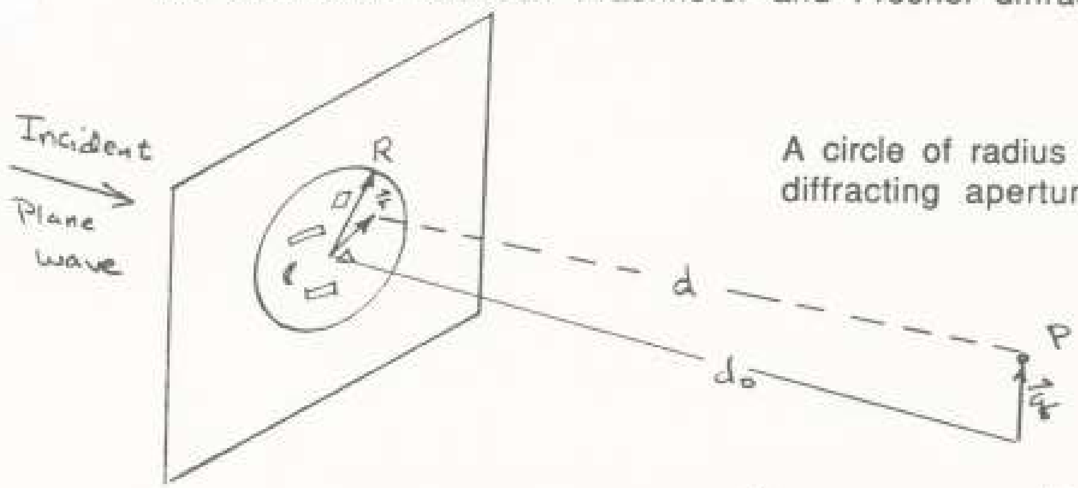
The distinction between Fraunhofer and Fresnel diffraction



A circle of radius R encloses all diffracting apertures in plane

$$kd = k[d_0^2 + |\vec{r}-\vec{q}|^2]^{1/2} \approx kd_0 + \frac{1}{2}\frac{k}{d_0}(q^2 - 2\vec{r}\cdot\vec{q} + r^2) + \cdots$$

assuming r, q are small compared with d_0

$$kd = k\left(d_0 + \frac{1}{2}\frac{q^2}{d_0}\right) - k\frac{\vec{r}\cdot\vec{q}}{d_0} + \frac{1}{2}\frac{kr^2}{d_0}$$

If the maximum value for the quadratic term in r is $\frac{1}{2}k\frac{R^2}{d_0}$,

then Fraunhofer diffraction occurs when $\frac{1}{2}k\frac{R^2}{d_0} \ll \pi$ (far field)

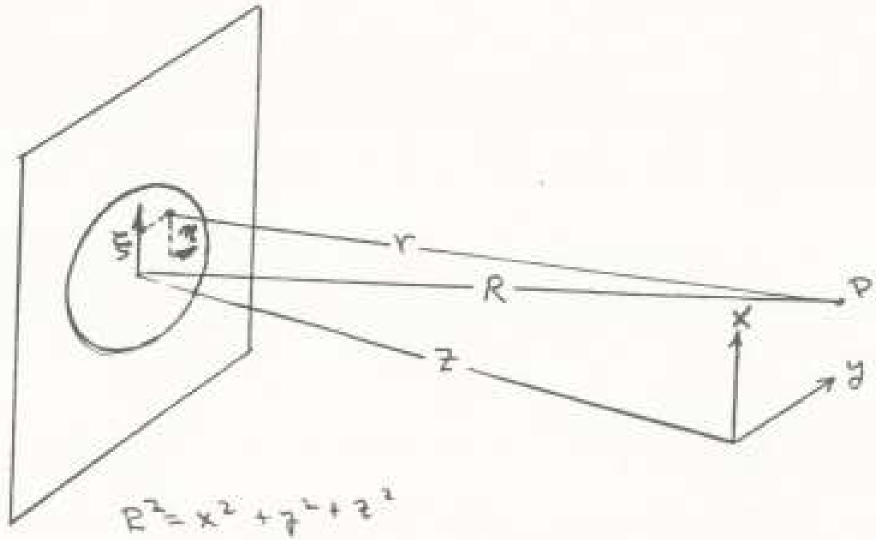
and Fresnel diffraction occurs when $\frac{1}{2}k\frac{R^2}{d_0} \gg \pi$ (near field)

<u>Thus</u>

Fresnel: $R^2 \gg \lambda d_0$ (near field)

Fraunhofer: $R^2 \ll \lambda d_0$ (far field)

For example if R=2mm, $\lambda = 5\times10^{-4}$mm, then Fresnel pattern for $d_0 \ll 2$m and Fraunhofer patterns for $d_0 \gg 2$m.



$$R^2 = x^2 + y^2 + z^2$$

$$r^2 = (x-\xi)^2 + (y-\eta)^2 + z^2$$

$$= \underbrace{x^2+y^2+z^2}_{R^2} - 2x\xi - 2y\eta + \xi^2 + \eta^2$$

$$r^2 = R^2\left[1 - \frac{2(x\xi + y\eta)}{R^2} + \frac{\xi^2+\eta^2}{R^2}\right]$$

$$\text{let } \boxed{\alpha \equiv x/R, \quad \beta \equiv y/R}$$

$$r = R\left[1 - 2(\alpha\xi + \beta\eta)/R + (\xi^2+\eta^2)/R^2\right]^{1/2} \approx R\left[1 - (\alpha\xi+\beta\eta)/R + (\xi^2+\eta^2)/2R^2\right.$$

$$r \approx R - (\alpha\xi + \beta\eta) + \frac{\xi^2+\eta^2}{2R} + \cdots$$

Fraunhofer Diffraction:

neglect $\dfrac{\xi^2+\eta^2}{2R}$ term

$$r = R - (\alpha\xi + \beta\eta)$$

$$\boxed{\psi(P) = \frac{-iA}{\lambda R}\, e^{ikR} \int_\sigma e^{-ik(\alpha\xi+\beta\eta)}\, d\xi\, d\eta}$$

Fraunhofer Diffraction Formula

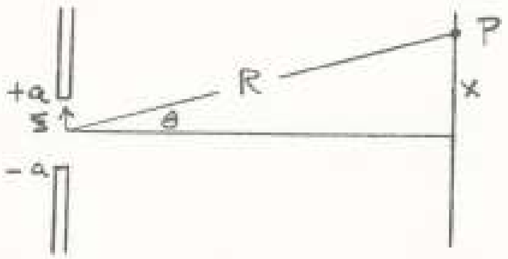
[A= ψ_0 amplitude of incident wave]

When $\dfrac{\xi^2+\eta^2}{2R}$ cannot be neglected, we have <u>Fresnel Diffraction</u>

Fraunhofer Diffraction

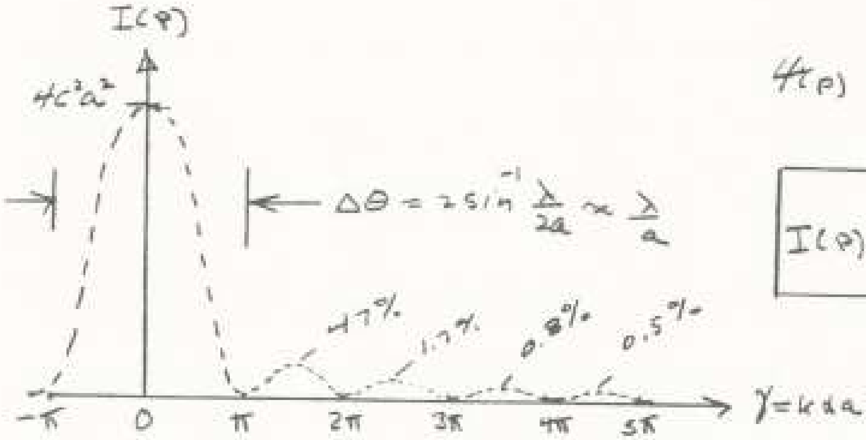
$$\boxed{\psi(P) = -i\,\psi_0 \frac{e^{ikR}}{\lambda R} \int_\sigma e^{-ik(\alpha\xi+\beta\eta)}\, d\xi\, d\eta}$$

Example 1 - an infinite slit



$$\psi(P) = c\int_{-a}^{a} e^{-ik\alpha\xi}\, d\xi$$

$$= \frac{ci}{k\alpha} e^{-ik\alpha\xi}\Big|_{-a}^{a} = \frac{ic}{k\alpha}\left(e^{-ik\alpha a} - e^{ik\alpha a}\right)$$

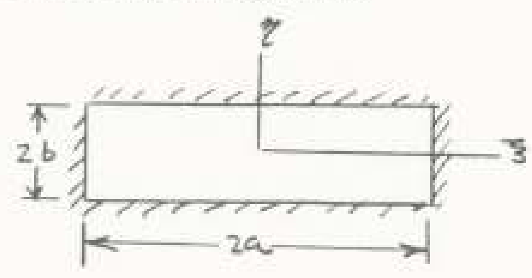


$$\psi(P) = \frac{2c}{k\alpha} \sin k\alpha a$$

$$\boxed{I(P) = |\psi(P)|^2 = 4c^2a^2\left(\frac{\sin\gamma}{\gamma}\right)^2}$$

where $\gamma = k\alpha a = \frac{2\pi}{\lambda} a \sin\theta$

Example 2 - a rectangular aperture

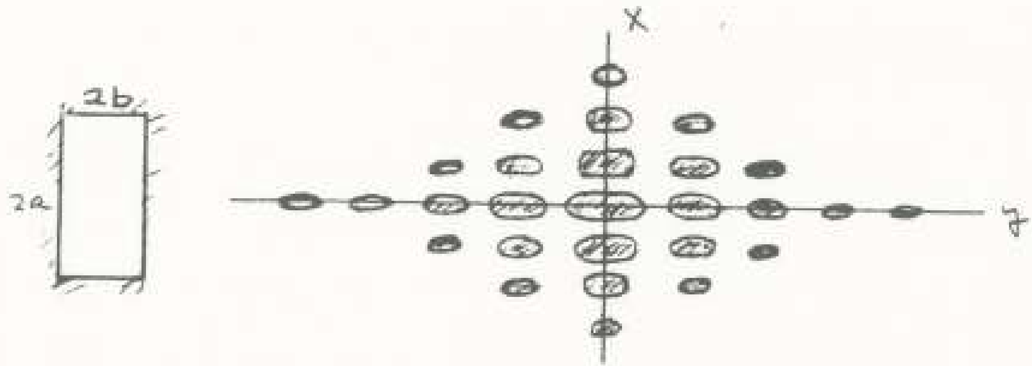


$$\psi(P) = c\int_{-a}^{a} e^{-ik\alpha\xi}\,d\xi \int_{-b}^{b} e^{-ik\beta\eta}\,d\eta$$

$$= 4cab\left(\frac{\sin k\alpha a}{k\alpha a}\right)\left(\frac{\sin k\beta b}{k\beta b}\right)$$

$$\gamma_a = k\alpha a, \quad \gamma_b = k\beta b$$

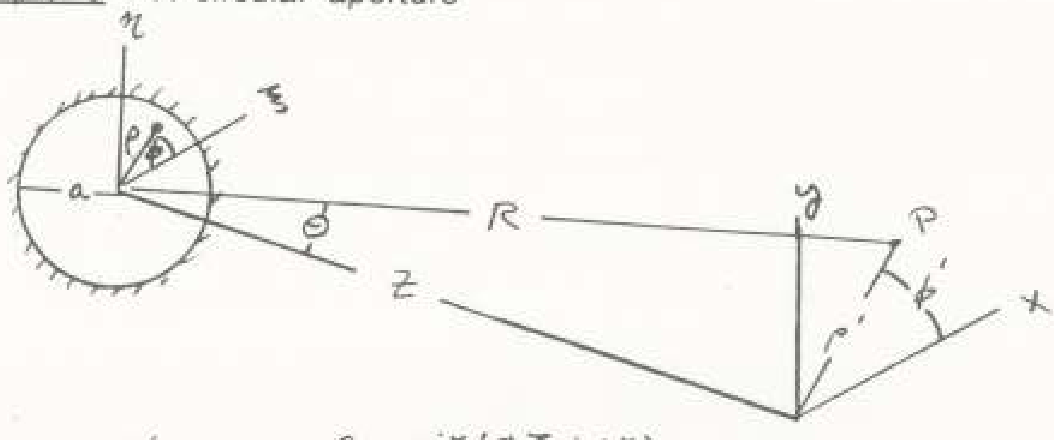
$$I(P) = 16c^2a^2b^2\left(\frac{\sin\gamma_a}{\gamma_a}\right)^2\left(\frac{\sin\gamma_b}{\gamma_b}\right)^2$$



maxima at $k\alpha a = (m+\frac{1}{2})\pi$, $k\beta b = (n+\frac{1}{2})\pi$

$m, n, = 0, \pm 1, \pm 2, \pm 3 \ldots$

Example 3 - A circular aperture



$$\psi(P) = c\int_{\sigma} e^{-ik(\alpha\xi+\beta\eta)}\,d\xi\,d\eta$$

$$\xi = \rho\cos\phi \qquad \eta = \rho\sin\phi$$

$$x = \rho'\cos\phi' \qquad y = \rho'\sin\phi'$$

$$\alpha = \frac{x}{R} = \frac{\rho'}{R}\cos\phi' \quad \beta = \frac{y}{R} = \frac{\rho'}{R}\sin\phi'$$

if θ is small,

$$\frac{\rho'}{R} = \sin\theta \approx \theta$$

$$\alpha\xi + \beta\eta = \frac{\rho\rho'}{R}(\cos\phi\cos\phi' + \sin\phi\sin\phi')$$
$$= \rho\theta\cos(\phi - \phi')$$

Because of cylindrical symmetry, result must be independent of ϕ'; choose $\phi' = 0$

$$\psi(p) = c\int_0^a \rho\, d\rho \int_0^{2\pi} e^{-ik\rho\theta\cos\phi}\, d\phi$$

$$\text{now} \quad J_n(u) = \frac{1}{2\pi i^n}\int_0^{2\pi} e^{iu\cos\phi} e^{in\phi}\, d\phi$$

$$J_0(u) = \frac{1}{2\pi}\int_0^{2\pi} e^{iu\cos\phi}\, d\phi$$

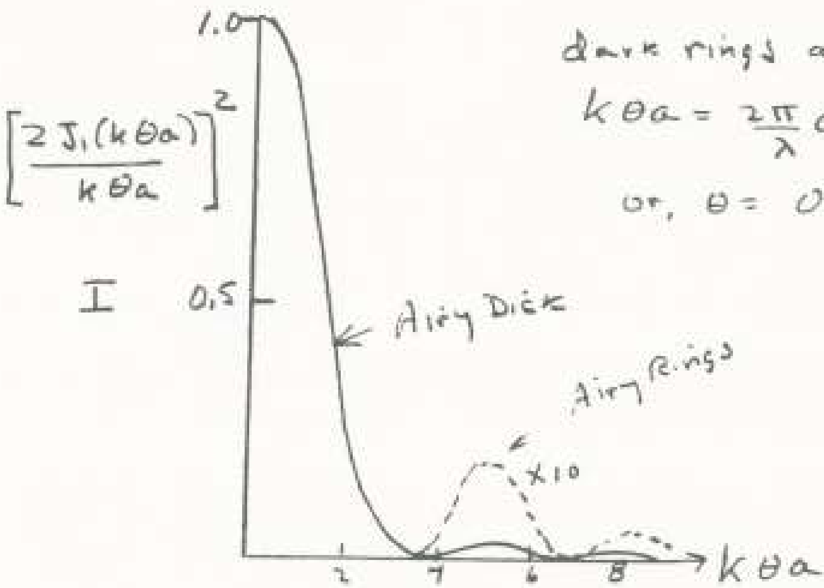
$$J_0(u) = J_0(-k\rho\theta) = J_0(k\rho\theta)$$

$$\psi(p) = 2\pi c\int_0^a J_0(k\rho\theta)\,\rho\, d\rho$$

$$\int u J_0(u)\, du = u J_1(u)$$

$$\psi(p) = 2\pi c a^2 \frac{J_1(k\theta a)}{k\theta a}$$

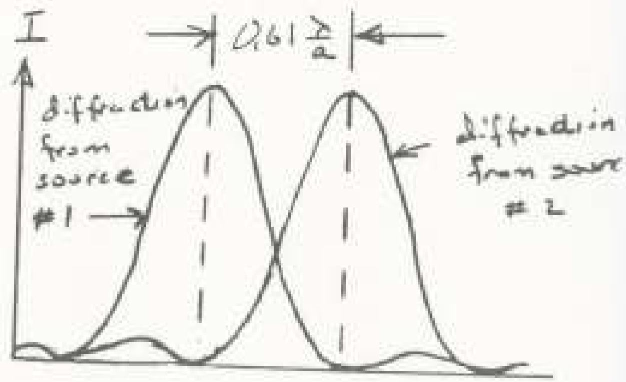
$$\boxed{I(p) = \pi^2 c^2 a^4 \left(\frac{2J_1(k\theta a)}{k\theta a}\right)^2}$$



dark rings at*

$k\theta a = \frac{2\pi}{\lambda}\theta a = 3.83, 7.02, \dots$

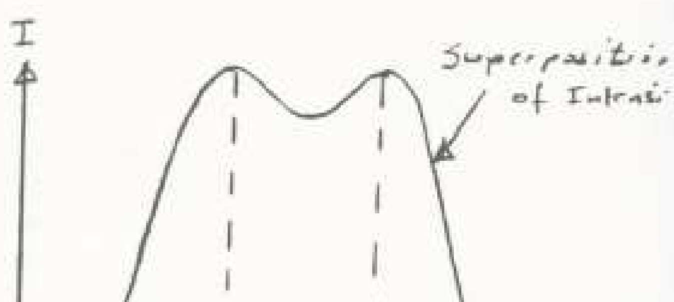
or, $\theta = 0.61\frac{\lambda}{a}, 1.12\frac{\lambda}{a}, \dots$



Angular Resolution:

The Rayleigh Criterion:

2 point sources of same wavelength are just resolved if the maximum intensity of one source occurs at the position of the first diffraction minimum of the second source.



Angular Resolution is 0.61 λ/a -- require telescopes of large aperture to resolve objects that have a small separation.

Limited by the diffraction of the lens system

* radius of Airy Disk is $1.22\frac{\lambda}{D} f$

f = R = focal length of lens

D = lens diameter

Consider the Fraunhofer diffraction formula:

$$\psi(P) = -i\psi_0 \frac{e^{ikR}}{\lambda R} \int_\sigma e^{-ik(\alpha\xi+\beta\eta)} \, d\xi \, d\eta$$

The factor $-i\,\psi_0\,[\exp(ikR)/\lambda R]$ contains the amplitude and phase information of the diffracted monochromatic wave at the aperture. Let's write:

$$\psi(P) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\xi,\eta)\, e^{-ik(\alpha\xi+\beta\eta)} \, d\xi \, d\eta$$

where $A(\xi,\eta)\,d\xi\,d\eta$ may be thought of as proportional to the diffracted field emanating from the aperture element $d\xi\,d\eta$. Call $A(\xi,\eta)$ the aperture function.

Define the spatial frequencies k_x, k_y

$$k_x = k\alpha = (kx/R)$$

$$k_y = k\beta = (ky/R)$$

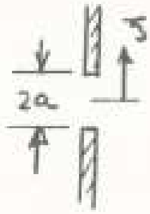
diffraction pattern $\Big\}$ $$\psi_p(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\xi,\eta)\, e^{-i(k_x\xi+k_y\eta)} \, d\xi \, d\eta$$

aperture function $$A(\xi,\eta) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(k_x,k_y)\, e^{i(k_x\xi+k_y\eta)} \, dk_x \, dk_y$$

The Fraunhofer diffraction pattern is the Fourier transform of the field distribution across the aperture (i.e., the aperture function). Or --

The Fraunhofer diffraction pattern (field distribution in the image plane) is the spatial frequency spectrum of the aperture function.

Consider again the single slit diffraction problem:



$$A(\xi) = c \qquad -a < \xi < a$$

$$A(\xi) = 0 \quad \text{elsewhere}$$

$$\psi_p(k_x) = \int_{-\infty}^{\infty} A(\xi)\, e^{-ik_x \xi}\, d\xi$$

$$= \int_{-\infty}^{\infty} A(\xi) \cos k_x \xi \, d\xi = \frac{c}{k_x}\, 2 \sin k_x a$$

$$= \frac{2c}{k\alpha} \sin k\alpha a$$

$$= 2ca \left[\frac{\sin k\alpha a}{k\alpha a} \right]$$

Fourier transform of δ-function

$$\psi(k_x) = \int_{-\infty}^{\infty} \delta(\xi - \xi_0)\, e^{-ik_x \xi}\, d\xi$$

$$\text{let } \xi - \xi_0 = \xi'$$

$$\psi(k_x) = \int_{-\infty}^{\infty} \delta(\xi')\, e^{-ik_x(\xi' + \xi_0)}\, d\xi' = e^{-ik_x \xi_0} \underbrace{\int_{-\infty}^{\infty} \delta(\xi')\, e^{-ik_x \xi'}\, d\xi'}_{1}$$

$$\boxed{\psi(k_x) = e^{-ik_x \xi_0}} \quad \delta\text{-function}$$

Consider the δ-functions at $\xi = \pm d$

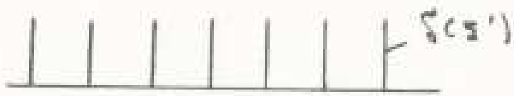
$$\psi(k_x) = e^{-ik_x d} + e^{ik_x d} = 2\cos k_x d$$

An array of N δ-functions: spacing d at $\xi = 0, d, 2d, 3d, \ldots$

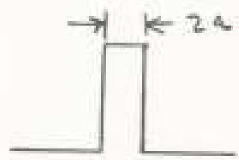
$$\psi(k_x) = \sum_{n=0}^{N-1} e^{-ik_x n d}$$

Consider the diffraction grating as a convolution:

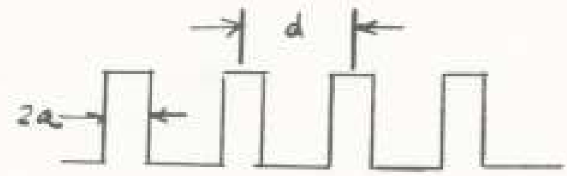
$$h(\xi) = \int_{-\infty}^{\infty} f(\xi')\, g(\xi-\xi')\, d\xi'$$

$f(\xi)$ 

array of δ-functions

$g(\xi)$ 

Single slit aperture function

$h(\xi)$ 

diffraction grating aperture function

$$\psi(k_x) = \int_{-\infty}^{\infty} h(\xi)\, e^{-ik_x \xi}\, d\xi$$

$$= \int_{-\infty}^{\infty} e^{-ik_x \xi} \int_{-\infty}^{\infty} f(\xi')\, g(\xi-\xi')\, d\xi'\, d\xi$$

$$= \int_{-\infty}^{\infty} f(\xi') \left[\int_{-\infty}^{\infty} e^{-ik\xi}\, g(\xi-\xi')\, d\xi \right] d\xi'$$

$x = \xi - \xi'$

$$\psi(k_x) = \int_{-\infty}^{\infty} f(\xi')\left[\int_{-\infty}^{\infty} e^{-ik_x(x+\xi')} g(x)\,dx\right] d\xi'$$

$$= \underbrace{\int_{-\infty}^{\infty} e^{-ik_x\xi'} f(\xi')\,d\xi'}_{\sum_{n=0}^{N-1} e^{-ik\alpha n d}} \underbrace{\int_{-\infty}^{\infty} e^{-ik_x x} g(x)\,dx}_{2ca\left(\frac{\sin k\alpha a}{k\alpha a}\right)}$$

$$\downarrow$$

$$\left\{\begin{array}{l}\text{first term is } 1 \\ \text{last term is } e^{-i\alpha k(N-1)d} \\ \text{common ratio is } e^{-i\alpha k d}\end{array}\right\} \quad \text{Sum} = \frac{1-e^{-i\alpha k N d}}{1-e^{-i\alpha k d}}$$

$$\psi(k_x) = \frac{\left(1-e^{-i\alpha k N d}\right)}{\left(1-e^{-i\alpha k d}\right)}\, 2ca\left(\frac{\sin k\alpha a}{k\alpha a}\right)$$

$$I_p = |\psi(k_x)|^2 = \left[\frac{1-e^{-i\alpha k N d}-e^{i\alpha k N d}+1}{1-e^{-i\alpha k d}-e^{i\alpha k d}+1}\right]\left(2ca\,\frac{\sin k\alpha a}{k\alpha a}\right)^2$$

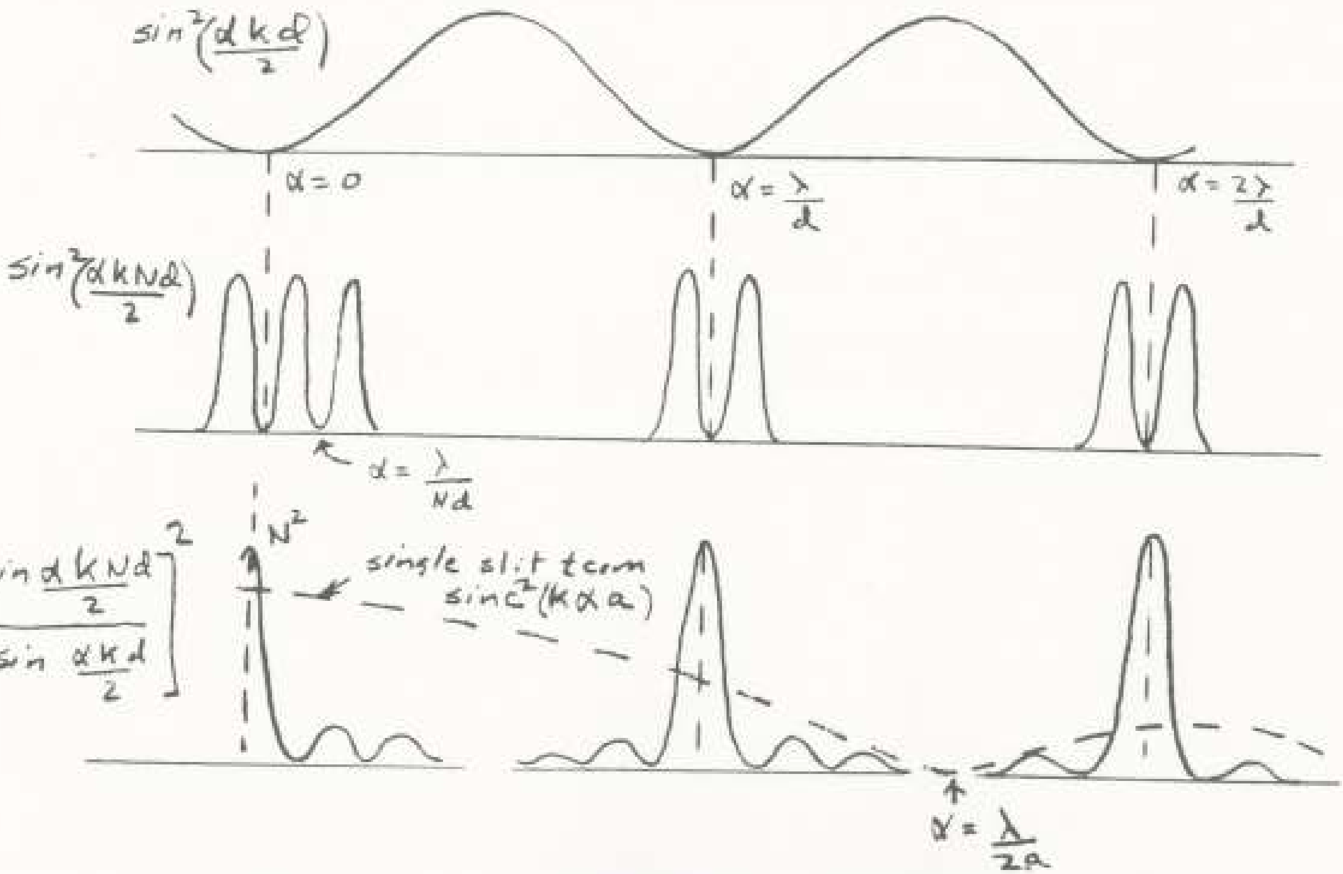
$$= \left(\frac{1-\cos\alpha k N d}{1-\cos\alpha k d}\right)\left[2ca\left(\frac{\sin k\alpha a}{k\alpha a}\right)\right]^2$$

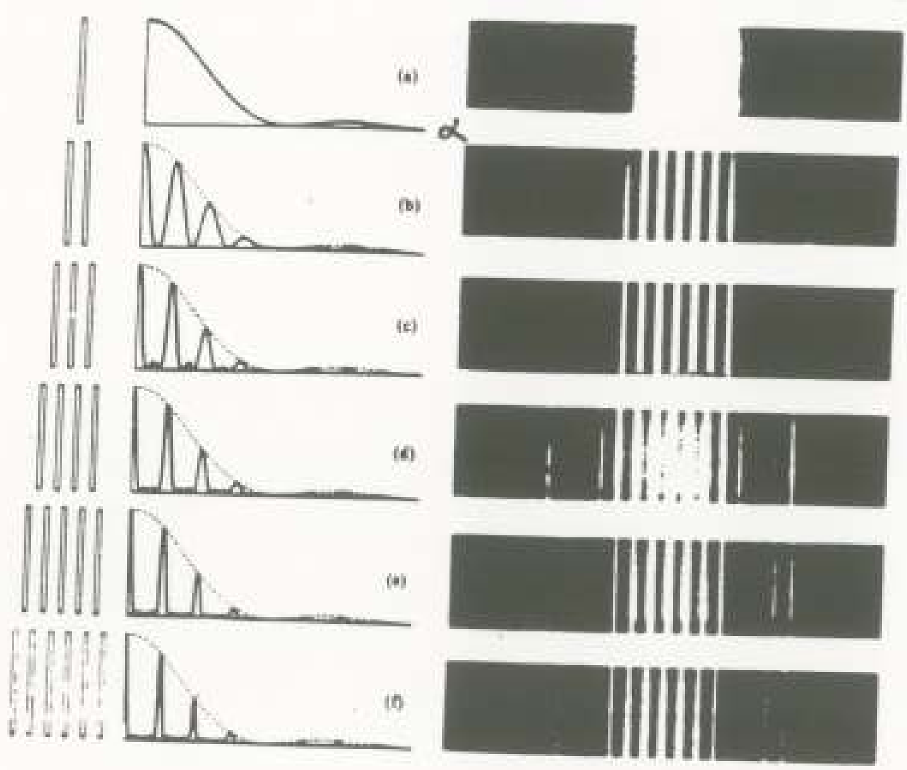
$$1-\cos 2\theta = 2\sin^2\theta$$

$$\boxed{I_p = 4c^2a^2\,\frac{\sin^2(\alpha k N d/2)}{\sin^2(\alpha k d/2)}\,\frac{\sin^2 k\alpha a}{(k\alpha a)^2} \qquad \text{N-slits}}$$

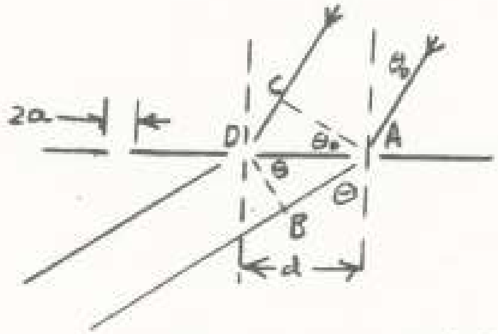
Two slits: $I_p = 4c^2a^2 \dfrac{\sin^2 k\alpha a}{(k\alpha a)^2} \, 4\cos^2 k\alpha d$

Look further at N-slit case:





More on the diffraction grating:



$$AB - CD = d(\sin\theta - \sin\theta_0)$$

$$\psi_P = \int_{-\infty}^{\infty} A(\xi)\, e^{-ik\alpha\xi}\, d\xi \quad \text{where } \alpha = \underbrace{\sin\theta}_{\frac{x}{R}\text{ (as before)}} - \sin\theta_0$$

$\sin\theta_0$: phase shift caused by inclination of the incident wave

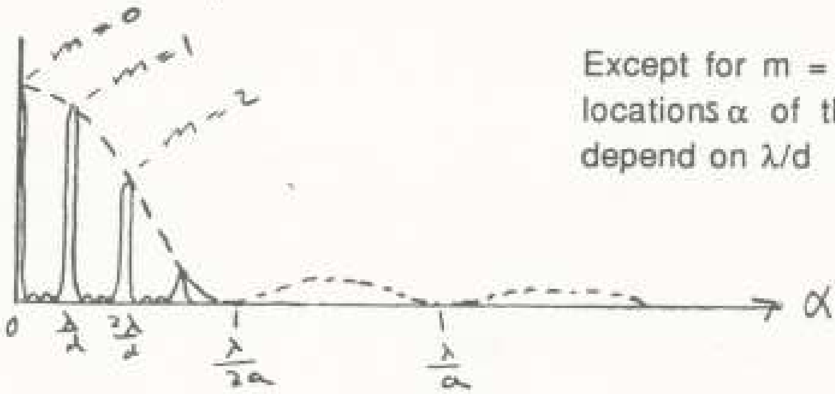
$$\psi_P = \left|\frac{\sin\frac{Nkd\alpha}{2}}{\sin\frac{kd\alpha}{2}}\right|^2 \left|\frac{\sin ka\alpha}{ka\alpha}\right|^2$$

N slits — single slit diffraction

We found principal maxima when $\frac{kd\alpha}{2} = m\pi \quad m = 0, \pm 1, \pm 2, \ldots$

$$\text{or } \alpha = \sin\theta - \sin\theta_0 = \frac{m\lambda}{d}$$

Each maximum was of a height N^2 modulated by the envelope of the single slit diffraction pattern



Except for m = 0 (zero order) the locations α of the principal maxima depend on λ/d

The points of maximum intensity are separated by points of zero intensity at $(Nkd\alpha/2) = n\pi$, $n = \pm 1, 2, \ldots$

or $\alpha = (n\lambda/Nd)$, $n = \pm 1, \pm 2, \pm 3$

The separation between a primary maximum of order m and a neighboring minimum is $\Delta\alpha = \lambda/Nd$.

If the wavelength is changed by an amount $\Delta\lambda$, the m^{th} order maximum is displaced by

$$\Delta\alpha' = \frac{|m|}{d}\Delta\lambda$$

Using the Rayleigh criterion, the lines of wavelength $\lambda = \pm (1/2)\Delta\lambda$ will be just resolved when the maximum of one wavelength coincides with the first minimum of the other wavelength

i.e., $\Delta\alpha = \Delta\alpha' \rightarrow \lambda/\Delta\lambda = |m|\ N$

The resolving power is the product of the order m of the diffraction and the number of grooves N

For the m^{th} order $d\alpha = d(\sin\theta - \sin\theta_0) = m\lambda$.

So $$\frac{\lambda}{\Delta\lambda} = \frac{Nd\,|\sin\theta - \sin\theta_0|}{\lambda}$$

The resolving power is thus equal to the number of wavelengths in the path difference between rays diffracted in the direction θ from opposite ends of a grating of width Nd.

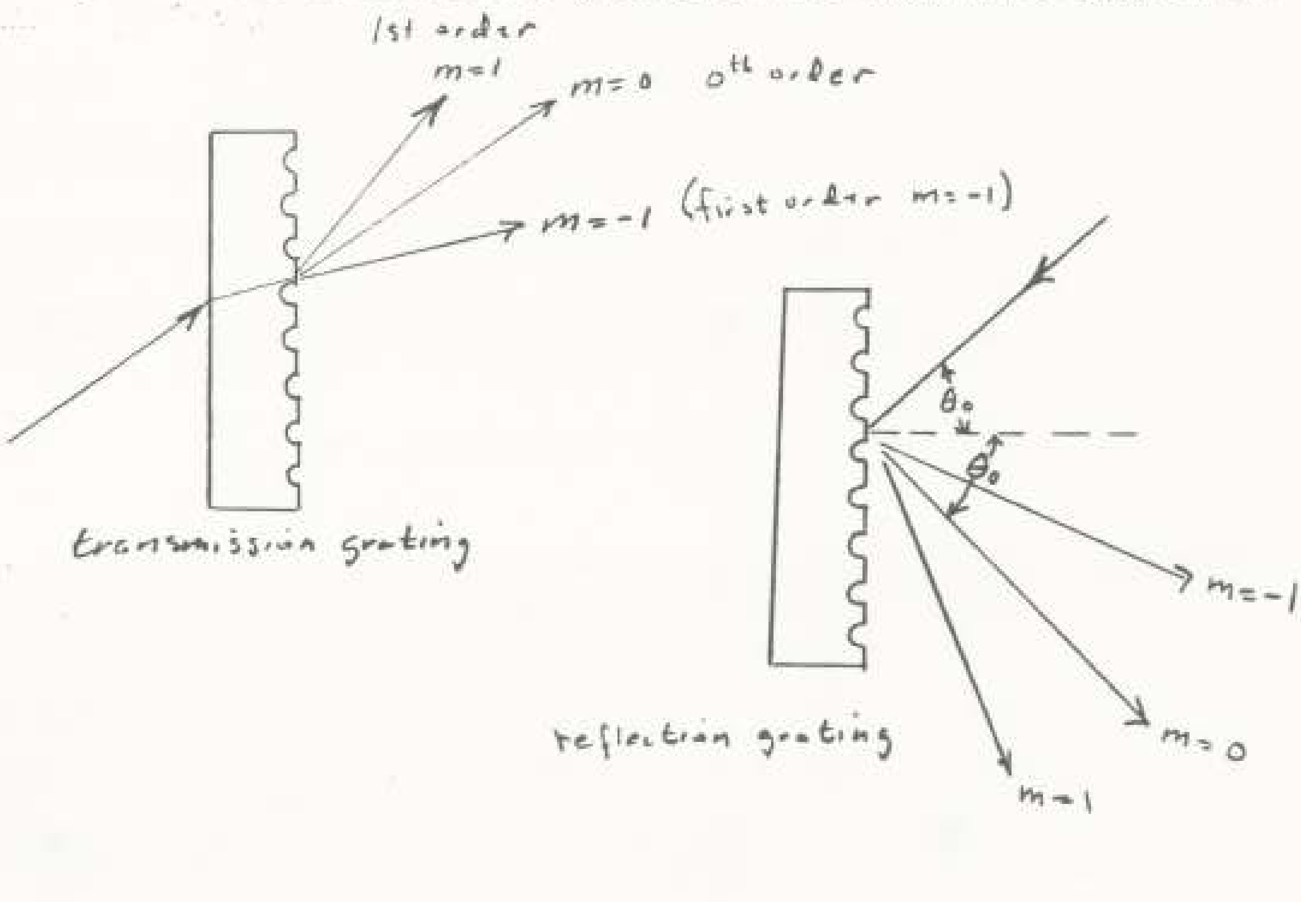
Since $(\sin\theta - \sin\theta_0)$ cannot exceed 2, the maximum resolving power cannot exceed

$\frac{2w}{\lambda}$ width of grating

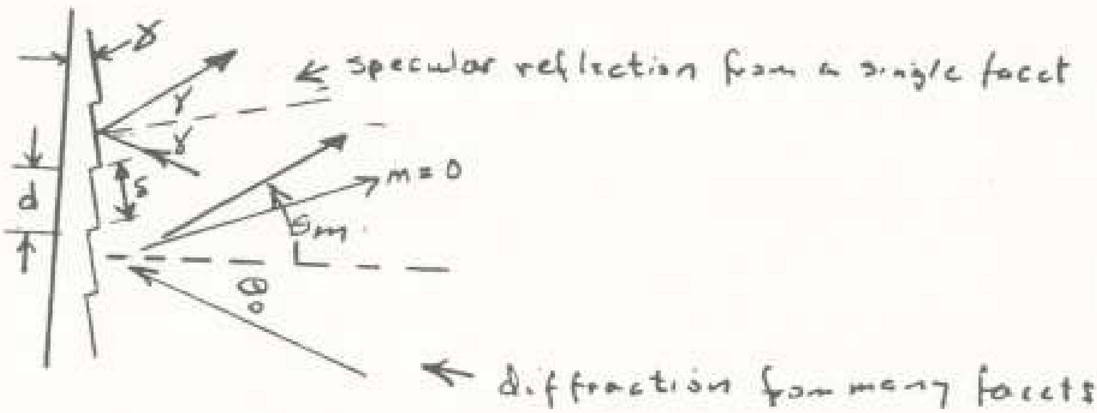
Example: resolve two lines 0.1Å apart near λ = 5000Å in second order (m = 2)

$$\frac{\lambda}{\Delta\lambda} = 2N \Rightarrow N \gtrsim \frac{\lambda}{2\Delta\lambda} = 25{,}000$$

Grating must have at least 25,000 grooves at 1200 lines/mm ⇒ 20.8 mm = W

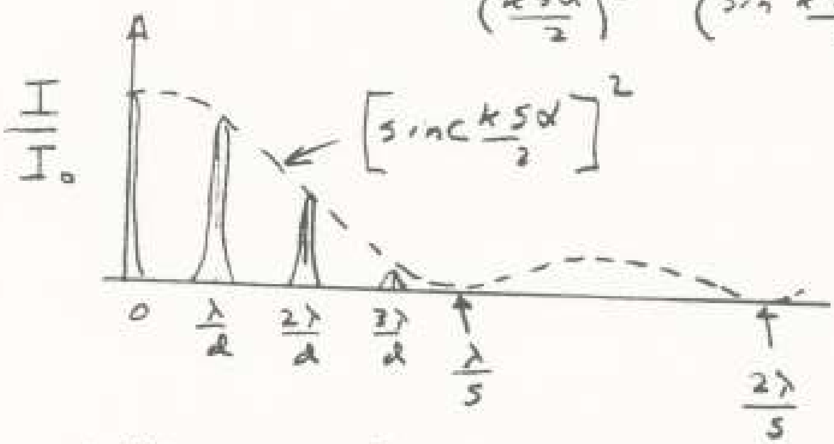


Blazed Diffraction Gratings

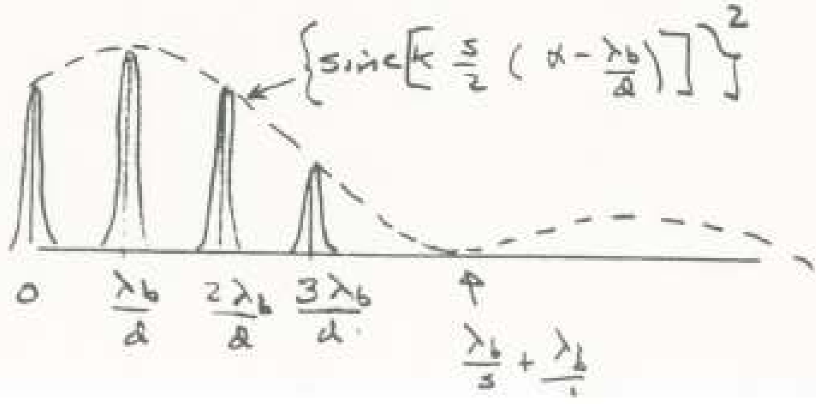


$$\alpha \equiv \sin\theta_m - \sin\theta_0 = \frac{m\lambda}{d}$$

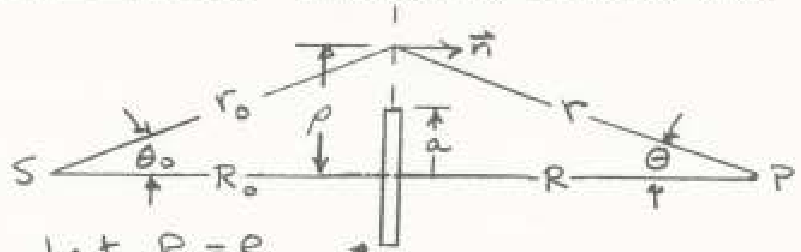
In general, $$\frac{I(\alpha)}{I_0} = \frac{\left(\sin \frac{ks\alpha}{2}\right)^2}{\left(\frac{ks\alpha}{2}\right)^2}\,\frac{\left(\sin Nk\frac{d}{2}\alpha\right)^2}{\left(\sin k\frac{d\alpha}{2}\right)^2}$$



Now, shift single facet pattern to be centered at $m=1$ when $\lambda = \lambda_b$ (blaze wavelength)



Babinet's Principle - Diffraction by a circular disk



let $R_0 = R$

Circular disk of radius a

$$\psi(P) = \frac{-iA}{2\lambda} \int_\sigma \frac{e^{ik(r+r_0)}}{r r_0} (\cos\theta_0 + \cos\theta)\, d\sigma$$

Kirchhoff Diffraction Formula

If $R_0 = R$, $\cos\theta_0 = \cos\theta$

$$\psi(P) = \frac{-iA}{\lambda} \int_0^{2\pi} d\phi \int_a^\infty \frac{e^{2ikr}}{r^2} \cos\theta\, \rho\, d\rho$$

$\cos\theta = \frac{R}{r}$ $\quad r^2 = R^2 + \rho^2$

$$= \frac{-i2\pi AR}{\lambda} \int_{\sqrt{R^2+a^2}}^\infty \frac{e^{2ikr}}{r^2}\, dr$$

Integrate by parts $\quad \int u\, dv = uv - \int v\, du$

$u = \frac{1}{r^2}$, $dv = e^{2ikr} dr$, $v = \frac{1}{2ik} e^{2ikr}$ $\quad du = \frac{-2}{r^3} dr$

$$\psi(P) = -ikAR \left\{ \left[\frac{e^{2ikr}}{2ik\, r^2} \right]_{\sqrt{R^2+a^2}}^\infty - \frac{1}{ik} \int_{\sqrt{R^2+a^2}}^\infty \frac{e^{ikr}}{r^3}\, dr \right\}$$

$$= -ikAR \left\{ -\frac{1}{2ik} \frac{\exp[2ik\sqrt{R^2+a^2}]}{R^2+a^2} + \frac{1}{2k^2} \frac{\exp[2ik\sqrt{R^2+a^2}]}{(R^2+a^2)^{3/2}} - \frac{3}{2k^2} \int_{\sqrt{R^2+a^2}}^\infty \frac{e^{2ikr}}{r^4}\, dr \right\}$$

$$\psi(P) = \frac{AR}{2} \frac{\exp[2ik\sqrt{R^2+a^2}]}{R^2+a^2} \left\{ 1 + \frac{i}{k} \frac{1}{\sqrt{R^2+a^2}} + \cdots \right\}$$

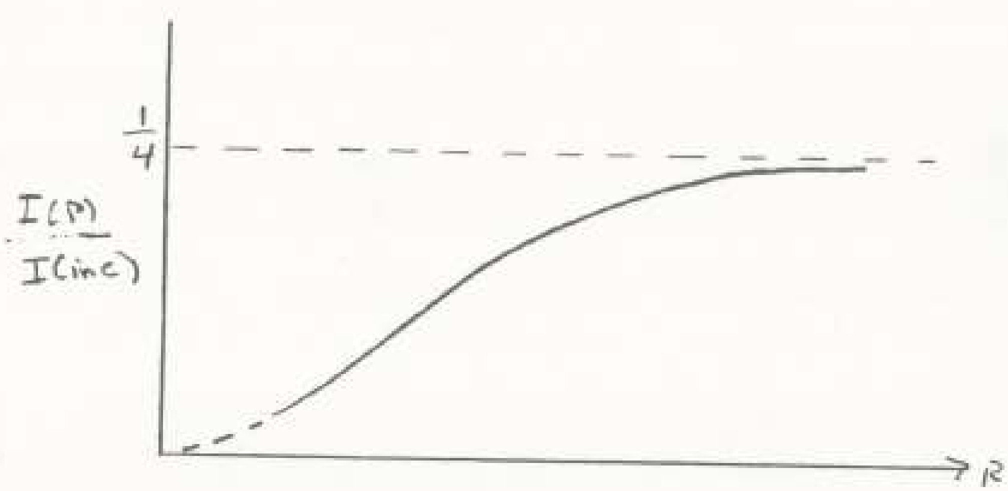
The 2nd term is of order $(1/kR) = (\lambda/2\pi R)$
Since we consider only $R \gg \lambda$, we drop this term.

$$\boxed{I(P) = |\psi(P)|^2 = \frac{A^2R^2}{4(R^2+a^2)^2}}$$

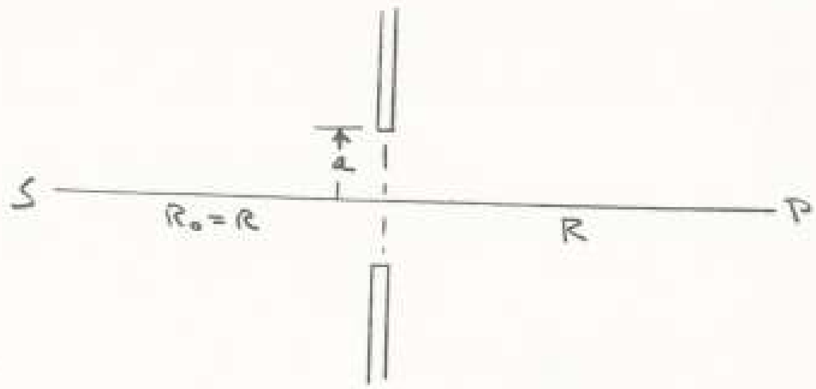
Note that the incident amplitude at the edge of the disk is

$$A \frac{\exp[ik\sqrt{R^2+a^2}]}{\sqrt{R^2+a^2}}$$

$$I_{inc} = |\psi_{inc}|^2 = \frac{A^2}{R^2+a^2}$$



Now consider the complementary situation:



Now the only difference is in the limits of integration

$$\psi(P) = -i\,\frac{2\pi A R}{\lambda}\int_{R}^{\sqrt{R^2+a^2}} \frac{e^{2ikr}}{r^2}\,dr$$

$$\psi(P) = -ikAR\left\{ \frac{1}{2ik}\,\frac{e^{2ikr}}{r^2}\Bigg|_{R}^{\sqrt{R^2+a^2}} + \text{terms of order } \frac{1}{kR} \text{ and higher} \right\}$$

$$\psi(P) = -\frac{AR}{2}\left\{ \frac{\exp\left[2ik\sqrt{R^2+a^2}\right]}{R^2+a^2} - \frac{e^{2ikR}}{R^2} \right\}$$

Notice that the sum of this amplitude and that of the previous calculation is

$$\psi(P) = \psi_1(P) + \psi_2(P) = \frac{AR}{2}\frac{e^{2ikR}}{R^2} = \frac{Ae^{2ikR}}{2R}$$

which is just the amplitude of the wave from the source in the absence of any diffracting screen. This is an illustration of <u>Babinet's principle</u>

$$\psi_0 = \psi_1 + \psi_2 \quad \text{complementary apertures}$$

Consider last calculation in limit R»a

$$\psi(P) = -\frac{AR}{2}\left\{ \frac{\exp\left[2ikR\sqrt{1+a^2/R^2}\,\right]}{R^2} - \frac{\exp\left[2ikR\right]}{R^2} \right\}$$

$$= -\frac{A}{2R}\exp\left[2ikR\right]\left\{ e^{\frac{ika^2}{R}+\cdots} - 1 \right\}$$

$$= -\frac{A}{2R}\exp\left[2ikR\right]\exp\left[\frac{ika^2}{2R}\right]\left(2i\sin\frac{ka^2}{2R}\right)$$

$$I(P) = |\psi(P)|^2 = \frac{A^2}{R^2}\sin^2\frac{ka^2}{2R}$$

or, in terms of I_{inc},

$$\boxed{I(P) = I_{inc}\sin^2\frac{ka^2}{2R}}$$

Intensity pattern is fundamentally different from that of a circular disk.