not be extinguished. A study of the fringe coincidences as described by eq. (6) leads to an estimate of the angular distance between the edges of the star. Michelson obtained the following values in seconds of arc:

- Betelgeuse: 0.047''
- Antares: 0.040''
- Arcturus: 0.022''

Since the distances of these stars from the solar system (their parallaxes) are known from other measurements, their linear diameters can be calculated. These diameters turn out to be of the order of $10^8$ km, which is about one hundred times the diameter of our sun and about equal to the diameter of the earth's orbit.

43. The Microscope

Helmholtz treated the resolution of the microscope in the same way as that of the telescope. For an object let us take two luminous points a distance $d$ apart and located in the lower focal plane $F_1$ of the objective; see fig. 89.

The objective produces an image of these points at infinity because the spherical waves emitted by the two points leave the objective as plane waves forming two bundles of parallel rays, the directions of which differ by an angle $\alpha$. If the medium on both sides of the objective is air, then this angle is the same as the angle between the two central rays from the objects to the optical center of the objective. The value of this angle is $\alpha d/f$, where $f$ is the lower focal length of the objective. If a medium of greater index of refraction $n > 1$ occupies the space between the object and the objective (immersion in oil), then for small angles of incidence the law of refraction gives $\alpha = n \alpha d/f$.

We assume that the objective is, from the point of view of geometrical optics, a perfectly corrected system of lenses. The rim of the objective and all other diaphragms are projected as geometrical images (real or virtual) by the succeeding lenses. The smallest of these images is called the "exit pupil" of the lens system and it limits the size of our ray bundles. In general the exit pupil is the virtual image of the rim of the front lens. Let the radius of the exit pupil be $a$.

$$A = \frac{a}{f}$$

is called the "numerical aperture"; from the elementary geometrical optics of lens systems (sine condition, Sec. 48) it follows that $a = f \sin \alpha$, where $2 \alpha$ is the angle of opening of the object ray cone. Hence, the definition (2) of $A$ becomes

$$A = n \sin \alpha$$

Each of our two ray bundles produces at infinity the Fraunhofer diffraction image of the exit pupil as described by eq. (36.9). Hence the microscope reproduces each of our luminous points in the form of a diffraction pattern (central field plus diffraction rings). In order to render these patterns observable at a finite distance, the eyepiece contains a converging lens on whose focal plane the diffraction patterns are reproduced. This image is observed through the eyepiece which acts as a magnifying glass.

This projection of the image upon a finite plane is of no concern to the theoretical investigation of resolution. Instead, the original diffraction pattern at infinity can be treated directly. As in the case of the telescope, eq. (42.1), we are then led to the result $n \alpha d/f = 0.61 \lambda / a$, which, according to the definition (2) of $A$, we can also write in the form

$$d = 0.61 \frac{\lambda}{A}$$

Two luminous points are resolved only if the distance between them is greater than that given by (3).

Recalling the meaning (2a) of $A$, we note that while the resolving power of the telescope depends on the size of the objective, the resolving power of the microscope depends on the angle $\alpha$ subtended by the objective at the specimen.
We can now explain the advantage of objective illumination. Let us assume, for instance, that the zeroth order spectrum falls on one edge of the exit pupil and that the first-order spectrum falls on the opposite edge; this situation will suffice to show the existence of a structure and to determine its correct spacing constant. But no other details. The angle between the two grating lines can be half that of before. In place of eqs. (6) and (7) we obtain

\[ d = \frac{\lambda}{\cos \theta} \]

(6a)

\[ \sin \theta = \frac{\lambda}{d} \]

(6b)

We find that the small improvement in the ratio 0.3333 over 0.3333 arises from the use of a larger angle between the grating lines. If, instead of the spectra of two still higher orders, we use two spectra of the same order, the direct field illumination of the second order does not enter the objective and while no object in place the field of view remains dark. For \( n \approx 1 \), \( \sin \theta \approx 1 \), the numerical estimate yields according to eq. (6a)

\[ d \approx \frac{\lambda}{2} \]

Small improvements can be resolved only by the use of shorter wavelengths: an ultraviolet microscope equipped with quartz and fluorite lenses which is effective down to \( \lambda = 0.2 \mu \) or an electron microscope. In the event of the structure of the diffraction image appears to be two-dimensional, the structure can always be seen as a superposition of cross grating. This structure is seen in another way, one is the “deflection-transform” of the other: see Vol. VI, eq. (4.13). This similarity transforms the structure of the original object. Also in the case from stopped-down reduces the similarity between the final image and the structure of the object.

43.4

ADDENDA, CHIEFLY TO THE THEORY OF DIFFRACTION

43.4

If, on the other hand, the object and the first lens of the objective are embedded in a medium of index of refraction \( n \) (immersion), then \( \lambda \) must be replaced by the smaller wavelength \( \lambda / n \). The diffraction gratings then must be

\[ \sin \theta = \frac{\lambda}{d} \]

(4a)

By (2.4) this means

\[ d = \frac{\lambda}{n \sin \theta} \]

(5)
B. Significance of phase gratings in microscopy

As an example, we shall consider the "laminary profile" of fig. 70. Like any pure phase grating, it is completely invisible if the image is perfect, for neither the retina nor the photographic plate can perceive phase differences. We would like to be able to see this grating as a set of bright and dark fringes, that is, as an amplitude grating. To accomplish this it suffices, according to the concluding remarks in Sec. 36 D, to shift the phase of the zeroth order spectrum by \( \pi/2 \) with respect to the phases of the spectra of higher orders. F. Zernicke\(^1\) produced this phase shift in the following way: a glass plate is placed in the focal plane of the objective. A thin layer of transparent material is attached to the center of this plate where for axis-parallel illumination the spectrum of zeroth order appears. While the spectra of higher orders are unaffected, the phase of the zeroth order spectrum is changed by an amount which depends on the thickness of the thin layer and its index of refraction relative to the surrounding medium. In order for the phase change to be \( \pi/2 \), the path difference must be \( \lambda/4 \) and the thickness

\[
d = \frac{2}{\lambda} (n - 1).
\]

This is a real "quarter wave plate"; the layer is less than 1 \( \mu \) thick in contrast to the "quarter wave plate" of crystal optics, see Sec. 36 B, where the thickness was determined by the small difference \( n_2 - n_1 \) between the indices of refraction in the two principal directions of oscillation rather than by the much larger difference \( n - 1 \) between the indices of refraction of the plate and its surroundings.

This phase contrast method of Zernicke is perhaps the most sensitive way of making very weak phase structures visible. Previously microscopists had been obliged to use more or less oblique illumination. The resulting loss of several spectra caused blurred images of the object. The Zernicke method, on the other hand, fully utilizes the tissue structure and makes it visible to the eye in the same way as an ideal staining method would do.

C. Luminous and illuminated objects

Because of the great successes of Abbe's theory, it was thought for a long time that Helmholtz's theory applied only to luminous sources, while in the treatment of illuminated objects only Abbe's theory was believed to hold\(^2\).

\(^2\)Such as in the review article by O. Lummer and F. Reiche: Die Lehre von der Bildentstehung im Mikroskop von Ernst Abbe, 1910.

44. On Young's Interpretation of Diffraction

However, Laue\(^3\) proved by means of a simple hypothetical experiment that the image of any small luminous object, for instance a glowing Wollaston wire, is perfectly complementary to the image of the same object when illuminated by an external source. If the wire is in a cavity of constant temperature, then according to the laws of radiation it is entirely invisible. The radiation emitted by the wire and that originating from the walls of the cavity and absorbed and re-emitted by the wire combine to produce the same radiation density as that of the background. The same holds for observations with a microscope: the images resulting from self-luminous objects and those resulting from homogeneous illumination of equal brightness of the same objects must structurally complement one another entirely. The resolution of neighboring objects must be the same in both cases.

To be sure, the illumination is not homogeneous in the case of a microscope. With the usual arrangement of a lower illuminating mirror, the illumination is merely as uniform as possible within the aperture. However, the illumination from above is missing, but this cannot make much difference as long as the object is not (or only slightly) reflecting. We may think of this illumination from above as added or we may just omit it. The rays which do not enter the aperture are equally insignificant. Therefore Laue was justified in applying the law of black-body radiation to the microscope.

The superiority of Abbe's point of view becomes apparent only with oblique illumination of the type used in dark field observations. For, Helmholtz's theory assumes that all points of the object radiate uniformly in all directions, and this is generally not true for the elements of a tissue structure.