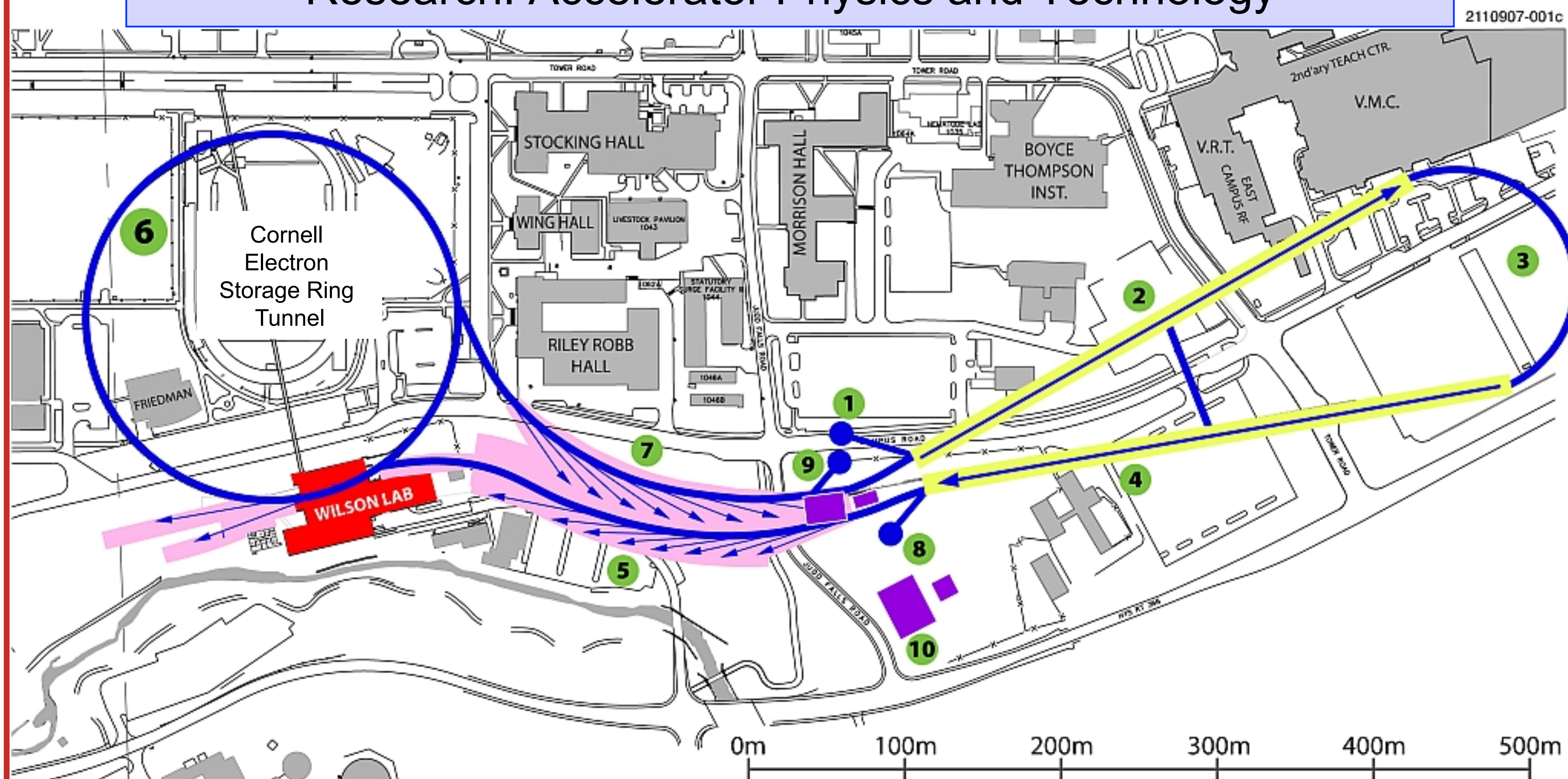
Errors, Error Propagation, Statistics

Georg H. Hoffstaetter

Cornell Physics Dep. / CLASSE

Research: Accelerator Physics and Technology



Why do we quote errors ?

Thanks to Anders Ryd for much of the presented material.

It provides information about the precision of the measurement.

For example the gravitational constant is measured to be
G_N=6.90 $10^{-11}m^3kg^{-1}s^{-2}$
The 'accepted' value is
G_N=6.6742(10) 10^{-11} $m^3kg^{-1}s^{-2}$.

Without quoting any errors we don't know if this was just a less precise measurement or a Nobel prize worthy discovery.

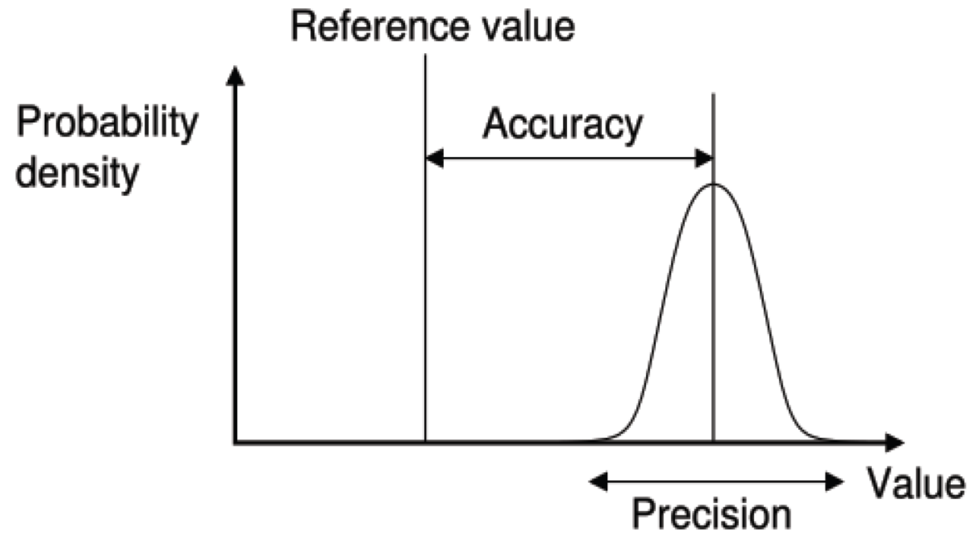
Example:
G_N=(6.90+/-0.25) 10^{-11} $m^3kg^{-1}s^{-2}$ would be in good agreement
G_N=(6.90+/-0.01) 10^{-11} $m^3kg^{-1}s^{-2}$ would be an interesting result …

Semantics

Accuracy vs. Precision

Accuracy is the degree to which a measurement agrees with the true value.

Precision is the repeatability of the measurement.



Accurate but not precise

Precise but not accurate

Error vs. Uncertainty

Error is the degree to which a Measurement agrees with the true value.

Uncertainty is an interval around the measurement in which repeated measurements will fall.

Sloppy and inconsistent language

Error is almost never what we are interested in. In science we typically do not know the 'true' value.

Rather we are interested in the **uncertainty**. This is what we need to quantify in any measurement.

We are often very sloppy and inconsistent in our language and call what is actually an uncertainty an error, **e.g. in the title of this lecture**.

Especially in High Energy Physics we try to get this straight when we write a paper, but in every day talk we are also sloppy and use the word error instead of uncertainty.

When we talk about measurement error,

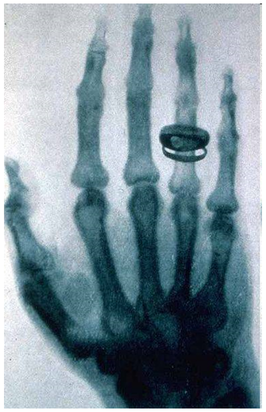
- It is not a blunder
- It is not an accident
- It is not due to incorrectly handling the equipment
- It is not the difference to an accepted value found in the literature

Importance of uncertainty

Example 1:
High fiber diets: A study in 1970 claimed that a high fiber diet reduces polyps forming in the colon, being precursors of cancer. A study in 2000 with more analyzed individuals showed no such effect. The **uncertainty** in the first study was too large an not properly accounted for. This left people eating lots of fibers for 30 years – yuck (just kidding).

Example 2:
A study in the late 60s found large levels of iron, which is required for red blood cell production, in spinach. Popular comics tried to promote spinach consumption. A study in the 90s showed that the original measurement had a reading error in the decimal point. The iron levels are a factor of 10 lower than claimed. The incorrect reading of the decimal was a **blunder**, not due to an uncertainty in the measurement. This left children eating lots of spinach for 30 years – yuck (not kidding)

Error analysis helps to limit bias



Fact of scientific life:
Scientists subconsciously bias data to their desired outcome,
even when they know about this tendency of their psyche.

Example: N-rays
X-rays discovered in 1895 by Roentgen with huge and fast success.
Another new type of radiation was reported in 1903:
Rene Blondlot (physicist, Nancy / F) discovered N-rays (with N for Nancy)
These became a matter of national pride to the French. Later several scientists, mostly French, claimed to have seen these rays.
100eds of papers published within about one year, 26 from Blondlot.
They **go through wood** and **metal** but are **blocked by water**.
They could be **stored in a brick**.
They are **emitted by rabbits, frogs and the human brain** (medical imaging)
Jean Becquerel (son of Henri who discovered radioactivity) found N-rays **transmitted over a wire** (brain scan per telephone …)
Robert Wood (John Hopkins) went to Blondlot's lab and secretly removed the sample.
Blondlot insisted he was still measuring N-rays.
Within months no one believed in N-rays any more.

Also: http://scienceblogs.com/drugmonkey/2010/08/harvard_found_marc_hauser_guil.php

Error a property of a measurement procedure

The **Uncertainty** (Error), in the lab, describes the distance from your measurement result within which your setup has determined that the true value is likely to lie. It describes therefore a property of your measurement procedure, when followed correctly.

Example: When you measure N_A you may obtain the literature value to 10^{-3}. This does not show that your setup has established that the true value of N_A is likely to be within 10^{-3} of your result. The 10^{-3} is therefore not the uncertainty of your experiment.

A new measurement following the same procedure will lead to a different measurement result, but usually the same uncertainty. The new and old result are likely to differ by an amount that is about as large as the uncertainty. The uncertainty is therefore a property of the measurement procedure, and building a good experiment means building an experiment with relatively small uncertainty.

Meaning of an Error

If we measure a voltage V_{sat} = 10.2 +/- 0.3 V, what does this mean?

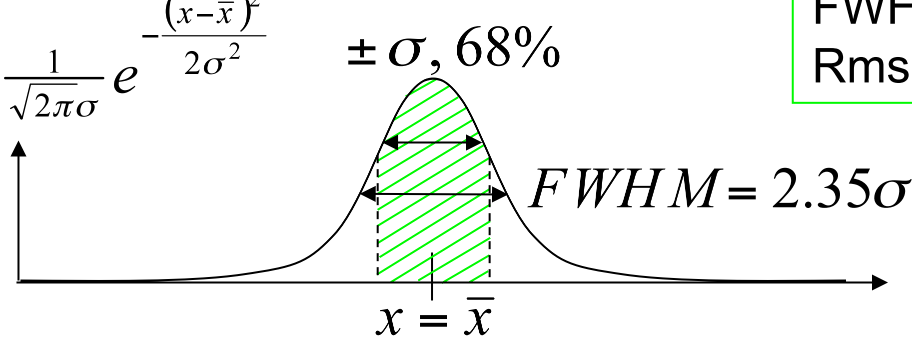
In general there are differences in different science disciplines.

In physics, a 1sigma error is generally used. If the measurements are normally distributed (Gaussian), this corresponds to a 68% confidence level (CL) interval. Or that 32% of the time the true value would be outside the quoted error range.

For statistical errors, this can be given a precise meaning. Many other errors are harder to estimate.

Gaussian distribution:

$$\rho(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$



FWHM	= full width at half maximum
Rms	= root of the mean square

$$rms = \sqrt{\left\langle (x-\bar{x})^2 \right\rangle} = \sigma$$

Different types of errors

CLASSE

- Statistical: From finite statistics, originates in the Poisson distribution.
- Systematic, e.g. how well can you measure a voltage, length, etc.
- Theory: For example, if the muon lifetime is measured by capturing muons in matter, there are corrections to the capture rate for mu- that comes from theory.

- Commonly quote these uncertainties separately:

τ_μ=(2.19+/-0.05$_{stat.}$+/-0.01$_{syst.}$+/-0.02$_{th.}$)μs

- Different notations are used for uncertainties, e.g.

τ_μ=(2.19(5)$_{stat.}$+/-(1)$_{syst.}$+/-(2)$_{th.}$)μs

- Errors are usually quoted as absolute errors, not relative errors.

Accurate arrival:
From 08:45 (Granville) to 15:55 (Montparnasse)
Drove only 3s/25800s = 0.01% too long.

Counting Statistics

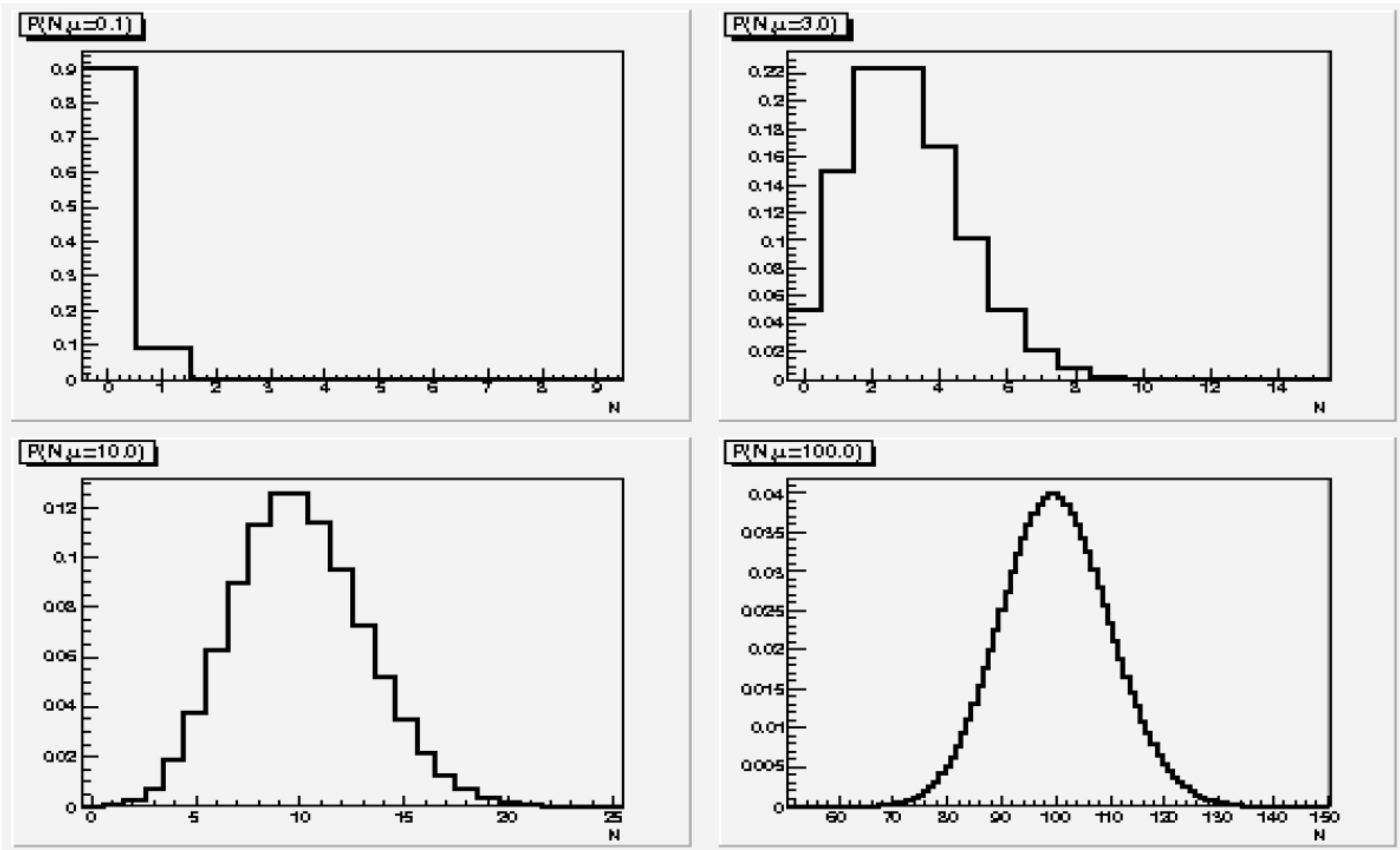
Imagine a situation where a number of events occur in a fixed period of time, where these events occur with a known average rate and independently of the time since the last event.

- E.g. a counting experiment is repeated 10 times, which of the 3 outcomes below would you expect ?

		mean	rms
a)	99, 100, 98, 101, 101, 99, 100, 101, 100, 99	99.8	1.0
b)	87, 105, 93, 108, 110, 90, 115, 82, 105, 97	99.2	10.4
c)	47, 115, 67, 97, 133, 103, 157, 78, 127, 94	101.8	31.4

The Poisson Distribution

For large <N> (μ>10), the Poisson distribution approaches a normal distribution.



Poisson Distribution

$$\boxed{P(N,\mu)=\frac{\mu^N}{N!}e^{-\mu}}$$

$$\sum_{N=0}^{\infty}P(N,\mu)=e^{-\mu}\sum_{N=0}^{\infty}\frac{\mu^N}{N!}$$

$$\langle N\rangle=\sum_{N=0}^{\infty}N\,P(N,\mu)=\sum_{N=1}^{\infty}N\frac{\mu^N}{N!}e^{-\mu}=\mu\sum_{N=1}^{\infty}\frac{\mu^{N-1}}{(N-1)!}e^{-\mu}=\mu$$

$$rms^2=\left\langle\left(N-\langle N\rangle\right)^2\right\rangle=\left\langle N^2-2N\langle N\rangle+\langle N\rangle^2\right\rangle=\left\langle N^2\right\rangle-\langle N\rangle^2$$

$$\left\langle N^2\right\rangle=\sum_{N=0}^{\infty}N^2\,P(N,\mu)=\sum_{N=1}^{\infty}N^2\frac{\mu^N}{N!}e^{-\mu}=\mu\sum_{N=1}^{\infty}N\frac{\mu^{N-1}}{(N-1)!}e^{-\mu}$$

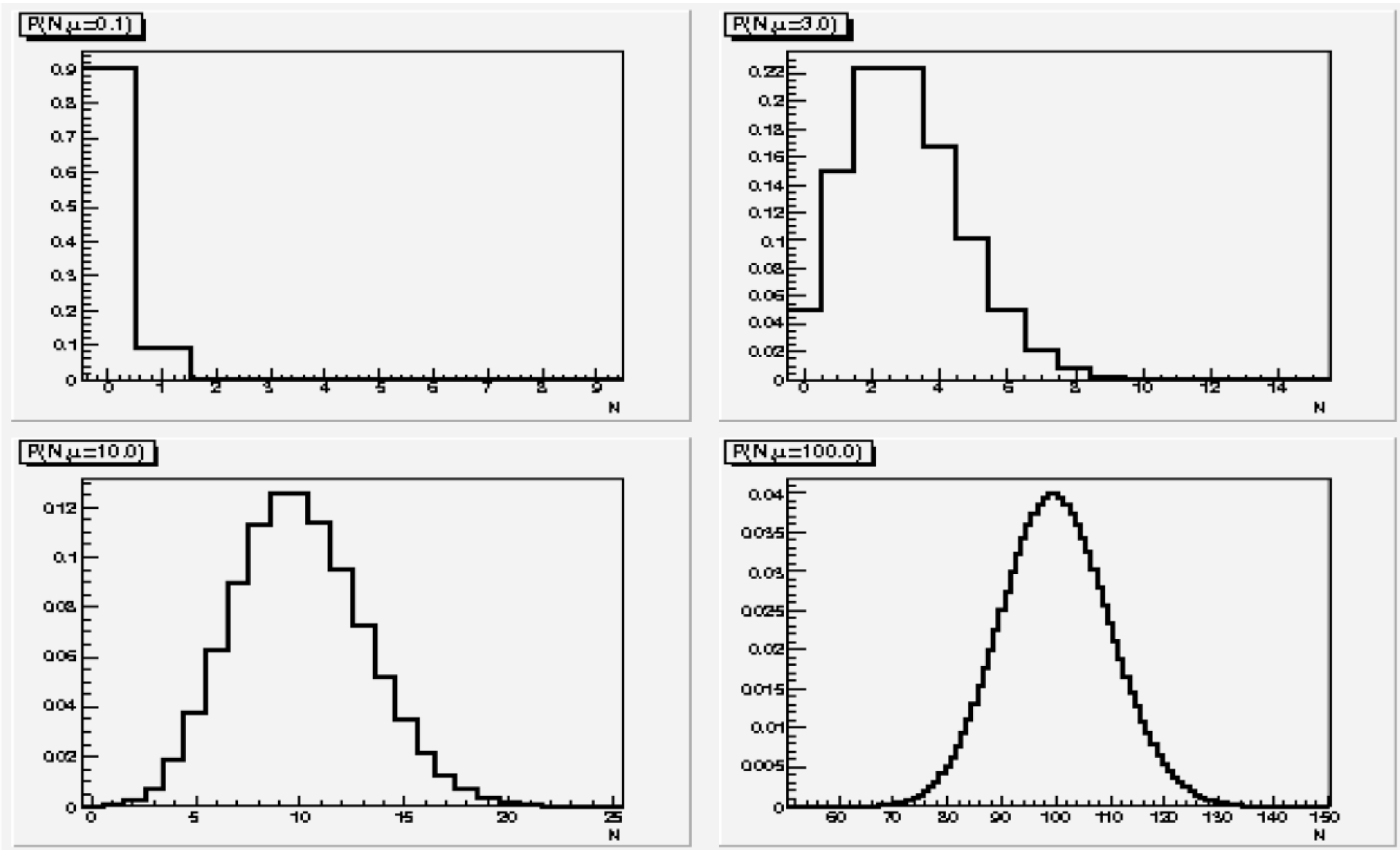
$$=\mu\sum_{N=1}^{\infty}\left[(N-1)\frac{\mu^{N-1}}{(N-1)!}e^{-\mu}+\frac{\mu^{N-1}}{(N-1)!}e^{-\mu}\right]=\mu(\mu+1)$$

$$rms^2=\mu\quad\Rightarrow\quad rms=\sqrt{\mu}\quad\Rightarrow\quad\frac{rms}{\langle N\rangle}=\frac{1}{\sqrt{\langle N\rangle}}$$

Limit of the Poisson Distribution

CLASSE

For large <N> (μ>10), the Poisson distribution approaches a normal distribution.



Use of the Poisson Distribution

- The Poisson distribution tells you how probable it is to obtain a given count if the mean is known.

  Typically we don't know the true mean, but our measured count serves as an estimate of the mean.

  We can now use this information to estimate the uncertainty.

- E.g. in a counting experiment we obtain 98 counts. We then assign the uncertainty of $98^{1/2}$ = 9.9 to say that the measurement leads to 98+/-10 counts.

Error Propagation

1) We have measured two yields N_1 and N_2. With uncertainties σ_{N1} and σ_{N2}.

2) Now we want $N=N_1-N_2$. What is the uncertainty of N ?

3) Let's consider a slightly more general case

$$c = f(a,b)$$

$$\bar{a} = \langle a \rangle, \quad \delta a = a - \bar{a}, \quad \sigma_a^2 = \langle \delta a^2 \rangle$$

$$\bar{b} = \langle b \rangle, \quad \delta b = b - \bar{b}, \quad \sigma_b^2 = \langle \delta b^2 \rangle$$

$$c = f(a,b) \approx f(\bar{a},\bar{b}) + \partial_a f(\bar{a},\bar{b}) \cdot \delta a + \partial_b f(\bar{a},\bar{b}) \cdot \delta b$$

$$\bar{c} = \langle f(\bar{a},\bar{b}) + \partial_a f(\bar{a},\bar{b}) \cdot \delta a + \partial_b f(\bar{a},\bar{b}) \cdot \delta b \rangle = f(\bar{a},\bar{b})$$

$$\sigma_c^2 = \langle \delta c^2 \rangle \approx \left\langle \left(\partial_a f(\bar{a},\bar{b}) \cdot \delta a + \partial_b f(\bar{a},\bar{b}) \cdot \delta b \right)^2 \right\rangle$$

$$= \left[\partial_a f(\bar{a},\bar{b}) \sigma_a \right]^2 + \left[\partial_b f(\bar{a},\bar{b}) \sigma_b \right]^2 + 2 \partial_a f(\bar{a},\bar{b}) \partial_b f(\bar{a},\bar{b}) \langle \delta a \delta b \rangle$$

Uncorrelated Errors

Assumption of uncorrelated errors:
Errors in variable a vary independently of those in variable b.

$$\langle \delta a \delta b \rangle = \langle \delta b \rangle \langle \delta a \rangle = 0$$

$$\sigma_c^2 = \left[\partial_a f(\overline{a},\overline{b})\sigma_a\right]^2 + \left[\partial_b f(\overline{a},\overline{b})\sigma_b\right]^2$$

Example: c = a - b

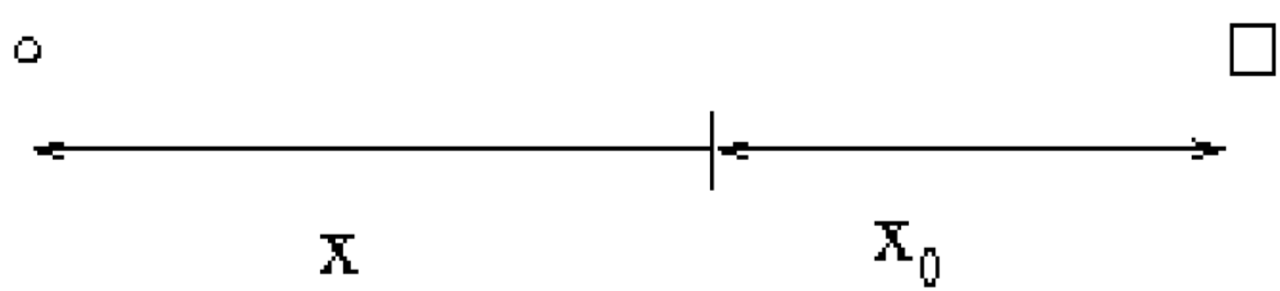
$$\sigma_c^2 = \sigma_a^2 + \sigma_b^2$$

$$\sigma_c = \sqrt{\sigma_a^2 + \sigma_b^2}$$

χ^2 Fitting

You have a source and detector, and you need to determine the distance $x+x_0$. You can change x, but do not have access to x_0.

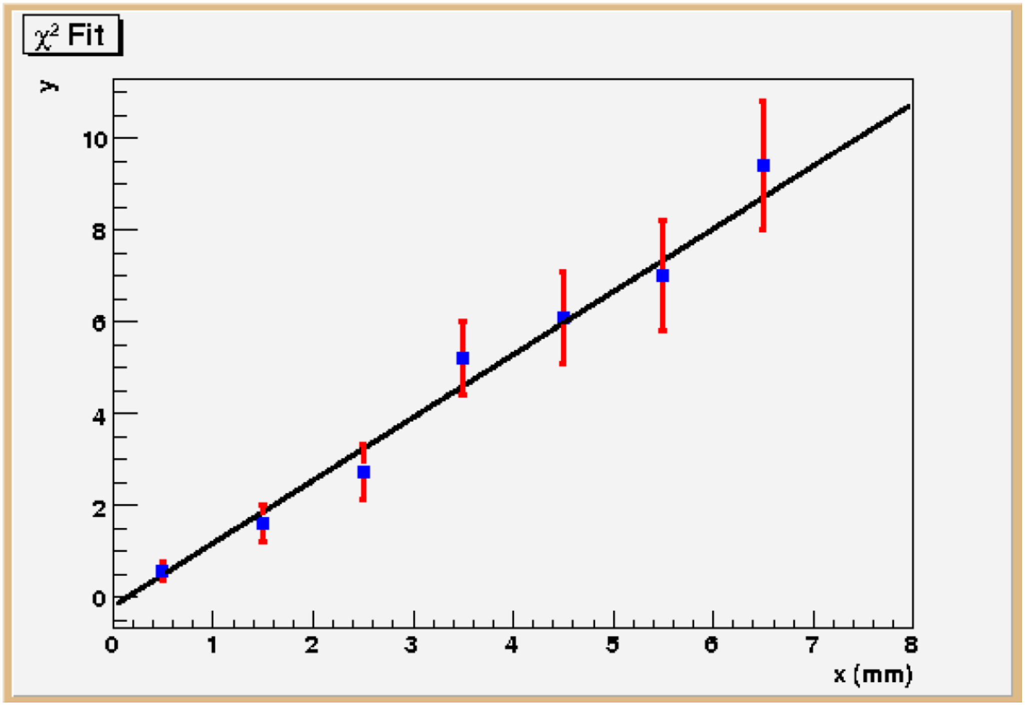
$$\text{Rate} \propto \frac{1}{(x+x_0)^2} \qquad y = \frac{1}{\sqrt{\text{Rate}}} \propto x + x_0$$



χ^2 Fit (Here for a line)

$$\chi^2(a,b) = \sum_{i=1}^{N} \frac{\left(y_i^{\text{meas}} - (a + b x_i)\right)^2}{\sigma_i^2}$$

$$\text{Minimize the } \chi^2 \textit{ i.e.} \text{ solve } \frac{\partial \chi^2}{\partial a} = \frac{\partial \chi^2}{\partial b} = 0$$



χ^2 Fit (here general linear function)

$$\chi^2 = \sum_{i=i}^{M} \left(\frac{y_i^{meas} - F_{ij}^{set}(\vec{x}_i) a_j}{\sigma_i} \right)^2 = \sum_{i=i}^{M} \left(y_i^n - F_{ij}(\vec{x}_i) a_j \right)^2 = \left(\vec{y} - \underline{F}\,\vec{a} \right)^2$$

Minimal χ^2 at $\frac{\partial}{\partial a_i} \chi^2 = 0$

$$\partial_{a_j} \chi^2 = -2\left(y_i - F_{ik} a_k \right) F_{ij} = 2\left(\underline{F}^T \underline{F}\,\vec{a} - \underline{F}^T \vec{y} \right)\Big|_j \quad \Rightarrow \quad \vec{a}_{\min} = \left(\underline{F}^T \underline{F} \right)^{-1} \underline{F}^T \vec{y}$$

Since the measurements y_i have a distribution, so have the parameters a_i:

$$\langle \vec{a} \rangle = \left(\underline{F}^T \underline{F} \right)^{-1} \underline{F}^T \langle \vec{y} \rangle$$

$$\left\langle \left(a_i - \langle a_i \rangle \right)^2 \right\rangle = \sum_{k,k'=1}^{M} \left(\underline{F}^T \underline{F} \right)^{-1} \underline{F}\Big|_{ik} \left(\underline{F}^T \underline{F} \right)^{-1} \underline{F}\Big|_{ik'} \left\langle \left(y_k - \langle y_k \rangle \right)\left(y_{k'} - \langle y_{k'} \rangle \right) \right\rangle$$

$$= \left(\underline{F}^T \underline{F} \right)^{-1} \underline{F}\,\underline{F}^T \left(\underline{F}\,\underline{F}^T \right)^{-1}\Big|_{ii} = \left(\underline{F}^T \underline{F} \right)^{-1}\Big|_{ii}$$

Error of a χ^2 Fit

The χ^2 fit allows for a simple estimate of the uncertainty on the extracted parameters:

$$I_{ij} = \tfrac{1}{2}\partial_{a_i}\partial_{a_j}\chi(\overline{a}), \quad \sigma_{a_i} = \sqrt{\left(I^{-1}\right)_{ii}}$$

In addition the χ^2 value at the minimum gives a measure of the goodness-of-fit: the χ^2/dof (Degrees of Freedom).

Example: with the 7 points and 2 parameters on the precious page dof=7-2=5.

Accumulating knowledge from measurements

If you measured a quantity by M independent procedures and obtained the values c_i with uncertainty σ_i, what is the best combined measurement and uncertainty ?

$$c = \frac{\sum_{i=1}^{M} a_i c_i}{\sum_{i=1}^{M} a_i}$$

The error propagation formula

$$\sigma_c^2 = \left[\partial_a f(\bar{a},\bar{b})\sigma_a\right]^2 + \left[\partial_b f(\bar{a},\bar{b})\sigma_b\right]^2$$

leads to

$$\sigma_c^2 = \frac{\sum_{i=1}^{M} (a_i\sigma_i)^2}{\left(\sum_{i=1}^{M} a_i\right)^2} \Rightarrow \frac{d(\sigma_c^2)}{d(a_k)} = \frac{2\sum_{i=1}^{M} a_i(a_k\sigma_k^2 - a_i\sigma_i^2)}{\left(\sum_{i=1}^{M} a_i\right)^3} = 0 \quad \text{if} \quad a_k\sigma_k^2 = const.$$

$$c = \frac{\sum_{i=1}^{M} c_i/\sigma_i^2}{\sum_{i=1}^{M} 1/\sigma_i^2} \quad , \quad \sigma_c^2 = \frac{1}{\sum_{i=1}^{M} 1/\sigma_i^2}$$

For M identical uncertainties:

$$\sigma_i = \sigma \qquad \sigma_c = \frac{\sigma}{\sqrt{M}}$$

Resources:

http://dcaps.library.cornell.edu/etitles/Frodesen/probabilitystatisticsparticlephysics.pdf

Particle Data Group (PDG): http://pdg.lbl.gov/