the three relations needed to compute the three circuit parameters. If \( \omega_B \), \( Q_B \), and \( R_B \) are measured, the circuit parameters can be found by solving Eq. (9.1), resulting in

\[
\begin{align*}
L &= \frac{R_B}{\omega_B Q_B} \\
C &= \frac{Q_B}{\omega_B R_B} \\
R_s &= \frac{R_B}{Q_B^2}
\end{align*}
\]  

(9.2)

This procedure indicates how the three microwave circuit parameters can be evaluated from the experimental study of the resonant circuit as a whole.

The study of the microwave resonant cavity differs from that of the low-frequency circuit in two respects: first, the equivalent-circuit parameters must be established separately for each mode under consideration; and second, the quantity \( R_s \), called the shunt resistance, is not uniquely defined due to the ambiguity in the meaning of the voltage and current. For convenience, \( R_s \) can be defined as

\[
R_s = \frac{1}{2} \frac{\langle |E| \rangle^2}{\text{power dissipated}} = \frac{\langle |E| \rangle^2}{2W}
\]  

(9.3)

where \( E \) is the peak electric field along the path of integration between some two points in the cavity and \( W \) is the power dissipated in the cavity.\(^1\)

For a few simple geometrical shapes the quantities \( \omega_B \), \( Q_B \), and \( R_B \) can be computed from the geometrical factors and the conductivity of the cavity walls. However, for most useful cavity shapes, mathematical computation is too difficult to be practical, and these quantities must be determined directly by experiment. Furthermore, both \( R_s \) and \( Q_B \) depend upon the particular sources of loss in the cavity and can be found only by experiment.

The knowledge of \( \omega_B \), \( Q_B \), and \( R_B \) is necessary and sufficient for the complete description of the resonant cavity in a given mode; in practice, these experimentally determined quantities form the set of cavity characteristics which are descriptive and sufficient for most applications. If necessary, the equivalent circuit parameters shown in Fig. 9.1 can be computed with the aid of Eq. (9.2).

In Chap. 7 experimental procedures for the determination of the resonant frequency of the cavity are discussed in connection with the measurement of wavelength. The experimental techniques used to measure \( Q_B \) are discussed in this chapter and those to determine \( R_s \) in Chap. 10.

\( a. \) Equivalent Circuits; Definitions of \( Q_B \), \( Q_L \), \( Q_m \), and Coupling Coefficient \( \beta \). A microwave cavity can be coupled to one, two, or more transmission lines. The cavity characteristics can be studied experimentally by measuring the self-impedance at one pair of input terminals, or by measuring the transfer of power from one set of terminals to another; the unused terminals, in either case, can be terminated in some known impedance. The complete description of the cavity characteristics and the effect of the coupled transmission lines can be evaluated by performing as many independent experiments as there are coupled transmission lines. By this process, the study of a cavity with multiple sets of input terminals can be reduced to the study of a system with only one or two sets of terminals.

The equivalent circuit of a cavity with two inputs is shown in Fig. 9.2. The coupling between the cavity and the transmission lines is symbolically represented by an iris which indicates some arbitrary method of exciting the cavity fields; it can be shown that the actual form of the coupling mechanism does not affect the equivalent circuit. The cavity resonance in a particular mode is represented by the parameters \( L \), \( C \), and \( R_s \). Two alternate forms of the equivalent circuit representing the coupling between the circuit and the transmission lines are shown in Fig. 9.2b and c; in the first, the coupling is represented by ideal transformers and in the second, by mutual inducances. In general, the coupling between the cavity and the transmission lines contains both reactive and resistive components; for example, a coupling loop has both self-inductance and resistive loss. Irrespective of the details of the coupling mechanism, the inducances \( L_1 \) and \( L_2 \) represent the self-inductances of the coupling elements which are due to the fringing field caused by the geometrical discontinuity at the junction of the transmission line and the cavity. These equivalent circuits can be simplified for analysis by referring the impedances of the three circuit loops to a single one, as indicated in Fig. 9.2d and e.

The characteristic behavior of the complete system can be studied in several ways. The simplest is to observe the variation in the power delivered to the load as the frequency of the signal source is varied or...
the cavity is tuned. In either case, the familiar resonance phenomenon occurs, causing the power output at resonance to differ substantially from the detuned condition. The exact behavior of the system depends upon the characteristics of the cavity and the degree of coupling between the cavity and the transmission lines.

Sometimes it is desired to measure the cavity parameters and the coefficients of coupling between the cavity and the transmission lines; sometimes it is merely desired to determine the unloaded cavity parameters, i.e., the characteristics of the cavity if it were not perturbed by the presence of the coupled transmission lines. For this purpose, it is convenient to measure the degree of coupling between the cavity and the transmission lines by specifying $Q_0$, $Q_L$, and $Q_{ext}$, the unloaded, the loaded, and the external $Q$ values, respectively, as defined in Chap. 7. These definitions are important because applications occur in which the effective coupling is of primary importance and also because in any given measurement the effect of residual coupling must be known if the meaning of the measurements is to be properly interpreted.

For convenience, the definitions of the $Q$ values given in Chap. 7 are repeated below. Consider a cavity coupled to a signal source whose internal impedance is equal to the characteristic impedance of the transmission line as indicated in Fig. 9.3a. The equivalent circuit of the cavity and the transmission line is shown in Fig. 9.3b, where the terminals of the coupling system (or network) are presumed to be located at some arbitrary position $a-a$ near the cavity. $L_1$
representing the self-inductance of the coupling mechanism and \( M \) the mutual inductance between it and the cavity inductance \( L \). The resistive losses in the coupling network are neglected (the effect of dissipation in the coupling network is included in the analysis given in Sec. 9.4). This circuit can be simplified further as shown in Fig. 9.3c and d; in the first, the cavity is shown as a coupled impedance in series with the primary; in the second, the primary is represented as a coupled impedance in series with the cavity parameters. The impedance coupled in series with the cavity parameters due to a matched generator is given by

\[
Z = \frac{(\omega M)^2}{Z_0 + j\omega L_1} = \frac{(\omega M)^2}{Z_0[1 + (\omega L_1/Z_0)^2]} \left( 1 - j\frac{\omega L_1}{Z_0} \right)
\]

Using definitions given by Eqs. (9.11) and (9.12), and \( X_1 = \omega L_1 \), Eq. (9.5) becomes

\[
Z = \beta R_t \left( 1 - \frac{jX_1}{Z_0} \right)
\]

The loaded \( Q \) value of the system is defined as the ratio of total reactance to total series loss. It is given by

\[
Q_L = \frac{\omega L - \beta R_t X_1/Z_0}{R_t(1 + \beta)} = \frac{\omega L}{R_t} \left( \frac{1}{1 + \beta} - \frac{(\beta R_t X_1/Z_0)(X_1/Z_0)}{1 + \beta} \right)
\]

The second term in the numerator of Eq. (9.8), representing the ratio of coupled reactance to the cavity reactance, is usually small compared to unity and can be neglected. Equation (9.8) then becomes

\[
Q_L = \frac{Q_0}{1 + \beta}
\]

where

\[
Q_0 = \frac{\omega L}{R_t}
\]

\[
\beta = \frac{(\omega M)^2}{Z_0 R_t} \left( 1 + \frac{(X_1/Z_0)^2}{1 + (X_1/Z_0)^2} \right)
\]

where \( \beta_1 = (\omega M)^2/Z_0 R_t \) is the ratio of the coupled resistance to the cavity resistance \( R_t \). When \( \beta_1 = 1 \), the coupled resistance and cavity losses are equal, and the cavity is said to be critically coupled. When \( \beta_1 < 1 \), the cavity is said to be undercoupled; when \( \beta_1 > 1 \), the cavity is called

Overcoupled. Under most circumstances, the second term in Eq. (9.12) is nearly equal to unity and \( \beta \approx \beta_1 \). Thus, at critical coupling \( Q_L = Q_0/2 \).

Equation (9.9) can be written as

\[
\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{\beta}{Q_0}
\]

or

\[
\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_{ext}}
\]

where

\[
Q_{ext} = \frac{Q_0}{\beta}
\]

or

\[
\beta = \frac{Q_0}{Q_{ext}}
\]

b. \( Q \) Circles. The solution of many problems involving resonant cavities can be simplified by considering, either graphically or analytically, the cavity input impedance in the complex impedance plane.

The impedance at the terminals of the coupling network \( o-o \) in Fig. 9.3b is equal to

\[
Z_{oo} = jX_1 + \frac{(\omega M)^2}{R_t + \frac{1}{1 + j(\omega L - 1/\omega C)}}
\]

or

\[
Z_{oo} = jX_1 + \frac{1 + j(\omega L - 1/\omega C)}{R_t}
\]

where \( \omega_0^2 = 1/LC \). For cavities with high \( Q_0 \), \( \omega = \omega_0 \). Equation (9.18) can then be written as

\[
\delta = \frac{\omega - \omega_0}{\omega}
\]

where

\[
\delta = \frac{2Q_0}{\omega}
\]

The quantity \( \delta \) is called the frequency-tuning parameter. For high-\( Q \) systems, \( \delta \) can be considered to be the variable, irrespective of whether the frequency of the signal \( \omega \) or the resonant frequency of the cavity \( \omega_0 \) is changed.

Consider a plot of Eq. (9.19) in the rectangular impedance plane. The second term of this equation corresponds to an equation of a circle, representing the impedance of a shunt resonance circuit with a resonant impedance \( \beta_1 Z_0 \). The effect of the self-reactance of the coupling system, as expressed by the first term in Eq. (9.19) is to displace the circle along the imaginary axis, as indicated in Fig. 9.4a.

The analysis and interpretation of certain experiments can be aided by choosing special reference planes along the transmission line at which the term representing the self-reactance of the coupling system disappears. This happens at a series of singular locations, half a wavelength apart, which are called the detuned-short positions. At frequencies
far off resonance, the second term in Eq. (9.19) vanishes, leaving the
self-reactance of the coupling system as the terminating load. This
reactive termination produces a complete reflection of the incident signal
and results in a series of voltage nodes which can be found by means of
the standing-wave detector.

Fig. 9.4. Input impedance of the resonant cavity. (a) The impedance referred to
some arbitrary position near the cavity; (b) the impedance locus referred to the
detuned short position.  

The impedance at the detuned short position, looking toward the
cavity, is essentially simple, having the form of a simple resonant shunt
circuit. This can be demonstrated analytically as follows: Let the
terminals b-b be selected at a distance l away from the terminals a-a.*

The arbitrary choice of the original terminals at a-a determines the effective value
of the coupling reactance Lc. In practice, the exact location of the terminals a-a is
of no concern because all of the quantities can be referred directly to the unambiguous
location of the detuned short.

Using Eq. (4.64), the cavity impedance at a-a can be transformed to the
terminals b-b. Therefore,

\[ \frac{Z_{bb}}{Z_0} = \frac{Z_{aa} + jZ_{ao} \tan \beta l}{Z_0 + jZ_{ao} \tan \beta l} \]  

(9.21)

The location of the terminals b-b can be chosen so that the impedance at
terminals b-b becomes zero when the cavity is detuned.

With the cavity detuned, \( Z_{ao} = jX_1 \). Hence, \( Z_{bb} = 0 \) when

\[ \tan \beta l = - \frac{X_1}{Z_0} \]  

(9.22)

or

\[ \beta l = \tan^{-1} \left( \frac{X_1}{Z_0} \right) \]  

(9.23)

Combining Eqs. (9.19), (9.21), and (9.23), the impedance at the detuned
short position for any value of \( \delta \) becomes

\[ \frac{Z_{bb}}{Z_0} = \frac{\beta}{1 + j2Q_0(\delta - \delta_b)} \]  

(9.24)

where

\[ \delta_b = \frac{\beta}{2Q_0} \left( \frac{X_1}{Z_0} \right) \]  

(9.25)

Equation (9.24) represents the impedance of a parallel resonant circuit
with a resonant impedance \( \beta Z_0 \); a graph of this location in the impedance
plane is shown in Fig. 9.4b. A comparison of Eqs. (9.19) and (9.24)
sows that the diameter of the resultant transformed circle is different
from the one that corresponds to the impedance at a-a. Also, the
resonant frequency of the circuit described by Eq. (9.24) no longer occurs
at the natural resonant frequency of the cavity but is altered by the
amount given by Eq. (9.25). However, these changes are not especially
significant and can be taken into account in the interpretation of the
results.

The shunt representation can be transformed into a series representa-
tion if the reference plane is chosen \( \lambda/4 \) away from the detuned-short
position. This position can be termed the "detuned-open" position. Tran-
sforming Eq. (9.24) through \( \lambda/4 \) with the aid of Eq. (4.64) leads to

\[ \frac{Z_{ao}}{Z_0} = \frac{1}{\beta} \left[ 1 + j2Q_0(\delta - \delta_b) \right] \]  

(9.26)

This is an equation of a series resonant circuit; the impedance at
resonance is \( Z_0/\beta \). Both the shunt and series representations are indicated
in Fig. 9.5.

* Unfortunately, the symbol \( \delta \) has two different meanings in the following equations.
When used in the expression \( \tan \beta l \) it represents the propagation constant in the trans-
mision line; when \( \delta \) is used alone, the meaning is the principal one in this chapter,
i.e., the coupling coefficient.
c. **Typ. Q Measurements.** The Q values of a resonant cavity can be determined experimentally in many ways. These can be divided into four groups:

1. Transmission method
2. Impedance measurement
3. Transient decay or the decrement method
4. Dynamic methods

In the first of these, the cavity with input and output terminals is used as a transmission device. The output signal is measured as a function of frequency, resulting in the conventional resonance curve from whose bandwidth the Q value can be computed. Although it is simple conceptually, there are practical difficulties in its application which make it necessary to pay considerable attention to several details to obtain accurate results. The use of this method is described in Sec. 9.2.

The second method, discussed in Sec. 9.3, is based upon the observation of the variation of the cavity input impedance with frequency. If the impedance of the cavity is measured as a function of frequency,

![Diagram](image)

**Fig. 9.5.** Shunt and series representation of the resonant cavity: The equivalent impedance (a) at the detuned-short position, (b) at the detuned-open position. The corresponding loci are shown in the complex impedance plane.

The impedance locus referred to the detuned-short position will lie on a circle; if referred to the detuned-open position, the locus will lie on a straight line. These data can be readily interpreted to provide the values of $$Q_0$$, $$Q_L$$, and $$Q_w$$. Since a circle can be defined by three points, it is necessary to make only three independent impedance measurements to describe completely the characteristics of the cavity and its coupling system. To improve the accuracy, additional data are usually taken to detect random, systematic, or accidental errors.

The impedance data can be interpreted by one of several methods. The standing-wave ratio can be used alone without the corresponding phase data. A plot of VSWR versus frequency contains all the necessary information; the use of this data is analogous to employing the universal resonance curve of a resonant circuit at low frequencies. Conversely, the phase data can be used without the VSWR data. The detailed discussion of these methods shows that sometimes one can choose between these methods in accordance with his preference. Sometimes, however, the choice of a particular method can lead to greater accuracy. Consider, for example, the Smith chart plot shown in Fig. 9.6 of the input impedance for three degrees of coupling. If the cavity is nearly critically coupled, the circle passes through the real axis near the point (1,0). In this case, both the VSWR and the phase information are equally important; the best accuracy is obtained for measuring the vector impedance at each frequency. If, however, the cavity is weakly coupled ($$\beta \ll 1$$), the resultant impedance locus is a very small circle. In this case, the phase data are not accurate because the entire circle is contained within a small range of phase angles. However, the VSWR varies substantially with frequency and the determination of the frequency interval between certain “half-power points” results in good accuracy. If the cavity is greatly overcoupled ($$\beta \gg 1$$), the resultant circle approaches the periphery of the Smith chart; the VSWR is high and does not change appreciably, but the phase angle changes rapidly and provides the needed information.

The decrement method described in Sec. 9.5, particularly applicable to high-Q cavities, uses the transient decay of the natural oscillations in the cavity. If the cavity under study is excited by a pulsed signal, during
the off peak the natural fields in the cavity decay exponentially with time and the time constant of the decay determines the Q. Figure 9.7a shows the equipment necessary for this method. A pulsed-modulated oscillator provides signals of sufficient duration to establish steady-state fields in the cavity. The output signal from the cavity is detected by a sensitive detector (preferably, a superheterodyne receiver), amplified by a wideband amplifier, and observed by an oscilloscope whose sweep is synchronized by the pulser. The typical response is indicated in Fig. 9.7b. The time constant is measured and the results interpreted as described in Sec. 9.5.

The decrement method is especially convenient for the high-Q systems because it does not need the high degree of frequency stability required in other types of measurements. This is apparent from the fact that in other methods the frequency must be sufficiently stable during the course of a given measurement; for example, in using VSWR measurements, the frequency must be constant at least during the time needed to measure a single standing-wave ratio. The measurement of Q values in excess of $10^4$ is especially simple; however, at lower values, the decay period is too short for convenient measurement.

The fourth group of methods described in Sec. 9.6 is based upon the dynamic observation of the cavity characteristics. These techniques are useful for two reasons: the frequency stability requirements of the signal source are reduced; and the Q values can be obtained more quickly, sometimes from a direct reading meter.

9.2 Transmission Method. The transmission method illustrated in Fig. 9.8a is the simplest phenomenological measurement of Q. A signal generator, preferably completely isolated from the load by a resistive pad or a ferrite isolator, is connected to the cavity through the input coupling system; a detector with a known response low is connected to the output coupling system. By varying the frequency of the signal generator, the transmission resonance curve shown in Fig. 9.8b can be observed from whose bandwidth the cavity Q can be determined. The resonance curve can be obtained also by tuning the cavity and keeping the frequency of the oscillator fixed. A choice between the two methods depends upon the details and ease of tuning and calibrating the apparatus under test.

The relation between the observed bandwidth of the resonance curve, the input and output coupling coefficients, $Q_i$ and $Q_o$, can be obtained as
follows: Assume that the load and generator impedances indicated in Fig. 9.2b are equal to the characteristic impedances of their respective lines. The losses of the coupling systems represented by \( R_1 \) and \( R_2 \) can be neglected or considered as parts of \( R_0 \) and \( R_L \), respectively. The self-inductances \( L_1 \) and \( L_2 \) can also be neglected; this approximation changes slightly the apparent resonant frequency of the system but for high-\( Q \) systems does not have other effects. With these approximations, Fig. 9.2b can be altered by referring both the primary and the tertiary loops into the middle loop, which results in the equivalent circuits shown in Fig. 9.2d and e. The loaded \( Q \) of the system is

\[
Q_L = \frac{\omega_0 L}{R_1 + n_1^2 Z_1 + n_2^2 Z_2}
\]  
(9.27)

or

\[
Q_L = \frac{\omega_0 L}{R_1 + (\omega M_1)^2 Z_1 + (\omega M_2)^2 Z_2}
\]  
(9.28)

where \( Z_1 \) and \( Z_2 \) represent the characteristic impedances of the input and output transmission lines, respectively. The input and output coupling coefficients are defined as

\[
\beta_1 = n_1^2 \frac{Z_1}{R_1}
\]  
(9.29)

\[
\beta_2 = n_2^2 \frac{Z_2}{R_2}
\]  
(9.30)

Using these, the relation between \( Q_0 \) and \( Q_L \) becomes

\[
Q_0 = Q_L (1 + \beta_1 + \beta_2)
\]  
(9.31)

The relation between the width of the resonance curve and the cavity \( Q \) is obtained as follows: The transmission loss \( T(\omega) \) through the cavity is defined as

\[
T(\omega) = \frac{P_L}{P_0}
\]  
(9.32)

where \( P_L \) is the power delivered to a load and \( P_0 \) is the maximum power available from the generator (to a matched load). Computing \( P_L \) and \( P_0 \) for Fig. 9.2d,

\[
T(\omega) = \frac{4 \delta \beta_3}{(1 + \beta_1 + \beta_2)^2 + 4 Q_0 \delta^2}
\]  
(9.33)

where \( \delta \) is the tuning parameter defined in Eq. (9.20). At resonance \( \delta = 0 \), and Eq. (9.33) becomes

\[
T(\omega_0) = \frac{4 \delta_3 \beta_3}{(1 + \beta_1 + \beta_2)^2}
\]  
(9.34)

Dividing Eq. (9.33) by Eq. (9.34), and using Eq. (9.31),

\[
T(\omega) = \frac{T(\omega_0)}{1 + 4 Q_0 \delta^2}
\]  
(9.35)

The half-power points of transmission occur when

\[
2 Q_0 \delta = \pm 1
\]  
(9.36)

or

\[
2 \delta = \pm \frac{1}{Q_0}
\]  
(9.37)

or

\[
2 \delta = \pm \frac{1}{1 + \beta_1 + \beta_2}
\]  
(9.38)

The quantity \( \Delta f \) is known as the half-power bandwidth of the resonance curve and is given by

\[
\frac{\Delta f}{f} = 2 \delta
\]  
(9.39)

Hence,

\[
Q_0 = \frac{f}{\Delta f} (1 + \beta_1 + \beta_2)
\]  
(9.40)

Thus, if the signal generator and detector impedances are both matched, the measured transmission curve shown determines \( Q_L \). The unloaded \( Q \) can be calculated if the coupling coefficients can be measured separately. However, the greatest value of the procedure lies in finding \( Q_0 \) by reducing the coupling coefficients sufficiently. This is usually done by reducing the coupling between detector and the cavity until it is found that further reduction in coupling no longer affects the measured resonance curve. A separate experiment must also be carried out to assure that the coupling between the cavity and the signal generator is sufficiently small. Depending upon practical circumstances, these procedures can be either simple or complicated. For example, if the coupling systems consist of inductive coupling loops, adjustable by rotation or withdrawal, the coupling coefficients can be reduced readily. If the coupling is provided by means of an iris, such procedures are generally impractical.

The transmission method suffers from the fact that a single measurement of the transmission curve alone, no matter how accurately made, does not give the \( Q \) values directly. For this reason, the unambiguous procedures involving the measurement of impedance, described in Sec. 9.3, are more commonly used. However, the transmission method, if carefully executed, is capable of producing good accuracy. It should be noted that the presence of loss in the coupling systems is not important, which is not the case in the impedance methods.

9.3. The Impedance Method. As explained in Sec. 9.1b, the cavity characteristics can be determined by measuring the input impedance of
a cavity as a function of frequency. The details of this method are considered in this section in several forms.

In Sec. 9.3a, procedures for determining the Q parameters are described for those conditions of coupling which permit accurate measurement of the impedance with frequency. If the coupling coefficient is very small, corresponding to the high values of $Q_L$, the standing-wave ratio alone can be measured, a procedure described in Sec. 9.3b. When the coupling coefficient is large compared to unity, corresponding to low values of $Q_L$, the phase of the input impedance becomes more meaningful and can be used independently as described in Sec. 9.3c.

The choice of the most appropriate method in a particular application is a matter of convenience, experience, and personal preference as there is no obvious division between the usefulness of the three methods of the methods mentioned. In some cases, a combination is more convenient; for example, the phase data can be used more readily if the standing-wave ratio at resonance is measured also. However, experience with the basic methods is most helpful in deciding upon the value of possible variations.

In the discussion of the basic methods the effect of loss in the coupling system is neglected; this is justified in nearly all cases. The more general case is discussed in Sec. 9.4 where this assumption is avoided.

a. Interpretation of the Impedance Data. The impedance method is used as follows. Figure 9.3 shows the cavity under study connected to a uniform transmission line through its cavity coupling system. A standing-wave detector is placed between the signal generator and the cavity to measure the input impedance. The measurement procedures are simplified if the relative tuning of the cavity and the signal generator are independent, i.e., if cavity tuning does not affect the output of the signal generator. It is particularly convenient to have the impedance of the signal source equal to $Z_0$.

The procedure begins with the determination of the detuned short position. The signal frequency is adjusted to the desired value and the cavity is detuned completely; this effectively terminates the transmission line in a pure reactance. The standing-wave detector is used to find a voltage node which locates the detuned short position; this location is recorded for future use (for convenience it can be marked on the standing-wave detector with a pencil). If the cavity is not tunable, the equivalent experiment can be performed by tuning the signal generator sufficiently far from the resonant frequency of the cavity. For high-Q systems, the change in frequency is not large, and the location of the detuned short determined in this manner is nearly the correct one. It is the approximate but not quite the correct value, as can be seen from Eq. (9.23). In case of doubt, it may be necessary to plot the position of the detuned short as a function of frequency and to refer further impedance mea-

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ments to a detuned short position appropriate for a particular frequency.

Next, it is necessary to determine the magnitude of the coupling coefficient, as the choice among the possible techniques depends upon its value. The probe of the standing-wave detector is placed at the detuned short position; Figure 9.5a shows that this locates the probe across the terminals of an equivalent-shunt resonant circuit. Tuning the cavity to produce the maximum voltage in the probe is equivalent to tuning the cavity to resonance, provided the source impedance is purely resistive. If this procedure is carried out correctly, the motion of the probe with respect to the detuned-short position will result in either a voltage maximum or voltage minimum at the detuned-short position since the cavity at resonance is a pure resistance. If the exploration of the standing-wave pattern results in a voltage minimum, the cavity is undercoupled; if it produces a maximum, the cavity is overcoupled. If the exploration indicates that a minimum (or a maximum) is not at the detuned-short position, some adjustment has been executed incorrectly. Assuming that the detuned-short position is located correctly, this means that the cavity is not exactly tuned to resonance, which is most likely if the source impedance is not purely resistive. The correct cavity tuning for resonance can be found by trial until either a minimum or a maximum occurs exactly at the detuned-short position.

The magnitude of the coupling coefficient is obtained by measuring VSWR at resonance. Since the impedances at the voltage minimum and maximum are $Z_0/r_0$ and $Z_0r_0$, respectively, where $r_0$ is the value of VSWR at resonance, Eq. (9.24) results in

Undercoupled case:

$$\beta = \frac{1}{r_0} \quad (9.41)$$

Overcoupled case:

$$\beta = r_0 \quad (9.42)$$

The evaluation of $\beta$ locates the intersection of the impedance circle with the real axis in an impedance plot shown in Fig. 9.6.

If the coupling coefficient $\beta$ is not greatly different from unity, the cavity tuning parameter $\delta$ can be changed in small increments and the impedance measured at each frequency. Typical experimental data are shown plotted in Figs. 9.9a and 9.9b; the first shows the impedance referred to the detuned-short position, and the second, to the detuned-open position.

Referring to Eq. (9.24), at certain frequencies the imaginary part of the denominator becomes equal to $\pm 1$. At these values of $\delta$, the input impedance becomes

$$\frac{Z_{th}}{Z_0} = \frac{\beta}{1 \pm j} \quad (9.43)$$
The locus of these points (corresponding to \( R = X \)) for all possible values of \( \beta \) shown in Fig. 9.10 is a circle with the center on the periphery of the Smith chart at 90° points and passes through the two endpoints of the resistive axis. The intersection of this circle with the circle representing the graph of the impedance as a function of frequency determines those frequencies at which

\[
2Q_0(\delta - \delta_0) = \pm 1
\]

Let these two values of \( \delta \) be called \( \delta_1 \) and \( \delta_2 \). Hence,

\[
2Q_0(\delta_1 - \delta_0) = 1
\]

\[
2Q_0(\delta_2 - \delta_0) = -1
\]

Subtracting and rearranging,

\[
Q_0 = \frac{1}{\delta_1 - \delta_2}
\]

or, in terms of frequency,

\[
Q_0 = \frac{f_0}{f_1 - f_0} = \frac{f}{\Delta f}
\]

Thus, the two frequencies at which the impedance locus passes through the points \( R = X \) determine the unloaded \( Q \) value. Frequencies \( f_1 \) and \( f_2 \) are called the half-power points.

Fig. 9.10. Identification of the half-power points from the Smith chart. \( Q_0 \) locus is given by \( X = R(B + G) \); \( Q_L \) by \( B = G + 1 \); \( Q_{ext} \) by \( B = 1 \).

The loaded and external \( Q \) values can be determined as follows: These are related to \( Q_0 \) through Eqs. (9.9) and (9.16). In terms of \( Q_L \), Eq. (9.24) becomes

\[
\frac{Z_{lb}}{Z_0} = \frac{\beta}{1 + j2Q_L(1 + \beta)(\delta - \delta_0)}
\]

In terms of \( Q_{ext} \),

\[
\frac{Z_{lb}}{Z_0} = \frac{\beta}{1 + j2Q_{ext}B(\delta - \delta_0)}
\]

Let \( \delta_{t1} \) and \( \delta_{t2} \) be the tuning parameters at which

\[
2Q_L(\delta - \delta_0) = \pm 1
\]

and \( \delta_{t1} \) and \( \delta_{t2} \) the tuning parameters at which

\[
2Q_{ext}(\delta - \delta_0) = \pm 1
\]

From these it is found, in a manner analogous to the derivation of Eq. (9.47), that

\[
Q_L = \frac{1}{\delta_{t1} - \delta_{t2}}
\]

\[
Q_{ext} = \frac{1}{\delta_{t1} - \delta_{t2}}
\]

By using the conditions given by Eqs. (9.50) and (9.51) the values of these tuning parameters can be identified from the graph of impedance.
plot with the aid of Eqs. (9.48) and (9.49). From these, the locus of points determining \( Q_L \) is given by

\[
\frac{Z_{sh}}{Z_0} = \frac{1}{1 + j(1 + \beta)}
\]  

(9.54)

The locus of points giving \( Q_{ext} \) is given by

\[
\frac{Z_{sh}}{Z_0} = \frac{1}{1 + j\beta}
\]  

(9.55)

These remarks are summarized in Fig. 9.10, which shows the location of the half-power points corresponding to \( Q_0, Q_L, \) and \( Q_{ext} \) with the aid of the defining loci given by Eqs. (9.43), (9.54), and (9.55).

![Fig. 9.11. Geometrical construction used to establish a linear frequency scale for the Q-circle impedance locus.](image)

The distribution of the measured points along the circular locus in the Smith chart is not linear with frequency; this makes impossible the accurate determination of the frequencies corresponding to the half-power points by interpolation between the measured points. An auxiliary scale of linear frequency, helpful in avoiding this difficulty, can be established by the construction shown in Fig. 9.11. The line \( AB \) is drawn perpendicularly to the resistive axis at any convenient location. The experimental impedance points such as \( a, b, c, d, \) and \( e \), corresponding to the frequencies \( f_a, f_b, f_c, f_d, \) and \( f_e \), respectively, are plotted. These points are projected upon the auxiliary frequency scale as indicated. Thus, the frequency of any point on the impedance locus whose frequency is not known can be found by projecting it to the frequency scale.

The \( Q \) values can be obtained also from the rectangular impedance chart shown in Fig. 9.12. The derivation of the defining loci for the three \( Q \) values is obtained as above, using Eq. (9.26) in place of Eq. (9.24). The use of the rectangular impedance chart is often advantageous because:

1. All loci are straight lines, which permits simple graphing.
2. The frequency scale along the impedance locus is linear.
3. The frequency scale along the impedance locus is independent of the position of the locus in the impedance plane.
4. The half-power points identifying the tuning parameters corresponding to \( Q_0, Q_L, \) and \( Q_{ext} \) are found at the intersection of the straight lines.
5. The change of the cavity \( Q \), due to changes in loading, displaces the impedance locus horizontally.

It is not necessary to use the half-power points to find \( Q \) values. A knowledge of the wavelength \( \lambda \) and two impedances, measured at two arbitrary values of \( \delta \), is sufficient to provide the necessary information. Consider, for example, points \( \delta_p \) and \( \delta_e \) shown in Figs. 9.10 and 9.12. If the laboratory data are plotted on the Smith chart, the resistance and reactance of the circuit can be determined. The reflection coefficient \( \Gamma \) corresponding to the impedance given by Eq. (9.24) can be computed from

\[
\Gamma = \frac{Z_{sh}/Z_0 - 1}{Z_{sh}/Z_0 + 1}
\]  

(9.56)

or

\[
\Gamma + 1 = \frac{-2\beta}{\beta + 1 + j2Q(\delta - \delta_e)}
\]  

(9.59)

The phase angle \( \phi \) of the radius vector originating from the origin \((O, O)\) in Fig. 9.11 is equal to the phase angle of the vector \((1 + \Gamma)\). Hence,

\[
\phi = \arctan \frac{2Q(\delta - \delta_e)}{\beta + 1}
\]  

(9.60)

Therefore, the intercept along the axis \( AB \), being proportional to \( \tan \phi \), is proportional to the frequency.
reactance components of the impedances can be obtained directly from the chart coordinates. [Alternately, they can be computed from the measured VSWR and the position of the minimum using Eq. (4.78).]

If $X$ and $R$ are the imaginary and the real parts of the impedance given by Eq. (9.24), the ratios $X/R$ for $\delta_p$ and $\delta_q$ are

$$
\left(\frac{X}{R}\right)_e = -\frac{1}{2} Q_0 (\delta_q - \delta_e)
$$

$$
\left(\frac{X}{R}\right)_p = -\frac{1}{2} Q_0 (\delta_p - \delta_e)
$$

(9.62)

Subtracting and rearranging,

$$
Q_e = \frac{1}{2} \frac{1}{\delta_p - \delta_q} \left[ \left(\frac{X}{R}\right)_e - \left(\frac{X}{R}\right)_p \right]
$$

(9.63)

which reduces to Eqs. (9.46) and (9.47) if $(X/R)_p = 1$, and $(X/R)_p = -1$.

If the laboratory data are plotted in the rectangular coordinates, as shown in Fig. 9.12, the quantity $X/R$ is recognized as the angle $\theta$ of the radius vector to a point on the impedance locus. Thus, if $\tan \theta_p = (X/R)_p$, $\tan \theta_q = (X/R)_q$, using Eq. (9.62),

$$
Q_e = \frac{1}{2} \frac{1}{\delta_p - \delta_q} \left( \tan \theta_p - \tan \theta_q \right)
$$

(9.64)

$$
= \frac{1}{2} \frac{1}{f_1 - f_2} \left( \tan \theta_p - \tan \theta_q \right)
$$

(9.65)

When $X = \pm R$, $\theta = \pm 45^\circ$, which again reduces to Eq. (9.47).

To summarize, this method is used as follows: The detuned-short position is found, the cavity tuned to resonance, and the value of the coupling coefficient $\beta$ measured by finding the VSWR $r_0$ at resonance. Additional measurements of VSWR and phase are then made at two other frequencies (which are also measured), and at as many other frequencies as are considered necessary for accuracy. Referring the measured impedances to the detuned-short position, the impedance locus is plotted together with the construction lines necessary to identify the half-power points, as shown in Fig. 9.10. The intersection of the impedance locus with the construction lines identifies the three half-power points whose frequencies can be found from the auxiliary linear-frequency scale and the construction shown in Fig. 9.11. Alternatively, the measured impedance can be referred to the detuned-open position and the VSWR and phase information transformed into data suitable for plotting in the rectangular impedance chart. The experimental points will lie along a straight line perpendicular to the resistive axis; the intersection of this locus with the construction lines shown in Fig. 9.12 locates the ordinates which correspond to the three half-power points. Since the vertical scale is linear in frequency, the ordinate intervals can be transformed into the units of frequency using the scale calibration provided by the experimental points.1

b. The Standing-wave Ratio Method. The method for interpreting the impedance data, described in the preceding section, is the most accurate of the three discussed in this section. However, it is time-consuming because it requires the measurement of phase and VSWR at each frequency as well as subsequent calculations to convert the laboratory data for plotting in the impedance plane. This can be simplified substantially by recording only the VSWR data as a function of frequency. Accuracy is not sacrificed by this simplification because a considerable quantity of data can be taken in a short time.

The experimental data required for the determination of $Q$ values con-

---

1 It is possible to improve the accuracy of the linear-frequency scale by arithmetically averaging the frequency intervals or by plotting the linear distance $X$ vs. frequency and graphically finding the average slope.
sist of the plot of VSWR as a functional frequency and the value of the coupling coefficient $\beta$ which is found as described in Sec. 9.3a by exploring the standing-wave pattern at the detuned-short position with the cavity tuned to resonance. Using this information, the $Q$ values are determined as follows:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure}
\caption{Determination of the half-power points from the VSWR data. $r_{13}$ points are computed from Eqs. (9.66), (9.67), and (9.68) for $Q_b$, $Q_L$, and $Q_{ext}$, respectively. The curve shown represents typical data for $f_s = 3,000$ Mc, $Q_b = 2,180$, $\beta = 0.78$.}
\end{figure}

The variation of the input impedance of a cavity resonator referred to the detuned-short position is shown in Fig. 9.6. It can be seen that the radius vector measured from the center of the Smith chart, corresponding to the reflection coefficient, increases continuously with the frequency-tuning parameter $\delta$. Figure 9.13 shows the typical variation of VSWR with frequency. To find the $Q$ values, it is necessary to identify the specific values of the standing-wave ratio which correspond to the half-power points. These can be found either graphically or analytically.

The standing-wave ratios at half-power points, $(r_{13})_b$, $(r_{13})_L$, $(r_{13})_{ext}$, corresponding to $Q_b$, $Q_L$, and $Q_{ext}$, respectively, can be found graphically, using the construction shown in Fig. 9.10, which is included for clarity in Fig. 9.14. The known value of $\beta$ [see Eqs. (9.41) and (9.42)] establishes the intercept between the circular impedance locus and the resistive axis, thus permitting the circle to be drawn. The construction lines corresponding to $Q_b$, $Q_L$, and $Q_{ext}$ loci can be drawn which define the impedances at the three half-power points. The VSWR corresponding to these

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure}
\caption{Graphical determination of the half-power VSWR’s. The experimental value of $\beta$ locates the circular impedance locus; its intersection with the half-power construction lines determines the half-power VSWR values by projection to the resistive axis as indicated.}
\end{figure}

are found by drawing the dashed circular arcs, as indicated, and reading their intercept along the resistive axis.

To obtain greater accuracy, it may be desirable to find the half-power VSWR points analytically. The input impedance at the detuned short at the frequencies given by Eqs. (9.44), (9.50), and (9.51) are given by Eqs. (9.43), (9.54), and (9.55), respectively. The half-power values of VSWR at these frequencies can be found by substituting these impedance values into Eq. (4.70), giving

\begin{align}
\text{For } Q_b: \quad (r_{13})_b &= \frac{2 + \beta^2 + \sqrt{4 + \beta^4}}{2\beta} \\
\text{For } Q_L: \quad (r_{13})_L &= \frac{1 + \beta + \beta^2 + (1 + \beta) \sqrt{1 + \beta^2}}{\beta} \\
\text{For } Q_{ext}: \quad (r_{13})_{ext} &= \frac{1 + 2\beta^2 + \sqrt{4\beta^4}}{2\beta}
\end{align}
The ratio of the half-power VSWR to VSWR at resonance $r_0$ is plotted for the three cases in Fig. 9.15. For small and large values of $\beta$, this ratio is

For $\beta \ll 1$:

\[
\begin{align*}
\left( \frac{r_{14}}{r_0} \right) & = 2 \\
\left( \frac{r_{14}}{r_0} \right)_L & = 2 \\
\left( \frac{r_{14}}{r_0} \right)_{ett} & = 1 \\
\end{align*}
\]  
(9.69)

For $\beta \gg 1$:

\[
\begin{align*}
\left( \frac{r_{14}}{r_0} \right) & = 1 \\
\left( \frac{r_{14}}{r_0} \right)_L & = 2 \\
\left( \frac{r_{14}}{r_0} \right)_{ett} & = 2 \\
\end{align*}
\]  
(9.70)

These relations show that for small $\beta$ the VSWR at the half-power points for $Q_0$ and $Q_L$ is twice the minimum value and can be measured easily. For $Q_{ett}$, however, the half-power VSWR becomes indistinguishable from the VSWR at resonance and cannot be used. For similar reasons, when $\beta$ is large, $Q_L$ and $Q_{ett}$ can be found, but $Q_0$ cannot. If $\beta$ is measured accurately, $Q_L$ and $Q_{ett}$ can be computed from $Q_0$ (and conversely) by using Eqs. (9.9) and (9.16).

The values of $Q_0$, $Q_L$, and $Q_{ett}$ are found from the frequencies at which the half-power VSWR occur. Calling these $\delta_1$ and $\delta_2$, $\delta_3$ and $\delta_4$, and $\delta_5$ and $\delta_6$, respectively,

\[
\begin{align*}
Q_0 & = \frac{1}{\delta_1 - \delta_2} = \frac{f_0}{f_1 - f_2} \\
Q_L & = \frac{1}{\delta_3 - \delta_4} = \frac{f_0}{f_3 - f_4} \\
Q_{ett} & = \frac{1}{\delta_5 - \delta_6} = \frac{f_0}{f_5 - f_6} \\
\end{align*}
\]  
(9.71)

In summary, the method is used as follows: The detuned-short position is found, the cavity carefully adjusted to resonance, and the standing-wave pattern explored to determine whether the cavity is overcoupled or undercoupled. The VSWR $r_0$ at resonance is measured and the value of the coupling coefficient $\beta$ calculated, using Eqs. (9.41) and (9.42), as appropriate. From Eqs. (9.66), (9.67), and (9.68), or Fig. 9.15, the VSWR at the half-power points is obtained. From the experimental graph of the VSWR vs. frequency (such as Fig. 9.13) the frequencies corresponding to the half-power VSWR are obtained. The $Q$ values are computed from Eqs. (9.71).

FIG. 9.15. Dependence of VSWR at half-power points upon $\beta$. $\beta = r_0$ or $1/r_0$, depending upon the degree of coupling.

c. The Phase Method. The phase method is based upon the measurement of the nodal position as a function of frequency. If $\beta \gg 1$, the standing-wave ratio is very high and difficult to measure, but the associated voltage nodes are sharp and they shift rapidly with tuning of the cavity or the source and are easily located accurately. In its basic form,
As the cavity is tuned close to resonance, the positions of the nodes change; plotting the position of one of the nodes results in the curves shown. Similar data are obtained if the cavity tuning is fixed and the signal frequency is changed. A typical curve is shown in Fig. 9.16b. With the cavity tuned far off resonance, the node position changes with frequency, as indicated by the dashed lines; the slope of these depends upon the distance of the measuring probe from the cavity. With the cavity tuned to resonance, tuning the signal source through the resonance frequency of the cavity results in the curve shown. The data can be altered, if desired, to appear as shown in Fig. 9.16a by subtracting from each point a distance corresponding to the displacement of the detuned short.

In the analysis of this procedure it is convenient to use the series representation of the cavity given by Eq. (9.26). The term \( \delta \) is not important as it merely represents a shift along the frequency scale and is neglected for brevity. Thus, Eq. (9.26) can be written as

\[
\frac{Z_{m}}{Z_{0}} = \frac{1}{\beta} (1 + j2Q_{0}\delta)
\]  

(9.72)

Figure 9.17 shows the impedance referred to the detuned-open position plotted on the Smith chart. Points A and B show the impedance at
resonance or the undercoupled and overcoupled cases, respectively. The complex reflection coefficient \( \Gamma \) measured at a distance \( l \) from the detuned-open position is obtained by substituting Eq. (9.72) into Eq. (4.69); thus,

\[
\Gamma = \left| \frac{Z_{o2}/Z_E}{Z_{o2}/Z_0} - 1 \right| / \phi - 2\beta l
\]

(9.73)

where \( \phi \) is the phase angle \( \Gamma \) at \( l = 0 \). For convenience, let

\[
y = 2Q_o \delta
\]

(9.74)

Substituting Eq. (9.72) into Eq. (9.73) and rationalizing, the phase angle \( \phi \) is

\[
\phi = \tan^{-1} \frac{2\beta y}{1 - \beta^2 + y^2}
\]

(9.75)

At resonance \((y = 0)\), from Fig. 9.17 or Eq. (9.75), the phase angle \( \phi = 0 \) and \( \phi = \pi \) for the undercoupled and overcoupled cases, respectively.

The voltage minimum occurs when the phase angle of \( \Gamma \) is \( n\pi \), or

\[
\phi - 2\beta l = \pm n\pi
\]

or

\[
2\beta l = \phi = n\pi
\]

(9.77)

with \( n = 1, 3, \ldots \). Equation (9.77) determines the distance \( l \) between the voltage node and the detuned-open position; the phase angle \( \phi \) is defined by the frequency through Eq. (9.75). Thus, the location of the voltage node is found from the Smith chart to be the electrical distance \( 2\beta l \) as indicated in Fig. 9.17.

Qualitatively, the relation between \( \delta \) and the location of the voltage node can be predicted by tracing a point along the impedance locus and observing the variation in \( 2\beta l \). Consider the two cases illustrated in Fig. 9.17. For \( \beta > 1 \), at resonance, the minimum occurs at point \( B \); as \( \delta \) increases from zero to infinity, \( \phi \) varies from \( \pi \) to zero. Consequently, \( 2\beta l \) begins at \( 0^\circ \) and progresses through negative angles to \( -180^\circ \), as can be verified by tracing the motion of the point \( C' \) when \( C \) moves from \( B \) to \( O \). When \( \beta < 1 \), at resonance the point \( A \) corresponds to a voltage maximum; the voltage minimum is found by adding \( 180^\circ \). Tracing the point \( D \) along the impedance locus as \( \delta \) increases from zero to infinity causes the point \( D' \) to begin at \( -180^\circ \), decrease toward \( -90^\circ \), and return again to \( -180^\circ \). Figure 9.18 shows the plot of \( 2\beta l \) vs. the tuning parameter \( \delta \) obtained by calculation from Eqs. (9.77) and (9.75) for three conditions of coupling and illustrates the data that can be obtained in the laboratory.

The characteristics of the curves shown in Fig. 9.18 depend upon \( Q_E \) and \( \beta \). For computation, the cardinal points can be selected in the following manner: The resonant frequency, \( \delta = 0 \), can be taken as the point of antisymmetry. Let the slope at this point be called \( S_o \). In the overcoupled case, let the frequencies at which the curve passes through \( \pm 90^\circ \) points be called \( \delta_1 \) and \( \delta_4 \). In the undercoupled case the frequencies corresponding to the points of zero slope are called \( \delta_1 \) and \( \delta_4 \). These values are identified in Fig. 9.18.

![Fig. 9.18. Computed curves showing the displacement of the voltage minimum with respect to the detuned-open position. The cardinal values of \( \delta \) used in the calculation of the \( Q \) values are shown.](image)

Analytic expressions connecting the \( \beta \) and \( Q \) values with \( \delta_1, \ldots, \delta_4 \) can be found as follows: The slope of the curves shown in Fig. 9.18 can be obtained by differentiating Eq. (9.77) with respect to \( \delta \):

\[
\frac{d}{d\delta} (2\beta l) = \frac{d\phi}{d\delta} = 2Q_o \frac{d\phi}{dy}
\]

(9.78)

Hence, differentiating Eq. (9.75) with respect to \( y \) leads to

\[
\frac{d\phi}{d\delta} = 4Q_o \beta \left( \frac{1 - \beta^2 + y^2}{(1 - \beta^2 + y^2)^2 + (2y\beta^2)^2} \right)
\]

(9.79)
At resonance,

\[ S_0 = \frac{d\phi}{d\delta} \bigg|_{\delta_0} \]  
(9.80)

\[ S_0 = \frac{4Q_0\beta}{1 - \beta^2} \]  
(9.81)

In the undercoupled case the points of zero slope are found from Eq. (9.79) by equating the numerator to zero. These occur at

\[ 2Q_0\delta = \sqrt{1 - \beta^2} \]

\[ 2Q_0\delta_2 = -\sqrt{1 - \beta^2} \]  
(9.82)

In the overcoupled case the curve passes through the points \( \pm 90^\circ \) at which \( 2\delta_2 = \pm \pi/2 \). Therefore, from Eq. (9.77), this occurs when \( \phi = \pi/2 \). Hence, \( \tan \phi = \infty \); this occurs when the denominator in Eq. (9.75) is zero. This leads to

\[ 2Q_0\delta = \sqrt{\beta^2 - 1} \]

\[ 2Q_0\delta_2 = -\sqrt{\beta^2 - 1} \]  
(9.83)

Equations (9.81), (9.82), and (9.83) contain the necessary information to evaluate the cavity parameters. Solving these for \( Q_0 \) and \( \beta \) results in

**Undercoupled case \((\beta < 1)\):**

\[ \beta = \frac{\delta_1 S_0}{2\sqrt{(\delta_1 S_0/2)^2 + 1}} \]  
(9.84)

\[ Q_0 = \frac{1}{2\delta_1 \sqrt{(\delta_1 S_0/2)^2 + 1}} \]  
(9.85)

**Overcoupled case \((\beta > 1)\):**

\[ \beta = \frac{S_0\delta_1}{2\sqrt{(S_0\delta_1/2)^2 - 1}} \]  
(9.86)

\[ Q_0 = \frac{1}{2\delta_1 \sqrt{(S_0\delta_1/2)^2 - 1}} \]  
(9.87)

\[ Q_{ext} = \frac{Q_0}{\beta} = \frac{1}{S_0\delta_1^2} \]  
(9.88)

\[ Q_L = \frac{Q_0}{1 + \beta} = \frac{1}{2\delta_1 [\sqrt{(S_0\delta_1/2)^2 - 1} + (S_0\delta_1/2)]} \]  
(9.89)

If \( \beta \gg 1 \), from Eq. (9.81), \( S_0 = 4Q_0/\beta \), or

\[ Q_{ext} = \frac{S_0}{4} \]  
(9.90)

Alternatively, combining Eqs. (9.88) and (9.90), for \( \beta \gg 1 \),

\[ Q_{ext} = \frac{1}{\delta_1} \]  
(9.91)

or

\[ Q_{ext} = \frac{1}{\delta_1 - \delta_1} = \frac{f_0}{f_1 - f_2} \]  
(9.92)

Eq. 9.32, Measurement of cavity characteristics—measurement:

Equating Eqs. (9.90) and (9.91) shows that the quantity \( S_0\delta_1 \) is approximately equal to 2 for the overcoupled case. An examination of Eqs. (9.86) through (9.89) shows that the denominators contain the difference of two nearly equal numbers. For this reason, the values of \( \beta \) and \( Q_0 \) cannot be accurately determined by this method, although it is accurate for measuring \( Q_{ext} \). For the undercoupled case, the method is not as useful as the VSWR method described in Sec. 9.3b, since it is difficult to find accurately the location of the minima due to their breadth at low VSWR; also, for \( \beta \ll 1 \), the minima do not shift sufficiently for accurate measurement.

To summarize, this method is used as follows: With the cavity tuned far from resonance, a location of a voltage node is found. As either the cavity or the signal source is tuned toward resonance, the position of the node changes. A graph showing the node position vs. the tuning parameter \( \delta \) is plotted using the coordinate axes shown in Fig. 9.18. Using Eqs. (9.86) through (9.92), as appropriate, the \( Q \) values can be computed.

This method can be modified by measuring the VSWR at resonance. The displacement of the voltage node at the frequencies corresponding...
to the three half-power points can be found graphically from the construction lines shown in Fig. 9.10. These are repeated for convenience in Fig. 9.19, which shows the nodal displacements explicitly. The measured value of $\beta$ at resonance establishes the intercept of the impedance locus with the resistive axis and permits the circle to be drawn. The intersection of the construction lines with the impedance locus determines the displacement of the voltage node at the half-power frequencies.

![Fig. 9.20. Equivalent circuit used in the analysis of the effect of the coupling loss. (a) Equivalent circuit of the coupling system, (b) referred to the detuned-short position, $b-b$.](image)

From the experiment graph of $V_{\text{min}}$ vs. frequency, shown in Fig. 9.16, the frequencies corresponding to these displacements are found and the $Q$ values computed using Eqs. 9.71.

9.4. Impedance Method—Effect of Coupling Loss. The three impedance methods for measuring the $Q$ values, described in Sec. 9.3, are based upon the assumption that the cavity coupling network is lossless. In most practical cases this approximation is valid; however, if losses are present, the impedance methods presented cannot be applied directly and must be altered. For brevity, only the modification of the VSWR method is considered in detail, but the method of analysis can be extended to the remaining cases as well. It should be noted that the definitions of the loaded and external $Q$ values are also modified by the presence of loss.

Figure 9.20a is the equivalent circuit of a cavity coupled to the transmission line and corresponds to Fig. 9.3b with the addition of the series resistance $R_s$. It is assumed that the loss in the coupling element can be represented in this manner regardless of the actual cause of loss. The input impedance as a function of frequency results in a circular locus as shown in Fig. 9.21. The plot of VSWR vs. frequency is shown in Fig. 9.22. The characteristic feature of the two graphs is the fact that the VSWR far off resonance reaches a limiting value, $r_{\text{min}}$, instead of becoming infinite.

Qualitatively, the impedance information is interpreted as in the former case. Certain points on the impedance locus are found which correspond to the half-power points and the frequency interval between them defines $Q_0$. Due to the coupling loss, the identification of the half-power points is somewhat different from the lossless case. The desired relations between the characteristic points of the impedance locus and the frequencies corresponding to the half-power points can be obtained in the following manner:

Referring to Fig. 9.20a, the impedance at $a-a$ is

$$\frac{Z_{aa}}{Z_0} = \frac{R_1 + jX_0}{Z_0} + \frac{R_1}{1 + j2Q_0} \quad (9.93)$$

With the cavity tuned far off resonance, the transmission line is terminated by the self-impedance of the coupling element. As before, using