The distinction between Fraunhofer and Fresnel diffraction

A circle of radius R encloses all diffracting apertures in plane

\[ kA = k \left[ d_0^2 + \left( \frac{d}{d_0} \times \frac{R}{d_0} \right)^2 \right]^{1/2} \approx k \cdot d_0 + \frac{1}{2} \frac{k}{d_0} \left( \frac{R^2}{d_0^2} \right) + \cdots \]

assuming \( r \), \( \rho \) are small compared with \( d_0 \)

\[ kA = k \left( d_0 + \frac{1}{2} \frac{R^2}{d_0} \right) - k \frac{R^2}{d_0} + \frac{1}{2} k \frac{R r^2}{d_0} \]

If the maximum value for the quadratic term in \( r \) is \( \frac{1}{2} k \frac{R^2}{d_0} \),

then Fraunhofer diffraction occurs when \( \frac{1}{2} \frac{k R^2}{d_0} \ll \pi \) (far field)

and Fresnel diffraction occurs when \( \frac{1}{2} \frac{k R^2}{d_0} \gg \pi \) (near field)

Thus

Fresnel: \( R^2 \gg \lambda d_0 \) (near field)

Fraunhofer: \( R^2 \ll \lambda d_0 \) (far field)

For example if \( R=2\text{mm}, \lambda=5\times10^{-4}\text{mm} \), then Fresnel pattern for \( d_0 \ll 2\text{m} \) and Fraunhofer patterns for \( d_0 \gg 2\text{m} \).
\[ R^2 = x^2 + y^2 + z^2 \]
\[ r^2 = (x-x'_0)^2 + (y-y'_0)^2 + z^2 \]
\[ = \frac{x^2 + y^2 + z^2}{R^2} - 2 \frac{x x'_0}{R^2} - 2 \frac{y y'_0}{R^2} + \frac{z^2}{R^2} \]
\[ r^2 \approx R^2 \left[ 1 - 2 \frac{x x'_0}{R^2} + \frac{z^2}{R^2} \right] \]

Let \( \alpha = \frac{x}{R}, \beta = \frac{y}{R} \)

\[ r \approx R - \alpha x'_0 + \frac{\beta^2 + n^2}{2R} + \ldots \]

**Fraunhofer Diffraction:**

neglect \( \frac{\beta^2 + n^2}{2R} \) term

\[ r = R - (\alpha x'_0 + \beta y'_0) \]

\[ \psi (p) = -\frac{i A}{\lambda R} e^{ikR} \int e^{-ik(x x'_0 + y y'_0)} \, dx \, dy \]

**Fraunhofer Diffraction Formula**

\[ [A = \psi_0, \text{amplitude of incident wave}] \]

When \( \frac{\beta^2 + n^2}{2R} \) cannot be neglected, we have **Fresnel Diffraction**
Fraunhofer Diffraction

\[ f(p) = -i \cdot \frac{V_0 \cdot e^{ikR}}{\lambda R} \int e^{-ik(x \cos \phi + y \sin \phi)} dx \]

**Example 1** - an infinite slit

\[ f(p) = \int_{-a}^{a} e^{-ikx} \frac{dx}{x} \]

\[ = \frac{i}{kd} \left[ \frac{e^{-ikx}}{k} \right]_{-a}^{a} \]

\[ = \frac{i}{kd} (e^{-ika} - e^{ika}) \]

\[ 4f(p) = \frac{2}{kd} \sin kda \]

\[ I(\phi) = |4f(p)|^2 = 4c^2 a^2 \left( \frac{\sin \gamma}{\gamma} \right)^2 \]

where \( \gamma = kd \) and \( \gamma = \frac{2\pi \omega}{\lambda} \sin \theta \)

**Example 2** - a rectangular aperture
\[ Y(\rho) = c \int_{-a}^{a} e^{-i k a_1} a_1 \int_{-b}^{b} e^{-i k a_2} a_2 \]

\[ = 4 c a b (\frac{\sin k a_1}{k a_1}) (\frac{\sin k b_1}{k b_1}) \]

\[ a_1 = \kappa a, \quad b_1 = \kappa b \]

\[ I(\rho) = 16 c^2 a^2 b^2 \left( \frac{\sin y a}{y a} \right)^2 \left( \frac{\sin y b}{y b} \right)^2 \]

Maxima at \( \kappa a = (m + \frac{1}{2}) \pi, \quad \kappa b = (n + \frac{1}{2}) \pi \)

\( m, n, = 0, 1, 2, 3, \ldots \)

**Example 3 - A circular aperture**

\[ Y(\rho) = c \int_{-\pi}^{\pi} e^{-i k (\rho \cos \phi + \rho \sin \phi)} d\phi d\rho \]

\( \phi = \rho \cos \theta \quad \rho = \rho \sin \theta \]

\( x = \rho \cos \theta \quad y = \rho \sin \theta \)
\[ a = \frac{x}{R} = \frac{f'}{R} \cos \phi' \quad \beta = \frac{\gamma}{R} = \frac{f'}{R} \sin \phi' \]

if \( \theta \) is small,

\[ \frac{f'}{R} \approx \sin \theta \approx \theta \]

\[ \alpha^2 + \beta^2 = \frac{f'^2}{R^2} \left( \cos^2 \phi \cos^2 \phi' + \sin \phi \sin \phi' \right) = \rho \theta \cos (\phi - \phi') \]

Because of cylindrical symmetry, result must be independent of \( \phi' \); choose \( \phi' = 0 \)

\[ f(r) = \frac{1}{\sqrt{2\pi r}} \int_0^\infty e^{-ikr} J_0(k \rho \theta) \, dk \]

now \( J_0(q) = \frac{1}{2\pi} \int_0^{2\pi} e^{i \cos \phi} \, d\phi \)

\[ J_0(q) = J_0(-k \rho \theta) = J_0(k \rho \theta) \]

\[ f(r) = 2\pi c \int_0^\infty \frac{1}{\sqrt{2\pi r}} J_0(k \rho \theta) \rho \, d\rho \]

\[ \int_0^\infty \rho J_0(\rho) \, d\rho = \sqrt{\pi} J_1(\infty) \]

\[ f(r) = 2\pi c \frac{J_1(k \rho \theta)}{k \rho \theta} \]

\[ I(r) = \frac{\pi^2 c^2 a^4}{k \rho \theta} \left( \frac{2 J_1(k \rho \theta)}{k \rho \theta} \right)^2 \]
Angular Resolution:

The Rayleigh Criterion:

2 point sources of same wavelength are just resolved if the maximum intensity of one source occurs at the position of the first diffraction minimum of the second source.

Angular Resolution is 0.61 \( \lambda / a \) -- require telescopes of large aperture to resolve objects that have a small separation.

Limited by the diffraction of the lens system

\[
\text{Radius of Airy Disk} \approx 1.22 \frac{\lambda}{D} \frac{1}{f}
\]

\[
f = R = \text{focal length of lens}
\]

\[
D = \text{lens diameter}
\]
Consider the Fraunhofer diffraction formula:

$$
U(p) = -i \frac{\psi_0}{\lambda R} \int e^{-ik(x^2 + y^2)} e^{-ik(x \xi + y \eta)} d\xi d\eta
$$

The factor $-i \frac{\psi_0}{\lambda R} [\exp(ikR)/\lambda R]$ contains the amplitude and phase information of the diffracted monochromatic wave at the aperture. Let's write:

$$
U(p) = \sum_{\xi} \sum_{\eta} \mathcal{A}(\xi, \eta) e^{-ik(x^2 + y^2)}
$$

where $\mathcal{A}(\xi, \eta) d\xi d\eta$ may be thought of as proportional to the diffracted field emanating from the aperture element $d\xi d\eta$. Call $\mathcal{A}(\xi, \eta)$ the aperture function.

Define the spatial frequencies $k_x, k_y$

$$
k_x = k x = (k x / R)
$$

$$
k_y = k y = (k y / R)
$$

The diffraction pattern

$$
U_p(k_x, k_y) = \sum_{\xi} \sum_{\eta} \mathcal{A}(\xi, \eta) e^{-i(k_x \xi + k_y \eta)}
$$

aperture function

$$
\mathcal{A}(\xi, \eta) = \frac{1}{(2\pi)^2} \int \int U(k_x, k_y) e^{i(k_x \xi + k_y \eta)} d k_x d k_y
$$
The Fraunhofer diffraction pattern is the Fourier transform of the field distribution across the aperture (i.e., the aperture function). Or --

The Fraunhofer diffraction pattern (field distribution in the image plane) is the spatial frequency spectrum of the aperture function.

Consider again the single slit diffraction problem:

\[
\frac{1}{2a} A(s) = \begin{cases} \cos k \pi s/a & \text{if } -a < s < a \\ 0 & \text{elsewhere} \end{cases}
\]

\[
\Psi_p(k_x) = \int_{-\infty}^{\infty} A(s) e^{-i k x s} ds
\]

\[
= \int_{-\infty}^{\infty} \cos k \pi s/a \cos k x s ds = \frac{c}{k_x} 2 \sin k \pi a
\]

\[
= \frac{2 c}{k x a} \sin k x a
\]

\[
= 2 c a \left[ \frac{\sin k x a}{k x a} \right]
\]

Fourier transform of \(\delta\)-function

\[
\Psi(k_x) = \int_{-\infty}^{\infty} \delta(s - s_0) e^{-i k x s} ds = e^{-i k x s_0}
\]

let \(s - s_0 = s'\)

\[
\Psi(k_x) = \int_{-\infty}^{\infty} \delta(s') e^{-i k x s'} ds' = e^{-i k x s_0} \int_{-\infty}^{\infty} \delta(s') e^{-i k x s'} ds'
\]

\[
\Psi(k_x) = e^{-i k x s_0} \delta\text{-function}
\]
Consider the δ-functions at $\xi = \pm d$

$$\psi(k_x) = e^{-ik_x d} + e^{ik_x d} = 2 \cos k_x d$$

An array of $N$ δ-functions: spacing $d$ at $\xi = 0, d, 2d, 3d, \ldots$

$$\psi(k_x) = \sum_{n=0}^{N-1} e^{-ik_x n d}$$

Consider the diffraction grating as a convolution:

$$h(\xi) = \int_{-\infty}^{\infty} f(\xi') \delta(\xi - \xi') d\xi'$$

- $f(\xi')$ array of δ-functions
- $g(\xi)$ single slit aperture function
- $h(\xi)$ diffraction grating aperture function

$$\psi(k_x) = \int_{-\infty}^{\infty} h(\xi) e^{-ik_x \xi} d\xi$$

$$= \int_{-\infty}^{\infty} e^{-ik_x \xi} \left( \int_{-\infty}^{\infty} f(\xi') \delta(\xi - \xi') d\xi' \right) d\xi$$

$$= \int_{-\infty}^{\infty} f(\xi') \left[ \int_{-\infty}^{\infty} e^{-ik_x \xi} \delta(\xi - \xi') d\xi \right] d\xi'$$
\[ \lambda = \lambda' - \lambda'' \]

\[ 4(kx) = \sum_{n=0}^{N-1} \left[ \int_{-\infty}^{\infty} e^{-ikx} f(x) \, dx \right] e^{-i\lambda nd} + 2ca \left( \frac{\sin kxa}{kxa} \right) \]

\[ \frac{e^{-i\lambda nd}}{1 - e^{-i\lambda kd}} \]

\[ \text{First term} = 1 \]

\[ \text{Last term} = e^{-i\lambda k N d} \]

\[ \text{Sum} = \frac{1 - e^{-i\lambda kd}}{1 - e^{-i\lambda kd}} \]

\[ 4(kx) = \frac{(1 - e^{-i\lambda k N d})}{(1 - e^{-i\lambda kd})} \cdot 2ca \left( \frac{\sin kxa}{kxa} \right) \]

\[ I_p = \left| 4(kx) \right|^2 = \left[ \frac{1 - e^{-i\lambda k N d} - e^{-i\lambda kd}}{1 - e^{-i\lambda kd} + 1} \right]^2 \left( 2ca \left( \frac{\sin kxa}{kxa} \right) \right)^2 \]

\[ = \left( \frac{1 - \cos \lambda k N d}{1 - \cos \lambda kd} \right) \left[ 2ca \left( \frac{\sin kxa}{kxa} \right) \right]^2 \]

\[ 1 - \cos \theta = 2 \sin^2 \theta \]

\[ I_p = 4c^2 a^2 \frac{\sin^2 (\lambda k N d/2)}{\sin^2 (\lambda k a/2)} \frac{\sin^2 kxa}{(kxa)^2} \]

\[ N = n' a \]
Two slits: \( I_p = \frac{4e^{2}a^{2}}{(kda)^{2}} \sin^{2}kxa \sin^{2}\left(\frac{dkda}{2}\right) \)

Look further at N-slit case:

\[ \sin^{2}\left(\frac{dkNda}{2}\right) \]

\[ \left[ \sin \frac{dNda}{2} \right]^2 \sin^2 \left( k \frac{dN}{a} \right) \]

\[ \left( \sin \frac{dNda}{2} \right)^2 \sin^2 \left( k \frac{dN}{a} \right) \]

Single slit term:

\( \sin^{2}kxa \)
More on the diffraction grating:

\[ AB - CD = d (\sin \theta - \sin \theta_0) \]

\[
\frac{\lambda}{d} \text{ (as before)}
\]

\[
\psi_p = \sum_{n=-\infty}^{\infty} A(n) e^{-ik_0 z}
\]

where \( \alpha = \sin \theta - \sin \theta_0 \)

\[
\psi_p = \left| \frac{\sin Nk_0 d}{2} \right|^2 \left| \frac{\sin k_0 d}{2} \right|^2 \left| \frac{\sin k_0 d}{2} \right|^2
\]

\[ N \text{ slits} \quad \text{single slit} \quad \text{diffraction} \]

We found principal maxima when \( \frac{k_0 d}{2} = m \pi \quad m = 0, \pm 1, \pm 2, \ldots \)

or \( d = \frac{\lambda}{2 \sin \theta - 2 \sin \theta_0} = \frac{m \lambda}{\alpha} \)
Each maximum was of a height $N^2$ modulated by the envelope of the single slit diffraction pattern. Except for $m = 0$ (zero order) the locations $\alpha$ of the principal maxima depend on $\lambda/d$.

The points of maximum intensity are separated by points of zero intensity at $(Nkd\alpha/2) = n\pi$, $n = \pm 1, 2, \ldots$.

or $\alpha = (n\lambda/Nd)$, $n = \pm 1, \pm 2, \pm 3$

The separation between a primary maximum of order $m$ and a neighboring minimum is $\Delta\alpha = \lambda/Nd$.

If the wavelength is changed by an amount $\Delta\lambda$, the $m^{th}$ order maximum is displaced by

$$\Delta \alpha' = \left| \frac{m}{\lambda} \right| \Delta \lambda$$

Using the Rayleigh criterion, the lines of wavelength $\lambda = \pm (1/2)\Delta\lambda$ will be just resolved when the maximum of one wavelength coincides with the first minimum of the other wavelength

i.e., $\Delta\alpha = \Delta\alpha' \rightarrow \lambda/\Delta\lambda = |m|/N$.

The resolving power is the product of the order $m$ of the diffraction and the number of grooves $N$.

For the $m^{th}$ order $d\alpha = d(\sin\theta - \sin\theta_0) = m\lambda$. 

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So \[ \frac{\lambda}{\Delta \lambda} = \frac{N \Delta \lambda}{\sin \theta - \sin \theta_0} \]

The resolving power is thus equal to the number of wavelengths in the path difference between rays diffracted in the direction \( \theta \) from opposite ends of a grating of width \( N \lambda \).

Since \( (\sin \theta - \sin \theta_0) \) cannot exceed 2, the maximum resolving power cannot exceed

\[ \frac{2w}{\lambda} \] width of grating

Example: resolve two lines 0.1Å apart near \( \lambda = 5000\)Å in second order \( (m = 2) \)

\[ \frac{\lambda}{\Delta \lambda} = 2N \Rightarrow N > \frac{\lambda}{2 \Delta \lambda} = 25,000 \]

Grating must have at least 25,000 grooves at 1200 lines/mm \( \Rightarrow 20.8 \) mm = \( w \)

1st order
\[ m = 1 \]

0th order
\[ m = 0 \]

First order \( m = -1 \)

Transmission grating

Reflection grating
Blazed Diffraction Gratings

\[ \delta = \sin \theta_m - \sin \theta_0 = m \frac{\lambda}{d} \]

In general,

\[ \frac{I(\delta)}{I_0} = \frac{(\sin k5d)^2}{(\sin Nk \frac{d}{2} \delta)^2} \frac{(\sin k \frac{d}{2} \delta)^2}{(\sin k \frac{d}{2} \delta)^2} \]

Now, shift single facet pattern to be centered at \( m = 1 \) when \( \lambda = \lambda_b \) (blaze wavelength)

\[ \sin \left[ \frac{k}{2} \left( \delta - \frac{\lambda_b}{d} \right) \right]^2 \]
Babinet's Principle - Diffraction by a circular disk

Let $R_0 = R$

Circular disc of radius $a$

$$
y(p) = \frac{-iA}{2\lambda} \int \frac{e^{ik(r+R_0)}}{rr_0} (\cos \theta_0 + \cos \theta) \, d\theta
$$

Kirschhoff Diffraction Formula

If $R_0 = R$, $\cos \theta_0 = \cos \theta$

$$
y(p) = -\frac{iA}{\lambda} \int_{-\pi}^{\pi} \int_{0}^{2\pi} e^{i\phi} \frac{e^{i2kr}}{r^2 + a^2} \, d\phi \, dr
$$

Integrate by parts

$$
u = \frac{1}{r^2}, \quad dv = e^{i2kr} \, dr, \quad u = \frac{1}{2ik} \frac{e^{i2kr}}{r^2} \quad du = -\frac{2}{r} \, dr
$$

$$
y(p) = -iKA \left\{ \frac{e^{i2kr}}{2ikr} \right\} \int_{0}^{\infty} \frac{e^{i2kr}}{r^2 + a^2} \, dr
$$

$$
= -iKA \left\{ -\frac{1}{2ik} \frac{\exp \left[ \frac{2ikr}{r^2 + a^2} \right]}{r^2 + a^2} + \frac{1}{2k^2} \frac{\exp \left[ \frac{2ikr}{r^2 + a^2} \right]}{\left( r^2 + a^2 \right)^{3/2}} \right\}
$$

$$
- \frac{1}{2k^2} \int_{0}^{\infty} \frac{e^{2ikr}}{r^2 + a^2} \, dr
$$
\[ H(\mathbf{r}) = \frac{AR \exp \left[ \frac{2i \mathbf{k} \cdot (\mathbf{r}^2 + a^2)}{\mathbf{r}^2 + a^2} \right]}{\frac{1}{4}} \left\{ 1 + \frac{i}{\mathbf{k}} \frac{1}{\mathbf{r}^2 + a^2} \right\} \]

The 2nd term is of order \((1/kR) = (\lambda/2\pi R)\).

Since we consider only \(R \approx \lambda\), we drop this term.

\[ I(\mathbf{r}) = |H(\mathbf{r})|^2 = \frac{A^2 R^2}{4(\mathbf{r}^2 + a^2)^2} \]

Note that the incident amplitude at the edge of the disk is

\[ H = \exp \left[ \frac{i \mathbf{k} \cdot (\mathbf{r}^2 + a^2)}{\sqrt{\mathbf{r}^2 + a^2}} \right] \]

\[ I_{\text{inc}} = |H_{\text{inc}}|^2 = \frac{A^2}{\mathbf{r}^2 + a^2} \]
Now consider the complementary situation:

\[
\begin{align*}
4(p) &= -i \frac{2\pi AR}{\lambda} \int_{R}^{\sqrt{R^2 + a^2}} \frac{e^{2i\kappa r}}{r^2} dr \\
4(p) &= -i 4\pi k \lambda R \left\{ \frac{1}{2} \frac{e}{2i k} \right\} + \text{terms of order } \frac{1}{\kappa R} \text{ and higher} \\
4(p) &= -\frac{AR}{2} \left\{ \frac{\exp \left[ 2i k \sqrt{R^2 + a^2} \right]}{R^2 + a^2} - \frac{e}{R^2} \right\}
\end{align*}
\]
Notice that the sum of this amplitude and that of the previous calculation is

\[ 4(p) = 4_1(p) + 4_2(p) = \frac{AR}{2} \frac{2ikR}{R^2} = \frac{Ae^{2ikR}}{2R} \]

which is just the amplitude of the wave from the source in the absence of any diffracting screen. This is an illustration of Babinet's principle.

\[ 4_0 = 4_1 + 4_2 \]

Consider last calculation in limit \( R \to a \)

\[ 4(p) = -\frac{AR}{2} \left\{ \frac{\exp[2i(kR_1 + \delta_1)]}{R^2} - \frac{\exp[2i(kR_2)]}{R^2} \right\} \]

\[ = -\frac{A}{2R} \exp[2i(kR)] \left\{ e^{\frac{ikR^2}{a}} + \cdots - 1 \right\} \]

\[ = -\frac{A}{2R} \exp[2i(kR)] \exp[\frac{4ka^2}{2R}] \left( 2i \sin \frac{ka^2}{2R} \right) \]

\[ I(p) = |4(p)|^2 = \frac{A^2}{R^2} \sin^2 \frac{ka^2}{2R} \]

Or, in terms of \( I_\text{inc} \)

\[ I(p) = I_\text{inc} \sin^2 \frac{ka^2}{2R} \]
Intensity pattern is fundamentally different from that of a circular disk.

\[ \frac{I(r)}{I_{inc}} \]

Interference (light directly from source and light diffracted near edges)