nd (3.14) is to be complete, therefore, we must write:

$$
\begin{aligned}
&\text{(a)} \quad \frac{\psi}{2\pi} = r\,A = \frac{k\,\mu\,p}{s}\,V,\\
&\text{(b)} \quad j = p\,s^2\,d,\\
&\text{(c)} \quad \mu\,\mu_0 = \frac{1}{k},\\
&\text{(d)} \quad z = \frac{x}{s},\\
&\text{(e)} \quad \underbrace{r}_{\text{magnet}} = \underbrace{\frac{y}{s}}_{\text{network}}.
\end{aligned}
\tag{3.16}
$$

Measurement of V, therefore, gives the value of ψ directly from equation (3.16a); it will be expressed in webers. The magnitude of the induction, $|\boldsymbol{B}|$, can then be obtained as a function of the field $\boldsymbol{E}$ measured *on the network*; at a point associated with the value y,

$$|\boldsymbol{B}| = \frac{k\,\mu\,p\,s}{y}\,|\boldsymbol{E}| \quad \text{webers m}^{-2}.$$

The full calculation of the values which the resistances should be given both in the case when the scale is the same in the x and y directions and when an interesting region of the system is to be expanded are to be found in the article by Christensen.

Just as in the tank, coils are represented by currents which are fed into the network at the corresponding nodes. In complex systems, we take the sense of the currents in the coils with respect to the axis Ox into account by supplying the network from a current source with two polarities and earthing the axis of the network. An alternating current is usually employed; the detection device is commonly a valve voltmeter with a very high impedance, provided with a detector which indicates the polarity of the potential being measured with respect to earth (Wakefield, 1958).

Along the boundaries of the pole-pieces, of infinite permeability, $R = \infty$; the circuit is open, since the condition $\frac{\partial \psi}{\partial n} = 0$ must be satisfied.

Another type of network of a more complicated design exists with which the curves A_θ = constant can be obtained directly, just as they can in the conjugate tank when the system possesses a plane of symmetry (Liebmann, 1950b).

CHAPTER 4

THE OPTICAL PROPERTIES OF ELECTROSTATIC LENSES

4.1 IMAGE FORMATION

4.1.1 The Focusing Effect of the Radial Field

Having considered static electric and magnetic fields from a general point of view, we are now in a position to establish the properties which such fields must necessarily possess if they are to be capable of focusing into a point image all the electrons which emerge from a point object. In this chapter, only electrostatic lenses will be considered, and, as a simple case with which to begin, we select the three-electrode lens of Fig. 1. The

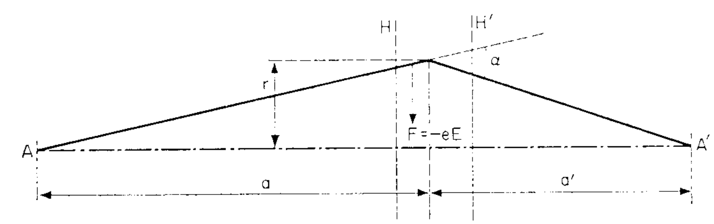

FIG. 16. Image-formation in a thin lens.

two outer grids are held at the accelerating potential of the electrons, while the central electrode is held at a quite different value. In this way, the electric field is localized between the two grids, and outside the lens the electrons move in straight lines. If all the electrons emitted by an (axial) point A are to pass through another (axial) point A', each ray must be deflected towards the axis through some angle α as it passes through the lens (Fig. 16). Since an electric field $\boldsymbol{E}$ produces a force $\boldsymbol{F} = e\,\boldsymbol{E}$ on each electron, the lens has to produce a radial field directed away from its centre. For a ray which reaches the lens at a distance r from the axis, the angular deviation α has to be proportional to r, thus

$$\alpha = r\left(\frac{1}{a} + \frac{1}{a'}\right). \tag{4.1}$$

As the deviation is proportional to the force, the intensity of the radial field must increase with the distance r from the axis according to a law of the form $E_r = K(z)\, r$, where $K(z)$ is either a constant or a function of z alone. In every lens, the field can be described by an expression of this kind, as we shall subsequently see.

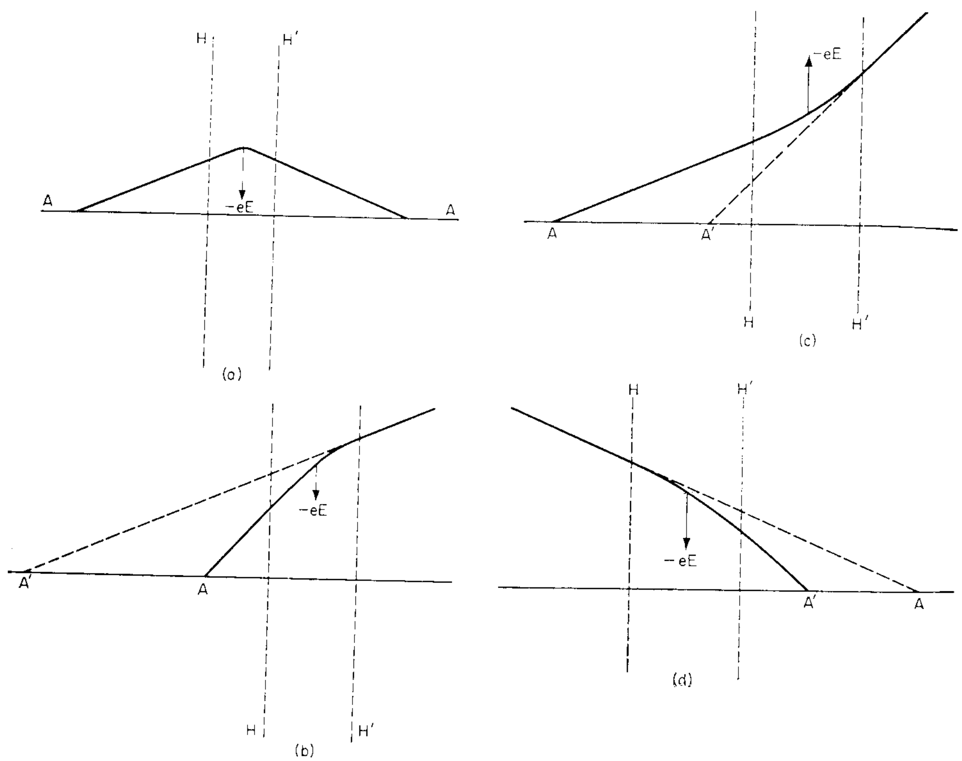

FIG. 17. (a) Convergent lens–real image. (b) Convergent lens–virtual image. (c) Divergent lens–virtual image. (d) Convergent lens–the real image of a virtual object.

The consequence is that for every ray, the ratio of α to r is a constant, $1/f$, such that the object and image distances satisfy the elementary lens relation

$$\frac{1}{a} + \frac{1}{a'} = \frac{1}{f}. \tag{4.1'}$$

If, therefore, a system of electrodes produces a *radial field, whose intensity at any point is proportional to the distance of the point from the axis,* it will focus charged particles just as an optical lens focuses light rays. The rays

will converge onto a point such as A', if the radial force is directed towards the axis, and if the lens is powerful enough; if both of these conditions are satisfied simultaneously, we shall have a converging lens producing a real image, as in Fig. 17a. In such a lens, however, it would be possible to reduce the field to such an extent that it became no longer capable of turning the rays which have originated at A through an angle sufficiently large for them to intersect the axis again at A', however distant. Nevertheless, if we produce backwards the rays which diverge rectilinearly beyond H' (a dotted line represents the produced section of an actual ray), the "virtual rays" thus created will converge into a point A'. The actual rays seem to have originated in A', which is called the "virtual" image of A (Fig. 17b). The same convention makes it possible to speak of "imaging" when the radial force repels the particles from the axis; the lens is called "divergent", in this case, and a real object always has a virtual image (see Fig. 17c). Figure 17d shows how this concept of virtuality can be extended to the object; a lens would have a virtual object if, for example, it were placed between another lens and the (real) image which this lens has produced.

For each field intensity, therefore, and for each particle velocity, an image is produced; these are the two factors which describe the action of the lens, or, optically speaking, its convergence. The slower the electrons and the stronger the radial field, the more convergent will be the lens—this must not be carried too far, however, as there is a point at which the lens turns into a mirror, as we shall see later.

4.1.2 The Impossibility of Separating the Longitudinal and Radial Field Components

The electrodes of such lenses as the three-electrode lens produce a satisfactory field configuration—at least in the axial region—but with a structure less simple than we have supposed in the previous section. In fact, *each radial component is associated with a longitudinal component of electric field, parallel to the axis.*

The map of lines of equal field strength shows that the radial component is appreciable only where the lines of force have a large curvature, and where, consequently, the longitudinal component of the field is changing rapidly over short distances. The laws which govern the electric field provide a relation between the radial and longitudinal field components in the neighbourhood of the axis; the flux across the surface of a small cylinder, whose axis coincides with that of the lens, is zero, so that for a field of axial symmetry,

$$E_r = -\frac{1}{2} r \frac{\partial E_z}{\partial z} = \frac{1}{2} r \frac{d^2\varphi}{dz^2}, \tag{4.2}$$

provided that second and higher order terms are negligible within the cylinder. We see that the radial component is a function of the rate at which the longitudinal component of the electric field is varying with z, or of the axial potential $\varphi(z)$. E_r depends upon r in the requisite fashion, and the factor $K(z)$ which was introduced earlier is proportional to the gradient of the longitudinal field component. This suggests that the extent of the lens should be defined as the region over which the derivative of E_z is still appreciable.

The effect of the longitudinal component is most important, as it retards or accelerates the electrons and thus has an indirect influence over the deviation of the beam which the radial component can produce; in short, it affects the convergence.

4.1.3 Ordinary Lenses without Grids

The longitudinal component has a far more pronounced effect in lenses without grids, which are much more common in practice, as lenses with grids are unsuitable for a number of reasons. Firstly, it is difficult

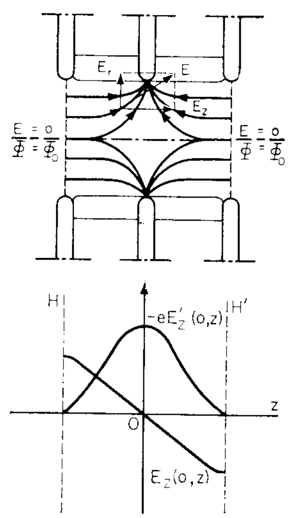

FIG. 18. The behaviour of the lines of force, the axial field, and the gradient of the axial field in a lens with two grids.

to construct a sufficiently transparent grid, combining extreme slimness of the meshes with mechanical rigidity; further, secondary electrons are produced at the wires, and their superfluous illumination at the final screen reduces the contrast of the image proper. In addition, the grid produces a slight haziness at the image.

So far, we have been assuming that the planes H and H' are effectively continuous conductors, at which lines of force will terminate normally (Fig. 18). In reality, when H and H' are grids, each line of force has a small

kink (Fig. 19) just before it is terminated; an estimate of the influence of this deformation can be made by comparing the variation of potential along a line of force with the potential difference along the bend near the grid. Small though this effect is, its effect is almost invariably harmful, however much we try to diminish it by reducing the mesh-length of the grid.

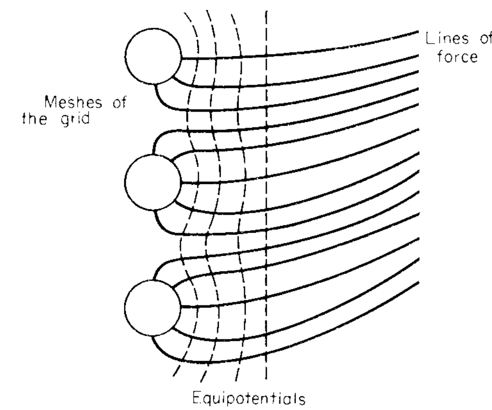

FIG. 19. The distortion of the field which the wires of the grids produce.

Subsequently, therefore, we shall consider only normal (grid-free) lenses, a typical example of which is the three-electrode lens, in which the grids are replaced by circular holes cut in plane electrodes ("Einzellinse" in German).

The longitudinal component now plays a considerably more important role. To make this more readily comprehensible, consider the symmetrical two-electrode lens of Fig. 20a.

In passing from the first half of the lens to the second, the sense of the radial field is reversed: if the field had a radial component only, it would have no overall effect. There is however, a longitudinal component which accelerates the electrons as they pass through the lens, which are as a result moving faster in the second half of the lens than in the first half. In the first half, therefore, the converging power is increased, and in the second, the diverging power diminished—the whole lens is convergent. In the three electrode lens, the action can be analysed similarly (see Fig. 20b) into an assembly consisting of a convergent unit between two divergent units.

Many of the features of ordinary electron lenses can be explained in terms of *the influence of this longitudinal component*. Lenses are, for example, *always convergent*. In order to explain this, we consider the shape of the lines of force of the electric field, which link the positive charges of one electrode with the negative charges of an adjacent electrode. The hole in the middle of each electrode make the lines of force curved, convex towards the axis, a consequence of their mutual repulsion. A region of convergence must always, therefore, be accompanied by a region in which the lens action is divergent; but electrons move more slowly in a convergent region, which slows them down, than in a divergent region, which accelerates them, so that the convergent effect always exceeds the divergent. A three-electrode lens remains convergent even though the central electrode be at a positive

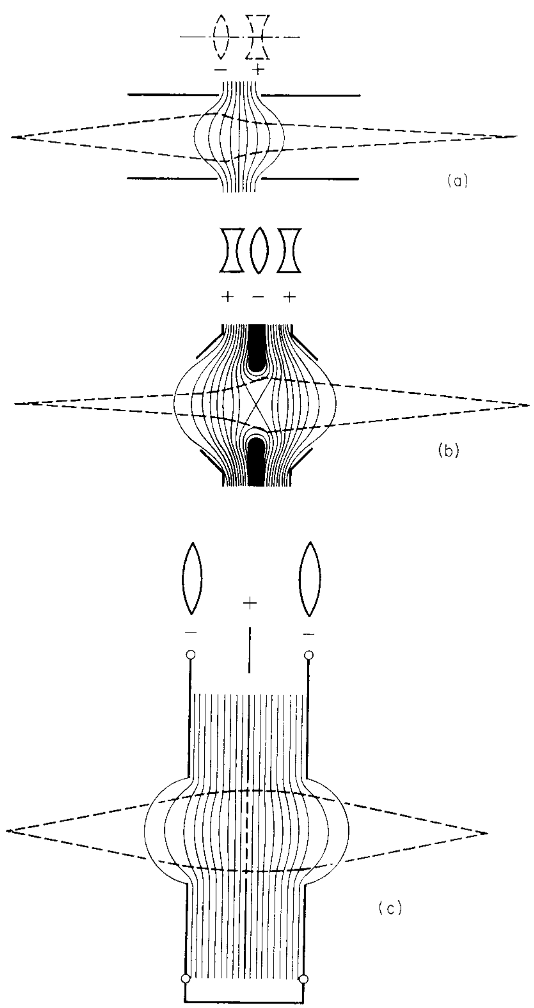

FIG. 20. Some electron lenses, with their glass optical counterparts. (a) A two-cylinder lens, which corresponds to a doublet. (b) A three-electrode lens, equivalent to a symmetrical triplet. (c) The practical example of a lens with a central mesh electrode, of which the analogue is a pair of convergent lenses.

potential with respect to the outer diaphragms. In the special case of a plane cathode near an accelerating diaphragm, however, the rule is no longer applicable, as the convergent region has disappeared—it is the only grid-free lens which is divergent, No application of these lenses, studied by Davisson and Calbick (1931, 1932) and MacNaughton (1952) has as yet been found.

Another feature of electron lenses which is explicable in terms of the effect of the longitudinal component is the fact that it is always found necessary to apply a potential difference across the two electrodes which is of the same order of magnitude as the beam accelerating potential, despite the differential retardation. This is quite different from the grid lens where the action of the field is everywhere uniform, and where only one tenth of the potential is necessary to produce the same convergence. Figure 20c shows a simple design for a lens with a single grid possessing the same optical properties as those of the version with two grids considered earlier. More details of these lenses, which are better than ordinary lenses in this respect, will be found in § 8.4.

4.1.4 The Electron Mirror; Minimum Focal Distance

We consider first a lens of the usual kind, with electrodes symmetrical about an axis $z'Oz$. The field in such a lens is then completely described by a set of equipotentials in a meridian plane, $\Phi(r, z) =$ constant. Our interest in Φ is caused by the fact that the velocity of the electrons (each of mass m and carrying charge $-e$) is given (in the non-relativistic case) by

$$v = \sqrt{\frac{2e\Phi}{m}} \tag{4.3}$$

provided the value of the potential at the cathode is chosen as origin ($V_c = 0$) since the electrons are emitted with a negligible velocity. Then

$$e/m = 1{\cdot}759 \times 10^{11}\ \mathrm{C\ kg^{-1}},$$

$$\sqrt{\frac{2e}{m}} = 0{\cdot}5932 \times 10^{6}\ \text{MKS units}.$$

The basic relation (4.3) suggests immediately how an electron lens can be converted into an electron mirror—all that we have to do is to apply a potential to the central electrode of a three electrode lens which is not only negative with respect to the neighbouring electrodes, but also with respect to the cathode. The field which ensues will produce a lens action just so long as the minimum potential which the electrons encounter, and which is situated in the middle of the central electrode ($r = z = 0$), is positive (if $\Phi(0, 0) > 0$, that is). A complete examination is to be found in § 8.2.

When the negative potential applied to the central electrode is numerically sufficiently large, however, the electrons are decelerated and begin to return towards their source without having reached the centre of the lens. This situation is called an "electron mirror"; its optical properties are analogous to those of glass optical mirrors.

When $\Phi(0, 0)$ is negative and very large, the electrons are turned about before reaching the central region of the lens where the field is convergent, and the mirror is divergent; if, on the other hand, $\Phi(0, 0)$ is negative but only slightly so, the electrons penetrate deep into the lens, and before they are reflected, they will have experienced the converging action of the field—the mirror is convergent.

If $\Phi(0, 0)$ is negative but very close to zero (within a few tenths of a volt) oscillations appear in the electron trajectories. This phenomenon is intermediate between the lens and mirror regions. The potential zone which produces oscillating rays (which are known as "transgaussian" rays) is virtually useless for image formation, though it has a few rather specialized applications.

$\Phi(0, 0)$, therefore, cannot be reduced below a certain limit, and a very high convergence is not, as a result, attainable. In practice, the least focal length that one can obtain is about as large as the radius of the central electrode when the inter-electrode spacing is also of the order of this radius, which is frequently the case.

A similar phenomenon appears when $\Phi(0, 0)$ approaches zero from the positive side—the convergence of an electron lens too, therefore, is bounded.

Should the electron beam cross-sectional area not be small relative to the area of the holes, it is possible that peripheral rays will be reflected, while axial rays experience only lens action—the trajectories, in such a case, become most involved.

4.1.5 The Lens Considered as an Assembly of Prisms. Similitude Properties

Just as a glass lens can be decomposed into an assembly of prisms, so can an electron lens, and although the method is more artificial than those which we have so far used, the analogy provides suggestive results. The basic electron prism is the parallel plate condenser, the standard type of deflector in cathode ray oscillographs. This deviating property suggests a simple description of the action of the radial component E_r of the electric field. Each volume element of the field is to be thought of as a small condenser, with plates distance d apart, of length l parallel to the axis, supporting a potential difference $U = E_r d$, and with a mean potential of Φ (see Fig. 21). This elementary condenser deflects the ray through a small angle α given by

$$\tan \alpha = \frac{1}{2} \frac{Ul}{\Phi d} \simeq \alpha. \tag{4.4}$$

We can draw a number of important conclusions from this law, which hold for both thick and thin lenses. Firstly, since neither the charge nor the mass of the particles appears in the deviation law, the ray path is independent of e and m, and the lens would focus no differently if an ion source replaced the electron source. This property provides the possibility of focusing ions, which is a most important operation in particle accelerators, and was brilliantly introduced into mass spectrography in France by Cartan (1937). Ion sources are also beginning to be introduced into microscopy (Gauzit, 1951, 1953, 1954; Magnan and Chanson, 1951).

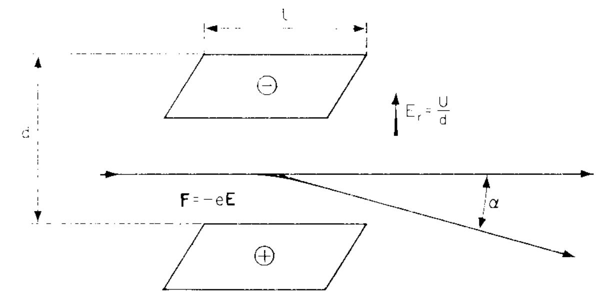

FIG. 21. The electrostatic prism.

It is worth mentioning in passing that electrostatic ion focusing can be hazardous in tubes which contain thermo- or photo-emissive cathodes, or luminous films, since an excessive bombardment by returning ions can deteriorate or destroy the delicate films (Broadway and Pierce, 1939; Schaeffer and Walcher, 1943). The positive ions which are produced all along the beam by the impacts of the electrons on the residual gas molecules retrace the electron paths and strike the cathode; negative ions, either produced by the cathode or originating in the residual gas, flow towards the fluorescent screen. In cathode ray tubes, these latter are caught in an ion trap, which consists of a pair of crossed deflecting fields, one magnetic, the other electrostatic, which act for a short distance on the beam. Their intensity is so adjusted that on electrons of mass m, their effects just cancel out; on ions with a mass M very different from m ($M \simeq 1840\, AM$, where A is the atomic number of the ion), the two fields have a large residual deflecting effect, and the ions are carried away either to an electrode or towards the wall of the tube.

Another consequence of equation (4.4) is that the trajectory is independent of the absolute dimensions of the lens: only the ratio l/d matters. Similarly, only the ratio U/Φ, the ratio of electrode potential to accelerating potential, intervenes in (4.4) and not the absolute value of either potential. The action of an electrostatic lens will not be affected by small variations of U and Φ, which can even be oscillatory provided their ratio remains unaltered. It seems as though it should be possible to connect the electrodes to a potentiometer fed by an unstable or even oscillatory supply. In certain of the cruder applications of electron optics, high tension X-ray tubes, for example, advantage is taken of this property by supplying the tube with the alternating voltage from a transformer. In more delicate applications, on the other hand, such as the electron microscope, only an extremely carefully stabilized high tension can be used, not varying by more than one per cent; so fine are the images which such an instrument produces that a defect due to relativity would otherwise appear. This is explained by the fact that for very fast electrons, equation (4.4) requires a relativistic correction, and becomes:

$$\alpha \simeq \frac{Ul}{2\Phi d}\frac{1+(e/m_0c^2)\Phi}{1+(e/2m_0c^2)\Phi} \qquad (4.5)$$

(where c is the velocity of light and m_0 is the rest mass of the electron). Equation (4.4) ceases to be linear as soon as $\frac{e\Phi}{2m_0c^2}$, which is equal to $0{\cdot}977\times 10^{-3}\,\Phi$ (Φ measured in kV), is no longer negligible (Ramberg, 1942). For the optical properties to remain constant it is no longer sufficient that such ratios as Φ_1/Φ_0, Φ_2/Φ_0 should be invariant, but the actual potentials Φ_1, Φ_2 too must be narrowly stabilized; otherwise, the image will be slightly hazy as a result of the slight variations of these potentials about their mean values. In fact, however, this relativistic effect is most difficult to observe, masked as it is by more ordinary defects caused by imperfections in the source of high tension; for example, the capacities at the high tension entry and between the electrodes may be appreciable, and, by destroying the validity of the proportionality law in the presence of a varying potential, may well dwarf the relativistic effect.

4.2 THE GAUSSIAN APPROXIMATION

4.2.1 Gaussian Conditions

We have seen that the necessary conditions for a good image to be formed by an electron optical system obtain only within a tubular surface of revolution enclosing the axis of the system rather closely. This is the situation in which we can neglect terms in r and $\mathrm{d}r/\mathrm{d}z$ of orders higher than the first in the differential equations of the electron trajectories. This first order approximation is known as "Gaussian optics", and the approximate trajectories which are obtained are called "Gaussian trajectories".

Equation (4.2) gives the lower order terms in the series expansion for E_r and E_z, which we shall write for convenience

$$E_z = -\varphi'(z), \qquad E_r = \frac{1}{2}r\,\varphi''(z); \qquad (4.6)$$

$\varphi(z)$, $\varphi'(z)$ and $\varphi''(z)$ represent the value, $\Phi(0, z)$, of the potential on the axis, and its derivatives with respect to z.

4.2.2 The Equations of Motion

Let us consider now the detailed calculation of trajectories to the Gaussian approximation for non-relativistic electrons. Applying Newton's law in the radial direction we obtain

$$f_r = m\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{1}{2}e\,r\,\varphi''(z) \qquad (4.7)$$

(remembering that the charge on the electron is $-e$).

The radial force has a focusing action when $\varphi''(z) > 0$ and a defocusing action when $\varphi''(z) < 0$. If the curve which represents $\varphi(z)$ is plotted with the cathode potential as the origin of coordinates ($\varphi = 0$), the lens will be locally convergent at a point with abscissa z if the curve is concave upwards at this point, and divergent if the curve is concave downwards (Fig. 22).

For the motion in the z direction, we obtain a first order differential equation directly from the conservation of energy:

$$\frac{1}{2}m\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 = e\,\varphi. \qquad (4.8)$$

With the aid of equations (4.7) and (4.8), we can study such electron trajectories as remain within a meridian plane; this is the case for all rays originating from an axial object point, and for a particular class of rays originating in an off-axial object point. For this case, we find, eliminating the time variable between the equations,

$$\sqrt{\varphi}\cdot\frac{\mathrm{d}}{\mathrm{d}z}\left(\sqrt{\varphi}\cdot\frac{\mathrm{d}r}{\mathrm{d}z}\right) = -\frac{1}{4}r\,\varphi'', \qquad (4.9)$$

or

$$\frac{\mathrm{d}^2 r}{\mathrm{d}z^2} + \frac{\varphi'}{2\varphi}\frac{\mathrm{d}r}{\mathrm{d}z} + \frac{\varphi''}{4\varphi}r = 0. \qquad (4.10)$$

Skew trajectories are, in fact, no problem as it is simple to transform their differential equations into an equation of the form (4.9) or (4.10). If Ox and Oy are mutually perpendicular axes in a plane perpendicular to Oz, we can describe quantities which depend linearly upon x and y in a condensed fashion by introducing the complex variable $u = x + jy$:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} + j\frac{\mathrm{d}y}{\mathrm{d}t},$$
$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + j\frac{\mathrm{d}^2y}{\mathrm{d}t^2}, \qquad (4.11)$$
$$E_u = E_x + jE_y = -\frac{1}{2}u\frac{\partial E_z}{\partial z}.$$

The equation which relates u and z in space relates r and z in a meridian plane. Writing u for r and interpreting the results with the aid of equations (4.11), oblique trajectories are found to have the same optical properties as the meridian trajectories to which we had previously restricted ourselves.

The similarity properties are immediately obvious if we notice that the equations are homogeneous and of zero degree with respect to φ, and that neither e nor m appears (we shall consider the effect of relativity upon the homogeneity later).

We now return to the Gaussian equation in the form (4.9), and integrate between object space (characterized by the index a) and image space (index b). We find

$$\left(\sqrt{\varphi}\frac{\mathrm{d}r}{\mathrm{d}z}\right)_b - \left(\sqrt{\varphi}\frac{\mathrm{d}r}{\mathrm{d}z}\right)_a = -\frac{1}{4}\int_a^b r\frac{\varphi''}{\sqrt{\varphi}}\,\mathrm{d}z.$$

For an incident trajectory parallel to the axis, $r'_a = 0$, and hence

$$r'_b = -\frac{1}{4\sqrt{\varphi_b}}\int_a^b r\frac{\varphi''}{\sqrt{\varphi}}\,\mathrm{d}z.$$

The image focal length, f_b, is given by

$$\frac{1}{f_b} = -\frac{r'_b}{r_a},$$

and hence

$$\frac{r_a}{f_b} = \frac{1}{4\sqrt{\varphi_b}}\int_a^b r\frac{\varphi''}{\sqrt{\varphi}}\,\mathrm{d}z. \tag{4.12a}$$

In a weak lens, r varies only slightly through the lens, and we can write approximately

$$r = r_a.$$

For a thin lens, therefore, we obtain the following expression for the image focal length, f_b:

$$\frac{1}{f_b} = \frac{1}{4\sqrt{\varphi_b}}\int_a^b \frac{\varphi''}{\sqrt{\varphi}}\,\mathrm{d}z. \tag{4.12b}$$

If φ' is zero on both sides of the lens, the second derivative can be eliminated on integrating by parts, and we obtain

$$\frac{1}{f_b} = \frac{1}{8\sqrt{\varphi_b}}\int_a^b \frac{\varphi'^2}{\varphi^{3/2}}\,\mathrm{d}z. \tag{4.12c}$$

4.2.3 The Reduced Equation

The Gaussian equation is difficult to use due to the presence of φ'', a for most lenses the only information available about the potential distr bution has been obtained from an electrolytic tank or a resistance networ and a second derivative determined graphically is far too inaccurate. In stead, therefore, it is often considered preferable to transform the Gaussia equation (Picht, 1939) by writing

$$R = r\,\varphi^{1/4}. \tag{4.13}$$

The curve $R(z)$ is called a "reduced" ray. Equation (4.10) then become

$$\frac{\mathrm{d}^2R}{\mathrm{d}z^2} + \frac{3}{16}\left(\frac{\varphi'}{\varphi}\right)^2 R = 0, \tag{4.14}$$

which, apart from the absence of an R' term, displays the interesting featur of possessing a single expression characteristic of the lens. All the Gaussian optical properties of the lens are determined by the characteristic functior

$$T(z) = \frac{\varphi'}{\varphi}.$$

For a weak lens, we can treat R in the same way as earlier we treated r. A ray incident parallel to the axis corresponds to R_a = constant and $R'_a = 0$. The "reduced focal length", F, is defined in terms of the quotient R'_b/R_a thus:

$$\frac{1}{F} = -\frac{R'_b}{R_a}.$$

If we assume that $R \simeq$ constant $\simeq R_a$ throughout the lens, we find

$$\frac{1}{F} = \frac{3}{16}\int_a^b \left(\frac{\varphi'}{\varphi}\right)^2 \mathrm{d}z. \tag{4.15a}$$

If the potentials are constant on both sides of the lens, we have

$$R'_b = r'_b\,\varphi_b^{1/4},$$

$$f_b = \frac{r'_b}{r_a} = \left(\frac{\varphi_b}{\varphi_a}\right)^{1/4} F_b,$$

and hence

$$\frac{1}{f_b} = \frac{3}{16}\left(\frac{\varphi_a}{\varphi_b}\right)^{1/4}\int_a^b \left(\frac{\varphi'}{\varphi}\right)^2 \mathrm{d}z, \tag{4.15b}$$

which reduces, if $\varphi_a = \varphi_b$, to the following expression:

$$\frac{1}{f_b} = \frac{3}{16}\int_a^b \left(\frac{\varphi'}{\varphi}\right)^2 \mathrm{d}z. \tag{4.15c}$$

Formula (4.15c) differs from formula (4.12c) even though *a priori* equivalent approximations have been made; in practice, (4.12c) is found very often