C-7: Nonlinear Oscillator

The purpose of this experiment is to discover the properties of a nonlinear circuit which closely approximates a driven pendulum. The references are:

1) "Chaotic States and Routes to Chaos in the Forced Pendulum," by D. D'Humieres, M.R. Beasley, B. A. Huberman, and A. Libchaber, Physical Review A26, 3483 (1982).

2) Our circuit is an improvement on the one presented by D'Humieres *et al.* It is described in "Analysis of the Phase Locked Loop Circuit".

3) General notes on phase lock loop circuits from The Art of Electronics by P . Horowitz and W. Hill (Cambridge, Carqbridge, 1980).

4) Excerpts from Order Within Chaos by P. Berge, Y. Pomeau, and C. Vidal (Hermann and Wiley, Paris, 1984). This reference includes an introduction to the modem physics of dynamical systems.

5) "Getting Started Guide for the Hewlett Packard 35660A Dynamic Signal Analyzer". 6) The data sheets for integrated circuits used in the experiment.

The following is a recommended sequence of steps for a long laboratory .Do not feel restricted to the operating parameters quoted in these notes, or even the methodology. Their main purpose is to help you get your feet wet. Please feel free to try out ideas as they occur to you.

Basics and Linear Response *Learning to use the analyzer* Scan over the "Getting Started Guide" for the HP 35660A Dynamic Signal Analyzer. Look at th~ spectrum-~9~-time series ~ for the signal source running as a sine wave, a periodic chirp, and a random noise source. Measure the frequency response of the low pass filter and the integrator modules of the oscillator. The procedure is described in Chapter 8 of the analyzer guide.

Measuring the VCO Properties Measure fo (= ro0/27t), the frequency of the VCO when it has no input, and k = -d~ These are important parameters for understanding the circuit. ";:: -tt *tJ]vj* ~V 8

Assembling the Phase Locked Loop Assemble the modules of the phase locked loop (PLL) following the block diagram. Leave one input of the integrator unused; the drive signal will go there later. First connect a short circuit across the sine wave input to the mixer and measure fL' the VCO frequency when the loop is closed.

Next connect the Wavetek 171 through a switch-controlled attenuator to the reference sine wave input of the mixer. It is useful to conn

ect the same signal to a digital AC voltmeter for monitoring during the experiment. Set the reference wave to an amplitude of -1.3 dBVrms (measured on the spectrum analyzer) or 860 mVrms (measured on the voltmeter) while the frequency, fs (= O>s/27t) is near fo.

Find the frequency over which the PLL locks as follows: Externally trigger an oscilloscope from the sync output of the reference wave generator. Monitor the output of the PLL by looking at the clock output of the VCO on the scope. Set the feedback resistance control on the integrator (R) to 770. Scan the frequency of the generator and note the range of locking. Scan the R control on the integrator, noting the range over which locking occurs. You should compare the locking range with that expected. There is substantial hysteresis near the edges of the locking range. If the PLL becomes unlocked because you took to large a step in reference frequency you should return to the center of the locking range and approach the edges more cautiously.

Reasonable settings for most of the measurements are: R = 770, reference wave amplitude = 860 mVrms, fs = fo. You might enjoy learning about PLL 's and their applications in the excerpt from The Art of Electronics.

Observing the Phase Space Picture of the Pendulum An oscilloscope can be connected so that you can watch the phase space motion of the pendulum being simulated. From D'Humieres *et al,* the integrator output represents the first time derivative of the pendulum's angular position de/dt. Connect it to the scope's y-input. The low pass filter output represents sine; connect it to the x-input. (Decrease the scope intensity before switching the time base to the xy mode to avoid burn\15e scope phosphor with the non-moving trace.) If the PLL is locked, the scope trace should be centered on the origin using the position controls. Throughout the lab, you may find it useful to descriptively label signal points and/or cables For example, the scope channels might be marked sine and de/dt. Masking tape works well for this purpose.

Linear Response At low drive levels the circuit is approximately linear just as the pendulum is approximately linear at small amplitudes. You can determine when the response is linear from the phase space picture. Figure 5a of D'Humierres *et al* shows the phase space when the circuit is linear. Measure the frequency response of the de/dt signal at low drive levels and for diff:r~~~of reference~ave amplitude an~ R. Compare the results for the Q and resonant frequency with that expected. Nonlinear Response

Harmonics Change the spectrum analyzer source to fixed frequency sine wave at the resonant frequency. Measure the spectra over a range of input amplitudes ranging from -40 dbVrms to -15 dbVrms. You should see harmonics of the drive frequency appearing as the drive amplitude is increased.

Nonlinear Frequency Response Set up the spectrum analyzer to measure the frequency response of the PLL following the procedure in Chapter 8 of the analyzer guide. Use a frequency scan of from 750 to 1150 Hz. Record the frequency response for several drive amplitudes between -60 dbVrms and about -10 dbVrms. For a good signal- to-noise ratio for low drive amplitudes, it is helpful to have the analyzer in the averaging mode. Keep the phase portrait of the PLL displayed on the scope. Note the shift in the peak response and the loss of symmetry of the frequency response curve with increasing drive amplitude. This is a indication of nonlinear response. Plot the maximum amplitude in the frequency response and the corresponding drive frequency as a function of the drive amplitude. Explain any qualitative indications you find for nonlinear behavior in these plots.

Exploring the State Diagram The idea is to vary the frequency and amplitude of the drive and to note and classify the various types of states that you encounter. Figure 3 of D'Humieres *et al* shows a phase diagram. You should encounter phase-locked *oscillatory states* and *rotating states* and chaotic regions.

The phase-locked states are easier to think about if you consider a rigid pendulum with the same equation of motion. An oscillatory state is one where the pendulum makes oscillations about the equilibrium and doesn't make any complete revolutions. States with complete revolutions are rotating states. D'Humieres *et al* denote oscillatory states as $n = O$ and rotating states as $n = 1,2,$ ...depending on the number of rotations. They give spectra and phase-space portraits for various states.

Set-Up Set the Integrator to R = 770. Set the reference signal amplitude to 860 m Vrms and the reference signal frequency to make the equilibrium phase <1>e = 7t/2. U se the spectrum analyzer signal source as the driving voltage, YE. You will want to get time series of sine and de/dt, so connect them to channels 1 and 2 of the spectrum analyzer, respectively. Set up the spectrum analyzer for measuring two time series simultaneously. Use a uniform window, full frequency span, force channel 1, and trigger on channel 2. This last setting is crucial to preserve the phase between the two channels. It will be convenient to use a top and bottom split screen display to view both time series simultaneously. Finally, in order to have a record of the phase space portraits, have a scope camera ready.

Frequency Scan First scan the drive frequency at a fixed drive amplitude of 5 dbVrms. Start with a drive frequency of at least 2000 Hz. Slowly decrease the frequency noting the first vibrating state. Note any evidence that makes you agree with or dispute the contention that this is indeed a low amplitude vibrating state. As you continue to lower the frequency, you should encounter a vibrating state with an amplitude that exceeds 900, as measured from straight down. Get a record of this state. Qualitatively interpret the time series.

Going very slowly now, at lower frequency you should encounter a period doubled state. Make records, again interpreting this state with the time series. If all goes well, you may get a second period doubling close at hand. Record and interpret it as well. By now you will be almost, or already into a chaotic state. It is easy to recognize from the phase portrait by its "noise", but impossible to discern from the time series. At this point, the states are incredibly densely spaced.

You need to be careful about hysteresis when making these measurements. Near the period doubling and chaotic regions the response at frequency at a given frequency depends on whether you approach it from above or below.

Rotating State Look around for a periodic rotating state and record it. U se the time series and phase portraits to confirm the classification.

Spectrum Measurements Go back to where you first encountered period doubling, and confirm this transition with spectra of de/dt. Use averaging to reduce noise. Study the second period doubling transition as well. Take more spectra to demonstrate the transition between periodic and chaotic states. Note in particular the enormous change in the continuous part of the spectrum. This is a hallmark of chaos. Do the same type of spectral measurement to see the comparison between a rotating state and nearby chaos. Examination of the Period-Doublin~ Transition

The last suggested experiment is a close examination of the period-doubling transition. We wish to test one aspect of the hypothesis that it is a transition of the "pitch-fork" variety mentioned in D'Humieres *et al* on page 3486, and described in the appendix for **Order Within** Chaos. Don't bother with a lot of details. The main point is the square root behavior of the strength of the new state as a function of the distance of control parameter from threshold.

In this experiment the spectral strength of the period doubling subharmonic is measured as a function of the drive frequency. Set the drive force amplitude at -1.3 dbVrms. Connect the de/dt signal to channel 1. Use a flat top window and force channel 1. Be sure to get measurements above as well as below the threshold frequency for the transition.

To interpret the data, first assign the threshold frequency (label it fc) by examining a plot of subharmonic amplitude (A) versus drive frequency (f). Then, plot In(A) vs In(f -fc) in order to estimate the exponent ~ in A -(f -fc)~ .

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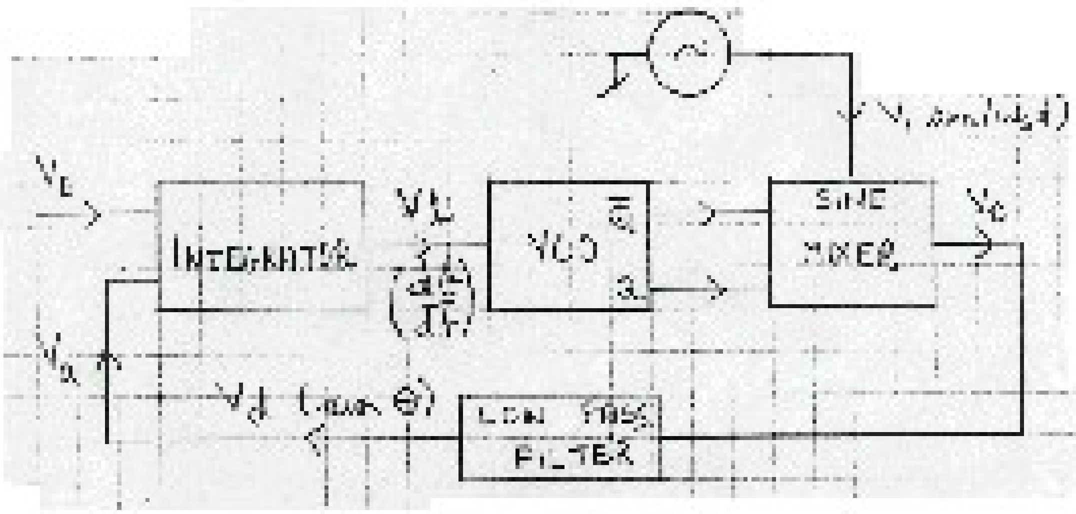
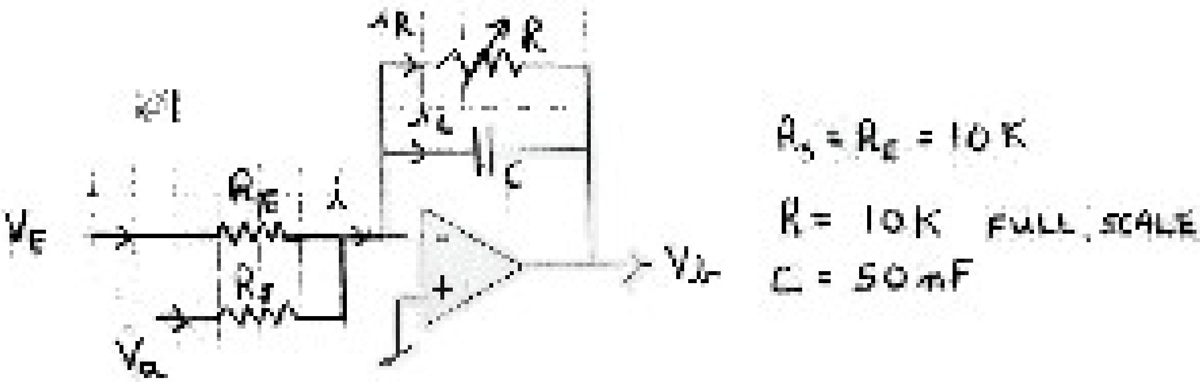
Analysis of the Phase Locked Loop Circuit

The block diagram of the circuit is below, and there is a detailed schematic on the next page. The circuit is analyzed in this note by considering the individual sections first and then closing the loop. Circuit elements such as buffers and voltage followers, pin numbers, and power connections are not included below, but they are in the schematic.

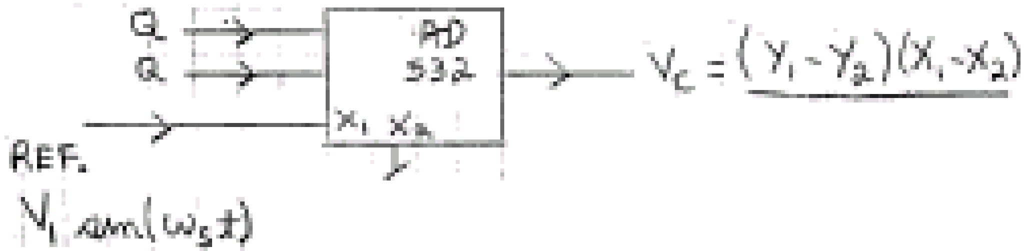


Integrator



This is a standard op-amp integrator. The input current is $I = V_a/R_S + V_E/R_E$. Current is conserved, $I = i_C + i_R$, and the voltage drop across R must equal the voltage drop across C, $i_C/C = R\, di_R/dt$. The output voltage is $V_L = -i_R R$. Combining equations

$$V_L + RC\frac{dV_L}{dt} = -R\left(\frac{V_a}{R_S} + \frac{V_E}{R_E}\right)$$



Mixing is done with an analog multiplier (AD 532). The inputs are the two square waves of the VCO and the sine wave reference oscillator. The output is a voltage V_C. The reference sine wave is at a fixed frequency, ω_s, that is related to the VCO frequency by $\omega_s = \omega_v + d\phi/dt$. We assume $d\phi/dt << \omega_v, \omega_s$. The output voltage of the mixer is

$$V_c = \frac{(Y_1 - Y_2)(X_1 - X_2)}{10V} \quad = \quad 2 \times g(t) \times V_1 \sin(\omega_s t + \phi)$$

where g(t) is

Expanding g(t) in a Fourier series gives

$$V_c = \frac{4.8}{\pi} V_1 \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin[(2n+1)\omega_v t] \sin(\omega_s t + \phi)$$

$$= \frac{2.4}{\pi} V_1 \sum_{n=0}^{\infty} \frac{1}{2n+1} \Big(\cos[2n\omega_v t - \phi] \; - \; \cos[(2n+2)\omega_v t + \phi] \Big)$$

V_c has terms with angular frequencies $\omega = d\phi/dt$, $\omega = 2\omega_v \pm d\phi/dt$ ($f \sim 50$ kHz), $\omega = 4\omega_v \pm d\phi/dt$, etc. The first of these is slowly varying compared to the others, and it is the only one that get through the low pass filter.

$$\frac{d^2\phi}{dt^2} + \frac{1}{RC}\frac{d\phi}{dt} - \frac{1.21V_1k}{R_SC}\cos\phi = \frac{k}{R_EC}(V_E+V_o) + \frac{1}{RC}(\omega_0-\omega_s) .$$

The equilibrium phase, ϕ_e, is determined by setting $V_E = 0$ (i.e. turning off the external driving force) and finding the angle at which the remaining force equals zero. It is given by

$$\cos\phi_e = \frac{R_S}{1.21V_1}\left(\frac{1}{kR}(\omega_s-\omega_0) - \frac{V_o}{R_E}\right)$$

If the absolute value of the right-hand side is greater than one, there is no equilibrium, i.e. the loop cannot lock. As shown below there is an additional requirement, $\sin\phi_e > 0$, for a stable equilibrium. Assuming there is a stable equilibrium, the equation of motion can be rewritten in terms of $\theta = \phi - \phi_e$, the deviation from equilibrium,

$$\frac{d^2\theta}{dt^2} + \frac{1}{RC}\frac{d\theta}{dt} - \frac{1.21V_1k}{R_SC}\left(\cos(\theta+\phi_e) - \cos\phi_e\right) = \frac{k}{R_EC}V_E .$$

This is the result of the circuit analysis.

When θ is small, $\cos(\theta + \phi_e) \cong \cos\phi_e - \theta\sin\phi_e$ and the equation becomes

$$\frac{d^2\theta}{dt^2} + \frac{1}{RC}\frac{d\theta}{dt} + \left(\frac{1.21V_1k}{R_SC}\sin\phi_e\right)\theta = \frac{k}{R_EC}V_E$$

When $\sin\phi_e < 0$, the coefficient multiplying θ is negative and the equilibrium is unstable, and when $\sin\phi_e > 0$, the coefficient is positive and the equilibrium is stable. The angular frequency, Ω, damping time, τ (D'Humieres *et al* have the wrong equation for τ), and quality factor, $Q = \Omega\tau/2$, for small oscillations can be determined from the above equation. They are

$$\Omega = \left(\frac{1.21V_1k}{R_SC}\sin\phi_e\right)^{1/2} \quad \tau = 2RC , \quad Q = \left(\frac{1.21V_1kR^2C}{R_S}\sin\phi_e\right)^{1/2}$$

When the nonlinearities are weak the equation can be analyzed by a perturbation method. Taylor expanding $[\cos(\theta + \phi_e) - \cos\phi_e]$ gives

$$\frac{d^2\theta}{dt^2} + \frac{2}{\tau}\frac{d\theta}{dt} + \Omega^2\theta = \frac{k}{R_EC}V_E + \Omega^2\left(\frac{\theta^3}{3!} - \frac{\cos\phi_e}{\sin\phi_e}\frac{\theta^2}{2}\right) + \ldots$$

The nonlinearity is in the $\theta^2, \theta^3, \ldots$ terms. The leading nonlinearity, θ^2, is multiplied by $\cos\phi_e$ (the $\sin\phi_e$ in the denominator cancels the $\sin\phi_e$ in the numerator of Ω^2).

Consider a special case where V_E is sinusoidal with frequency ω and there is only one power of θ on the right-hand-side

$$\frac{d^2\theta}{dt^2} + \frac{2}{\tau}\frac{d\theta}{dt} + \Omega^2\theta = Ae^{i\omega t} + B\theta^n ;$$

equation of motion becomes

This is eq. (A8) of D'Humieres *et al* with a different constant in front of the $\sin\theta$ term because of different mixing techniques.

Since $V_b = d\phi/dt = d\theta/dt$, $d\theta/dt$ can be measured by measuring V_b which is the output of the integrator. The output of the low pass filter is $V_d \propto \cos\phi = -\sin\theta$ ($\phi_e = \pi/2$ is assumed), and measuring V_d gives $\sin\theta$.