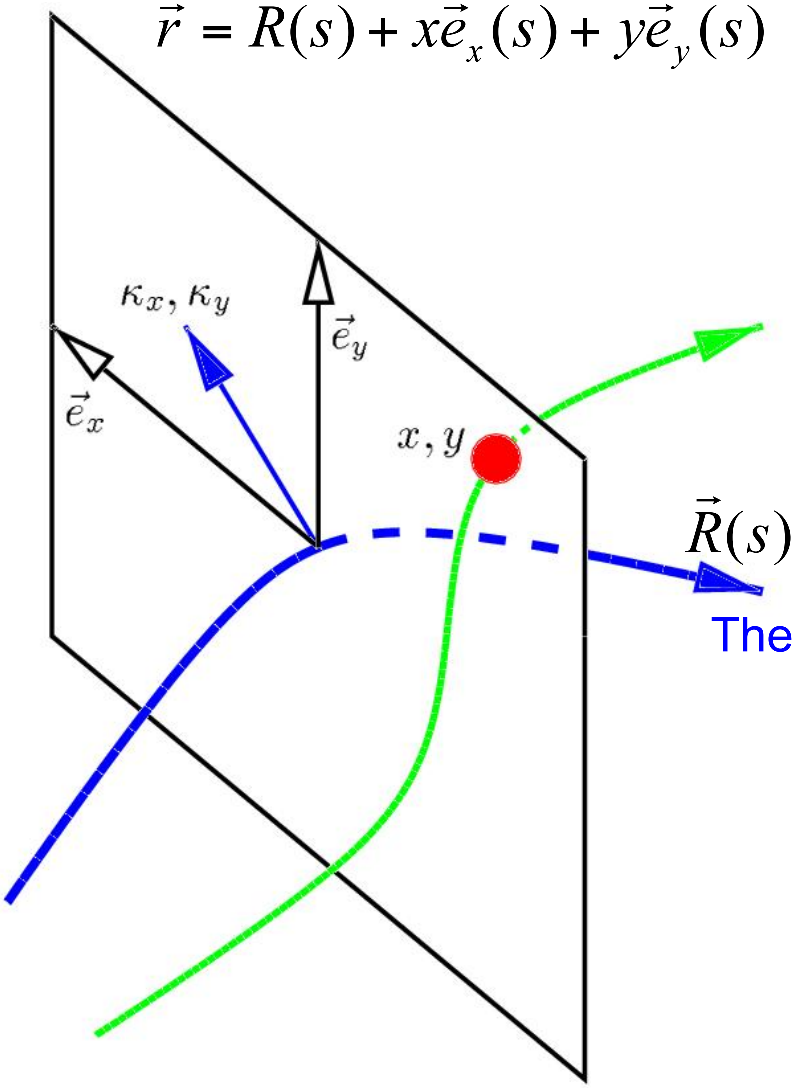
The comoving Coordinate System

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$

$$|d\vec{R}| = ds$$

$$\vec{e}_s \equiv \tfrac{d}{ds}\vec{R}(s)$$



The time dependence of a particle's motion is often not as interesting as the trajectory along the accelerator length "s".

The 4D Equation of Motion

$$\frac{d^2}{dt^2}\vec{r} = \vec{f}_r(\vec{r}, \tfrac{d}{dt}\vec{r}, t)$$

3 dimensional ODE of 2nd order can be changed to a

6 dimensional ODE of 1st order:

$$\left.\begin{aligned} \tfrac{d}{dt}\vec{r} &= \tfrac{1}{m\gamma}\vec{p} = \frac{c}{\sqrt{p^2-(mc)^2}}\vec{p} \\ \tfrac{d}{dt}\vec{p} &= \vec{F}(\vec{r},\vec{p},t) \end{aligned}\right\} \quad \tfrac{d}{dt}\vec{Z} = \vec{f}_Z(\vec{Z},t), \quad \vec{Z} = (\vec{r},\vec{p})$$

If the force does not depend on time, as in a typical beam line magnet, the energy is conserved so that one can reduce the dimension to 5. The equation of motion is then autonomous.

Furthermore, the time dependence is often not as interesting as the trajectory along the accelerator length “s”. Using “s” as the independent variable reduces the dimensions to 4. The equation of motion is then no longer autonomous.

$$\tfrac{d}{ds}\vec{z} = \vec{f}_z(\vec{z},s), \quad \vec{z} = (x, y, p_x, p_y)$$

The 6D Equation of Motion

Usually one prefers to compute the trajectory as a function of "s" along the accelerator even when the energy is not conserved, as when accelerating cavities are in the accelerator.

Then the energy "E" and the time "t" at which a particle arrives at the cavities are important. And the equations become 6 dimensional again:

$$\frac{d}{ds}\vec{z} = \vec{f}_z(\vec{z},s)\,, \quad \vec{z} = (x, y, p_x, p_y, -t, E)$$

But: $\vec{z} = (\vec{r}, \vec{p})$ is an especially suitable variable, since it is a phase space vector so that its equation of motion comes from a Hamiltonian, or by variation principle from a Lagrangian.

$$\delta \int \left[p_x \dot{x} + p_y \dot{y} + p_s \dot{s} - H(\vec{r},\vec{p},t) \right] dt = 0 \quad \Rightarrow \quad \text{Hamiltonian motion}$$

$$\delta \int \left[p_x x' + p_y y' - H\,t' + p_s(x, y, p_x, p_y, t, H) \right] ds = 0 \quad \Rightarrow \quad \text{Hamiltonian motion}$$

The new canonical coordinates are: $\vec{z} = (x, y, p_x, p_y, -t, E) \quad \text{with} \quad \mathrm{E} = \mathrm{H}$

The new Hamiltonian is: $K = -p_s(\vec{z}, s)$

Significance of Hamiltonian

The equations of motion can be determined by one function:

$$\tfrac{d}{ds}x = \partial_{p_x} H(\vec{z},s)\,, \quad \tfrac{d}{ds}p_x = -\partial_x H(\vec{z},s)\,, \quad \dots$$

$$\tfrac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial}H(\vec{z},s) = \vec{F}(\vec{z},s) \quad \text{with} \quad \underline{J} = \text{diag}(\underline{J}_2)\,, \quad \underline{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The force has a Hamiltonian Jacobi Matrix:

A linear force: $\vec{F}(\vec{z},s) = \underline{F}(s)\cdot\vec{z}$

The Jacobi Matrix of a linear force: $\underline{F}(s)$

The general Jacobi Matrix : $F_{ij} = \partial_{z_j} F_i$ or $\underline{F} = \left(\vec{\partial}\,\vec{F}^T\right)^T$

Hamiltonian Matrices: $\underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = 0$

Prove : $F_{ij} = \partial_{z_j} F_i = \partial_{z_j} J_{ik}\partial_{z_k} H = J_{ik}\,\partial_k\partial_j H \quad \Rightarrow \quad \underline{F} = \underline{J}\,\underline{D}\underline{H}$

$$\underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = \underline{J}\,\underline{D}\underline{J}\underline{H} + \underline{J}\,\underline{D}^T\,\underline{J}^T\,H = 0$$

H i Symplectic Flows

The flow of a Hamiltonian equation of motion has a symplectic Jacobi Matrix

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

A linear flow: $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0$

The Jacobi Matrix of a linear flow: $\underline{M}(s)$

The general Jacobi Matrix : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = (\vec{\partial}_0 \vec{M}^T)^T$

The Symplectic Group SP(2N) : $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

$$\tfrac{d}{ds}\vec{z} = \tfrac{d}{ds}\vec{M}(s, \vec{z}_0) = \underline{J}\vec{\nabla}H = \vec{F}$$

$$\tfrac{d}{ds} M_{ij} = \partial_{z_{0j}} F_i(\vec{z}, s) = \partial_{z_{0j}} M_k \partial_{z_k} F_i(\vec{z}, s)$$

$$\tfrac{d}{ds}\underline{M}(s, \vec{z}_0) = \underline{F}(\vec{z}, s)\underline{M}(s, \vec{z}_0)$$

$$K = \underline{M}\,\underline{J}\,\underline{M}^T$$

$$\tfrac{d}{ds}\underline{K} = \tfrac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T + \underline{M}\,\underline{J}\tfrac{d}{ds}\underline{M}^T = \underline{F}\,\underline{M}\,\underline{J}\,\underline{M}^T + \underline{M}\,\underline{J}\,\underline{M}^T\underline{F}^T = \underline{F}\,\underline{K} + \underline{K}\,\underline{F}^T$$

$\underline{K} = \underline{J}$ is a solution. Since this is a linear ODE , $\underline{K} = \underline{J}$ is the unique solution.

Symplectic Flows i H

For every symplectic transport map there is a Hamilton function

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

Force vector: $\vec{h}(\vec{z}, s) = -\underline{J}\left[\frac{d}{ds}\vec{M}(s, \vec{z}_0)\right]_{\vec{z}_0 = \vec{M}^{-1}(\vec{z}, s)}$

Since then: $\frac{d}{ds}\vec{z} = \underline{J}\vec{h}(\vec{z}, s)$

There is a Hamilton function H with: $\vec{h} = \vec{\partial} H$

If and only if: $\partial_{z_j} h_i = \partial_{z_i} h_j \quad \Rightarrow \quad \underline{h} = \underline{h}^T$

$$\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J} \quad \Rightarrow \quad \begin{cases} \frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T = -\underline{M}\,\underline{J}\frac{d}{ds}\underline{M}^T \\ \underline{M}^{-1} = -\underline{J}\,\underline{M}^T\underline{J} \end{cases}$$

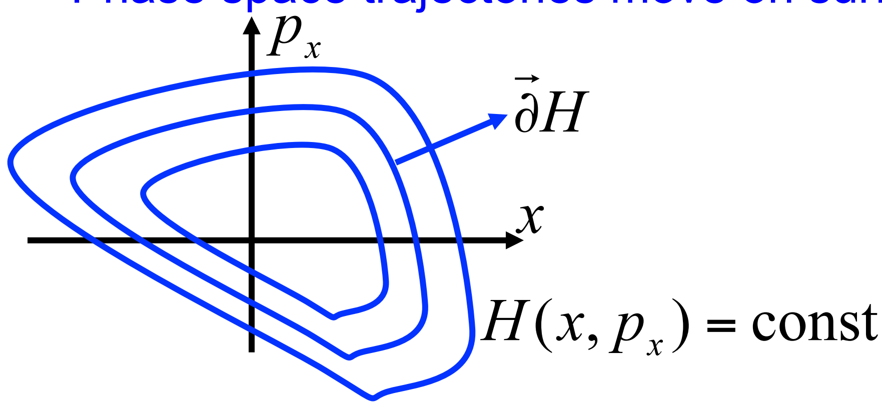
$$\vec{h} \circ \vec{M} = -\underline{J}\frac{d}{ds}\vec{M}$$

$$\underline{h}(\vec{M})\underline{M} = -\underline{J}\frac{d}{ds}\underline{M}$$

$$\underline{h}(\vec{M}) = -\underline{J}\frac{d}{ds}\underline{M}\,\underline{M}^{-1} = \underline{J}\frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T\underline{J} = -\underline{J}\,\underline{M}\,\underline{J}\frac{d}{ds}\underline{M}^T\underline{J} = \underline{M}^{-T}\frac{d}{ds}\underline{M}^T\underline{J} = \underline{h}^T$$

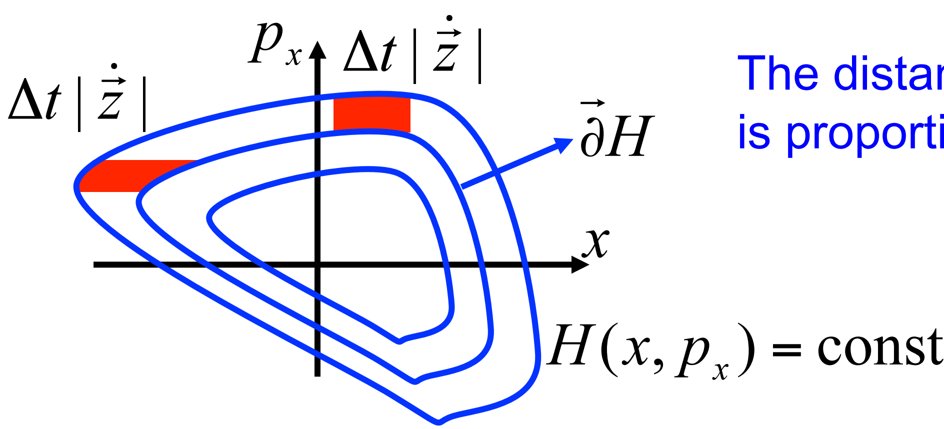
Phase space density in 2D

- Phase space trajectories move on surfaces of constant energy



$$\frac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial} H \quad \Rightarrow \quad \frac{d}{ds}\vec{z} \perp \vec{\partial} H$$

- A phase space volume does not change when it is transported by Hamiltonian motion.

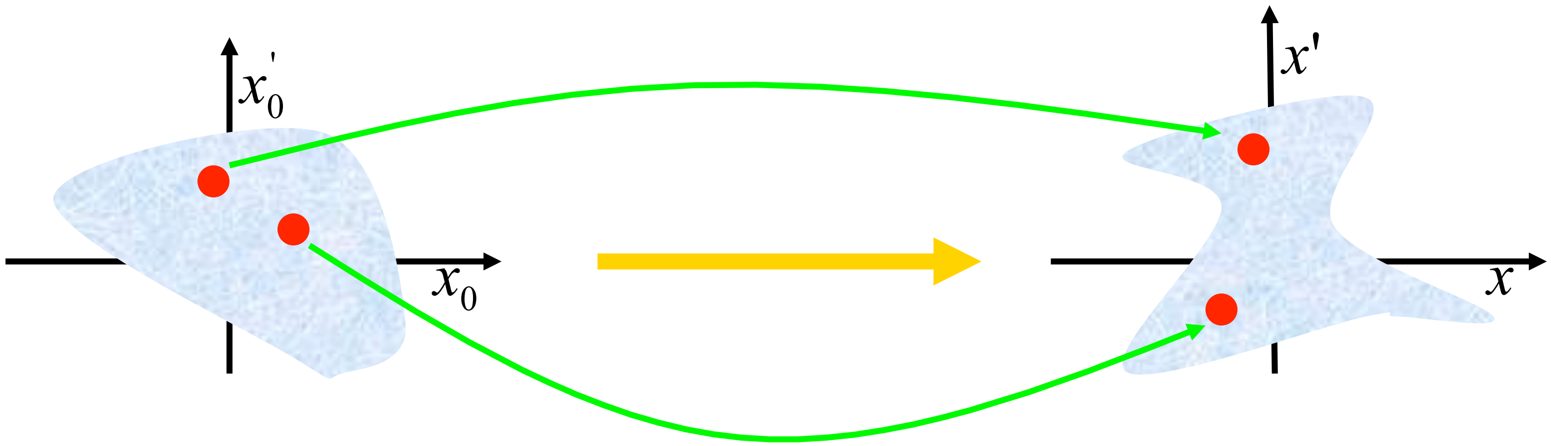


The distance d of lines with equal energy is proportional to $1/\,|\,\vec{\partial} H\,| \propto |\,\dot{\vec{z}}\,|^{-1}$

$$\boxed{d * \Delta t\,|\,\dot{\vec{z}}\,| = \mathrm{const}}$$

Lioville's Theorem

- A phase space volume does not change when it is transported by Hamiltonian motion. $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0$ with $\det[\underline{M}(s)] = +1$



Volume = $V = \int\!\!\int_V d^n\vec{z} = \int\!\!\int_{V_0} \left|\frac{\partial \vec{z}}{\partial \vec{z}_0}\right| d^n\vec{z}_0 = \int\!\!\int_{V_0} |\underline{M}| d^n\vec{z}_0 = \int\!\!\int_{V_0} d^n\vec{z}_0 = V_0$

Hamiltonian Motion → $V = V_0$

But Hamiltonian requires symplecticity, which is much more than just $\det[\underline{M}(s)] = +1$

Generating Functions

The motion of particles can be represented by Generating Functions

Each flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

With a Jacobi Matrix : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = (\vec{\partial}_0 \vec{M}^T)^T$

That is Symplectic: $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

Can be represented by a Generating Function:

$$F_1(\vec{q}, \vec{q}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1 \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1$$

$$F_2(\vec{p}, \vec{q}_0, s) \quad \text{with} \quad \vec{q} = \vec{\partial}_p F_2 \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_2$$

$$F_3(\vec{q}, \vec{p}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_3 \quad , \quad \vec{q}_0 = -\vec{\partial}_{p_0} F_3$$

$$F_4(\vec{p}, \vec{p}_0, s) \quad \text{with} \quad \vec{q} = \vec{\partial}_q F_4 \quad , \quad \vec{q}_0 = -\vec{\partial}_{p_0} F_4$$

6-dimensional motion needs only one function ! But to obtain the transport map this has to be inverted.

F i SP(2N) [for notes]

Generating Functions produce symplectic tranport maps

$$F_1(\vec{q},\vec{q}_0,s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s) \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s)$$

$$\vec{z} = \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \vec{q} \\ -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s) \end{pmatrix} = \vec{f}(\vec{Q},s)$$

$$\vec{z}_0 = \begin{pmatrix} \vec{q}_0 \\ \vec{p}_0 \end{pmatrix} = \begin{pmatrix} \vec{q}_0 \\ \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s) \end{pmatrix} = \vec{g}(\vec{Q},s)$$

$$\vec{z} = \vec{f}(\vec{g}^{-1}(\vec{z}_0,s),s)$$

$$\vec{M} = \vec{f} \circ \vec{g}^{-1}$$

(function concatenation)

Jacobi matrix of concatenated functions:

$$\vec{C}(\vec{z}_0) = \vec{A} \circ \vec{B}(\vec{z}_0)$$

$$C_{ij} = \partial_j C_i = \sum_k \partial_{z_{0j}} B_k(\vec{z}_0) \left[\partial_{z_k} A_i(\vec{z}) \right]_{\vec{z}=\vec{B}(\vec{z}_0)} \quad \Rightarrow \quad \underline{C} = \underline{A}(\vec{B})\underline{B}$$

$$\vec{M} \circ \vec{g} = \vec{f} \quad \Rightarrow \quad \underline{M}(\vec{g}) = \underline{F}\,\underline{G}^{-1}$$

F i SP(2N) [for notes]

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$$\vec{f}(\vec{Q},s) = \begin{pmatrix} \vec{q} \\ -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s) \end{pmatrix} \quad \Rightarrow \quad F = \begin{pmatrix} \underline{1} & \underline{0} \\ -\vec{\partial}_q \vec{\partial}_q^T F_1 & -\vec{\partial}_q \vec{\partial}_{q_0}^T F_1 \end{pmatrix}$$

$$\vec{g}(\vec{Q},s) = \begin{pmatrix} \vec{q}_0 \\ \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s) \end{pmatrix} \quad \Rightarrow \quad G = \begin{pmatrix} \underline{0} & \underline{1} \\ \vec{\partial}_{q_0} \vec{\partial}_q^T F_1 & \vec{\partial}_{q_0} \vec{\partial}_{q_0}^T F_1 \end{pmatrix}$$

$$F = \begin{pmatrix} 1 & 0 \\ -F_{11} & -F_{12} \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 1 \\ F_{21} & F_{22} \end{pmatrix} \quad \Rightarrow \quad G^{-1} = \begin{pmatrix} -F_{21}^{-1}F_{22} & F_{21}^{-1} \\ 1 & 0 \end{pmatrix}$$

$$\underline{M}(\vec{g}) = F\,G^{-1} = \begin{pmatrix} -F_{21}^{-1}F_{22} & F_{21}^{-1} \\ F_{11}F_{21}^{-1}F_{22} - F_{12} & -F_{11}F_{21}^{-1} \end{pmatrix}$$

$$\underline{M}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\underline{M}^T$$

The map from a generating function is symplectic.

$$= \begin{pmatrix} -F_{21}^{-1} & -F_{21}^{-1}F_{22} \\ F_{11}F_{21}^{-1} & F_{11}F_{21}^{-1}F_{22} - F_{12} \end{pmatrix}\begin{pmatrix} -F_{22}F_{12}^{-1} & F_{22}F_{12}^{-1}F_{11} - F_{21} \\ F_{12}^{-1} & -F_{12}^{-1}F_{11} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

SP(2N) i F [for notes]

Symplectic tranport maps have a Generating Functions

$$\vec{z} = \vec{M}(\vec{z}_0)$$

$$\begin{pmatrix} \vec{q} \\ \vec{q}_0 \end{pmatrix} = \begin{pmatrix} \vec{M}_1(\vec{z}_0) \\ \vec{q}_0 \end{pmatrix} = \vec{l}\,(\vec{z}_0)\,, \quad \begin{pmatrix} \vec{p}_0 \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \vec{p}_0 \\ \vec{M}_2(\vec{z}_0) \end{pmatrix} = \vec{h}(\vec{z}_0) = \underline{J}\left[\vec{\partial}F_1(\vec{q},\vec{q}_0)\right]_{\vec{l}\,(\vec{z}_0)}$$

$$\vec{\partial}F_1 = -\underline{J}\vec{h}\circ\vec{l}^{\,-1} = \vec{F}$$

For F_1 to exist it is necessary and sufficient that $\partial_i F_j = \partial_j F_i \quad \Rightarrow \quad \underline{F} = \underline{F}^T$

$$-\underline{J}\vec{h} = \vec{F}\circ\vec{l} \quad \Rightarrow \quad -\underline{J}\,\underline{h} = \underline{F}(\vec{l}\,)\,\underline{l}$$

Is $\underline{J}\ \underline{h}\ \underline{l}^{-1}$ symmetric ? Yes since:

$$\underline{J}\,\underline{h}\,\underline{l}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \vec{\partial}^T_{q_0}\vec{M}_2 & \vec{\partial}^T_{p_0}\vec{M}_2 \end{pmatrix}\begin{pmatrix} \vec{\partial}^T_{q_0}\vec{M}_1 & \vec{\partial}^T_{p_0}\vec{M}_1 \\ 1 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} M_{21} & M_{22} \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ M_{12}^{-1} & -M_{12}^{-1}M_{11} \end{pmatrix} = \begin{pmatrix} M_{22}M_{12}^{-1} & M_{21} - M_{22}M_{12}^{-1}M_{11} \\ M_{12}^{-1} & M_{12}^{-1}M_{11} \end{pmatrix}$$

SP(2N) i F [for notes]

$$\underline{J}\,\underline{h}\,\underline{l}^{-1} = \begin{pmatrix} M_{22}M_{12}^{-1} & M_{21} - M_{22}M_{12}^{-1}M_{11} \\ M_{12}^{-1} & M_{12}^{-1}M_{11} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$\vec{M}(\vec{z}_0) = \begin{pmatrix} \vec{M}_1(\vec{q}_0, \vec{p}_0) \\ \vec{M}_2(\vec{q}_0, \vec{p}_0) \end{pmatrix}, \quad \underline{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$\underline{M}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\underline{M}^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} -M_{12} & M_{11} \\ -M_{22} & M_{21} \end{pmatrix}\begin{pmatrix} M_{11}^T & M_{21}^T \\ M_{12}^T & M_{22}^T \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$M_{12}M_{11}^T = M_{11}M_{12}^T \qquad \Rightarrow \quad \left(M_{12}^{-1}M_{11}\right)^T = [M_{12}^{-1}M_{11}M_{12}^T]M_{12}^{-T} = M_{12}^{-1}M_{11}$$

$$M_{21}M_{22}^T = M_{22}M_{21}^T$$

$$M_{11}M_{22}^T - M_{12}M_{21}^T = 1$$

$$M_{22}M_{11}^T - M_{21}M_{12}^T = 1$$

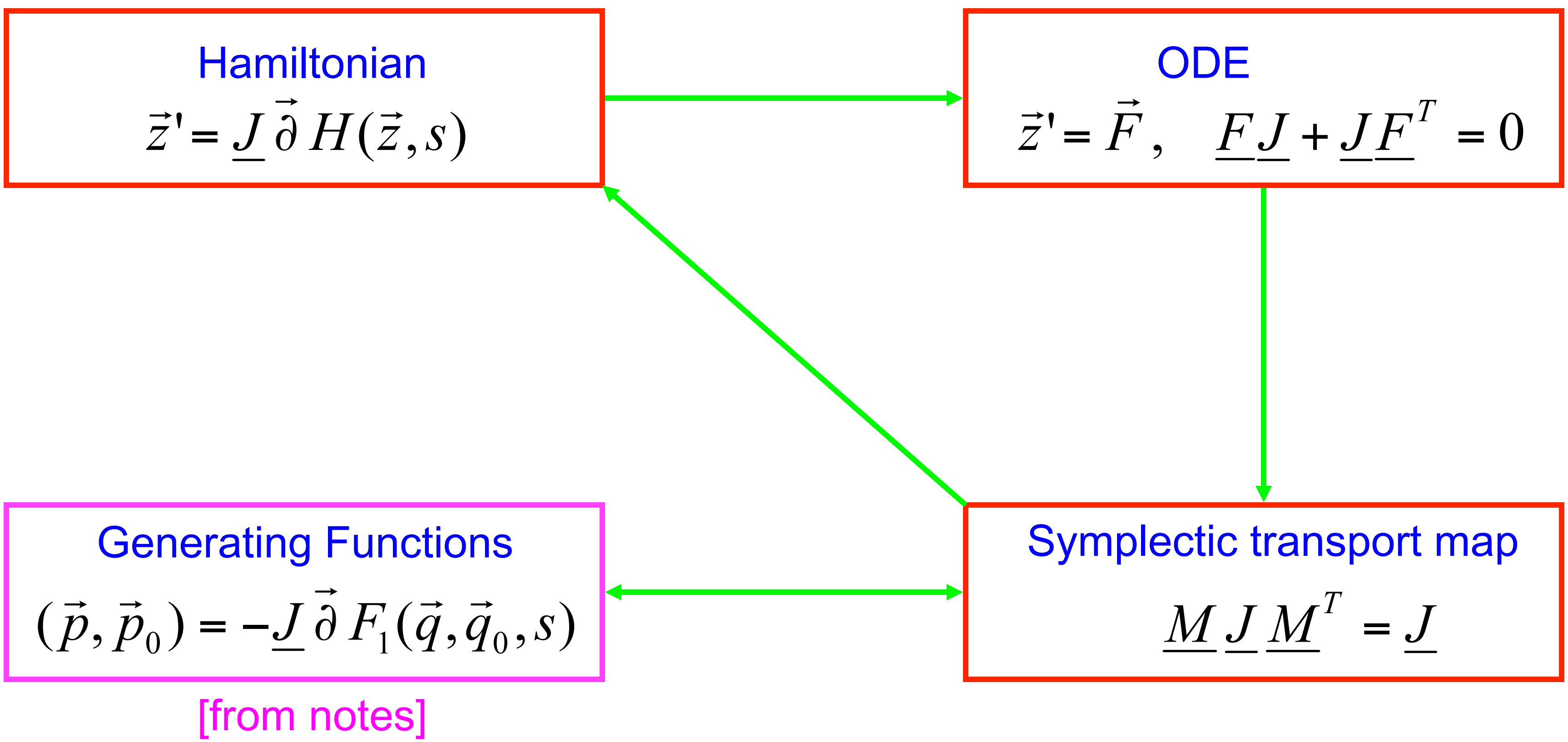
$$D = D^T$$

$$A = A^T$$

$$\left(M_{22}M_{12}^{-1}\right)^T = [M_{22}M_{11}^T M_{12}^{-T} - M_{21}]M_{22}^T = M_{22}[M_{12}^{-1}M_{11}M_{22}^T - M_{21}^T] = M_{22}M_{12}^{-1}$$

$$M_{21} - M_{22}M_{12}^{-1}M_{11} = M_{21} - M_{22}M_{11}M_{12}^{-T} = M_{12}^{-T} \longrightarrow B = C^T$$

Symplectic Representations



Advantages of Symplecticity

- Determinant of the transfer matrix of linear motion is 1:

$$\vec{z}(s) = \underline{M}(s)\cdot\vec{z}_0 \quad \text{with} \quad \det(\underline{M}(s)) = +1$$

- One function suffices to compute the total nonlinear transfer map:

$$F_1(\vec{q},\vec{q}_0,s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s) \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s)$$

$$\left.\begin{aligned}\vec{z} &= \begin{pmatrix}\vec{q}\\ \vec{p}\end{pmatrix} = \begin{pmatrix}\vec{q}\\ -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s)\end{pmatrix} = \vec{f}(\vec{Q},s)\\ \vec{z}_0 &= \begin{pmatrix}\vec{q}_0\\ \vec{p}_0\end{pmatrix} = \begin{pmatrix}\vec{q}_0\\ \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s)\end{pmatrix} = \vec{g}(\vec{Q},s)\end{aligned}\right\} \quad \begin{aligned}\vec{z} &= \vec{f}(\vec{g}^{-1}(\vec{z}_0,s),s)\\ \vec{M} &= \vec{f}\circ\vec{g}^{-1}\end{aligned}$$

- Therefore Taylor Expansion coefficients of the transport map are related.
- Computer codes can numerically approximate $\vec{M}(s,\vec{z}_0)$ with exact symplectic symmetry.
- Liouville's Theorem for phase space densities holds.

Eigenvalues of a Symplectic Matrix

For matrices with real coefficients:

If there is an eigenvector and eigenvalue: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$

then the complex conjugates are also eigenvector and eigenvalue: $\underline{M}\vec{v}_i^{\,*} = \lambda_i^* \vec{v}_i^{\,*}$

For symplectic matrices:

If there are eigenvectors and eigenvalues: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$ with $\underline{J} = \underline{M}^T \underline{J}\,\underline{M}$

then $\vec{v}_i^{\,T} \underline{J}\vec{v}_j = \vec{v}_i^{\,T} \underline{M}^T \underline{J}\,\underline{M}\vec{v}_j = \lambda_i \lambda_j \vec{v}_i^{\,T} \underline{J}\vec{v}_j \quad \Rightarrow \quad \vec{v}_i^{\,T} \underline{J}\vec{v}_j (\lambda_i \lambda_j - 1) = 0$

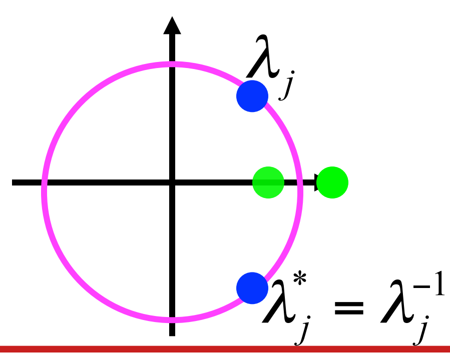
Therefore $\underline{J}\vec{v}_j$ is orthogonal to all eigenvectors with eigenvalues that are not $1/\lambda_j$. Since it cannot be orthogonal to all eigenvectors, there is at least one eigenvector with eigenvalue $1/\lambda_j$

Two dimensions: λ_j is eigenvalue

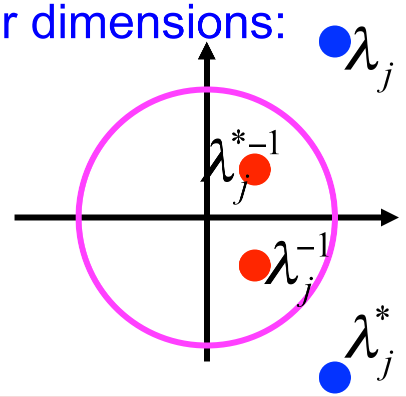
Then $1/\lambda_j$ and λ_j^* are eigenvalues

$\lambda_2 = 1/\lambda_1 = \lambda_1^* \quad \Rightarrow \quad |\lambda_j| = 1$

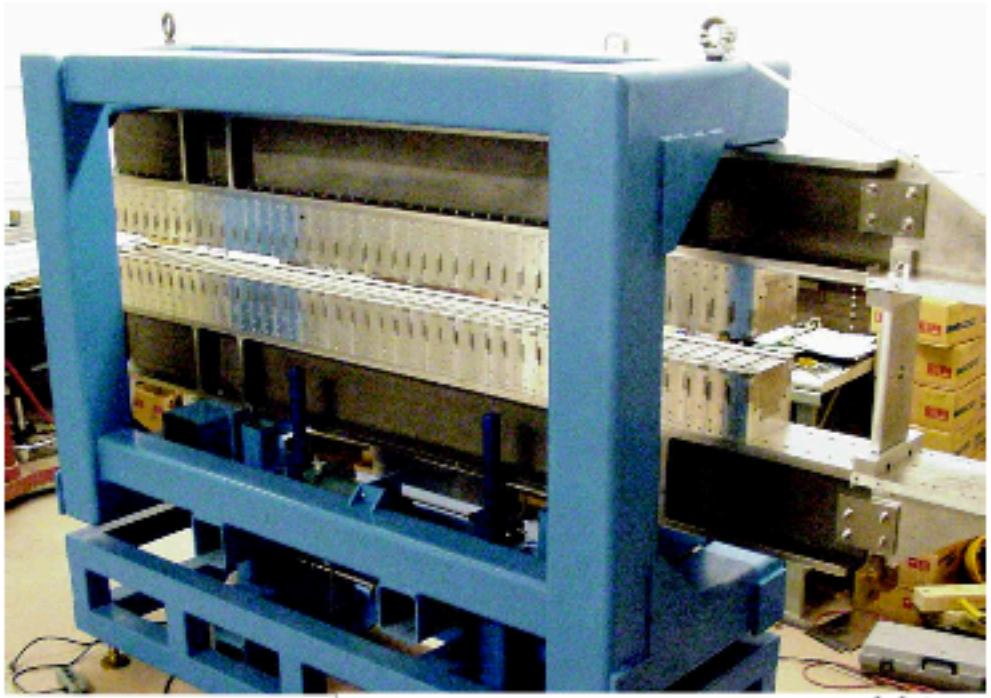
$\lambda_2 = 1/\lambda_1 = \lambda_2^*$

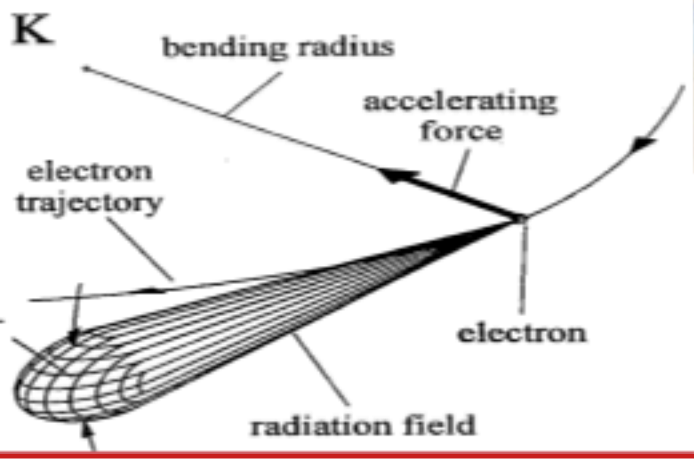


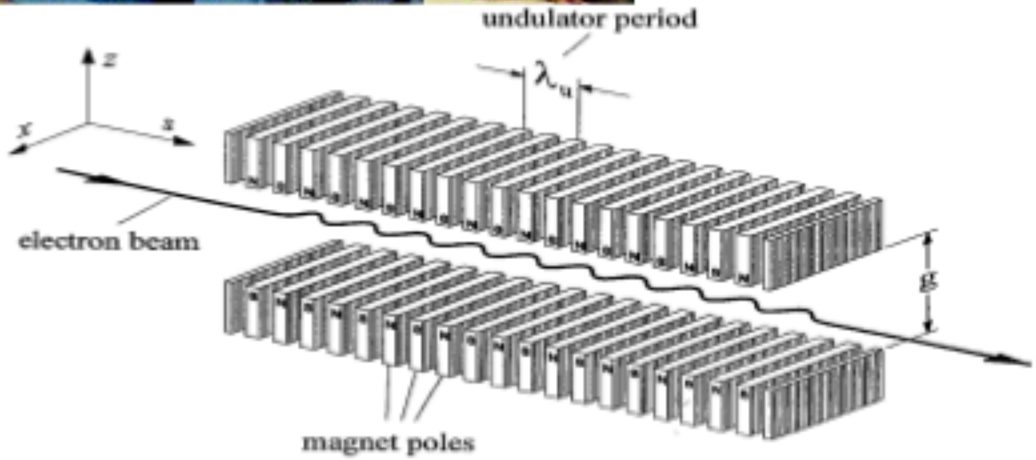
Four dimensions:



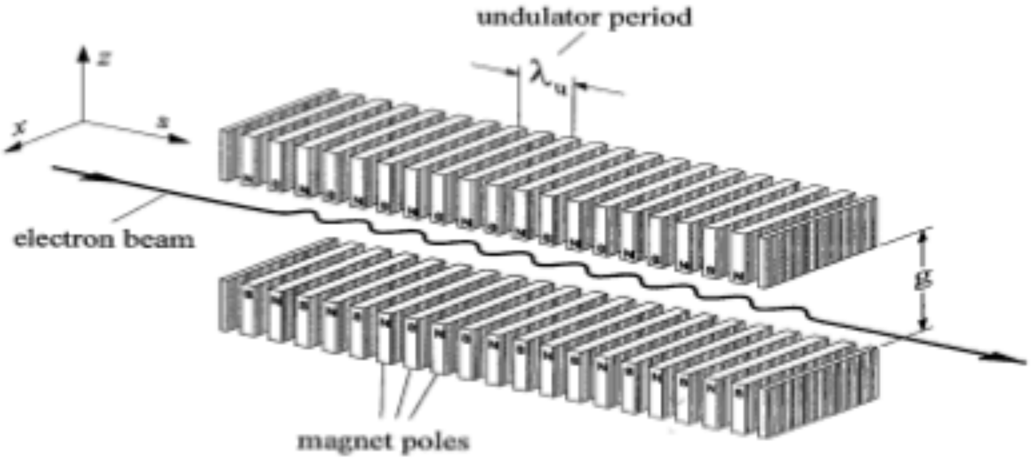
Radiation production



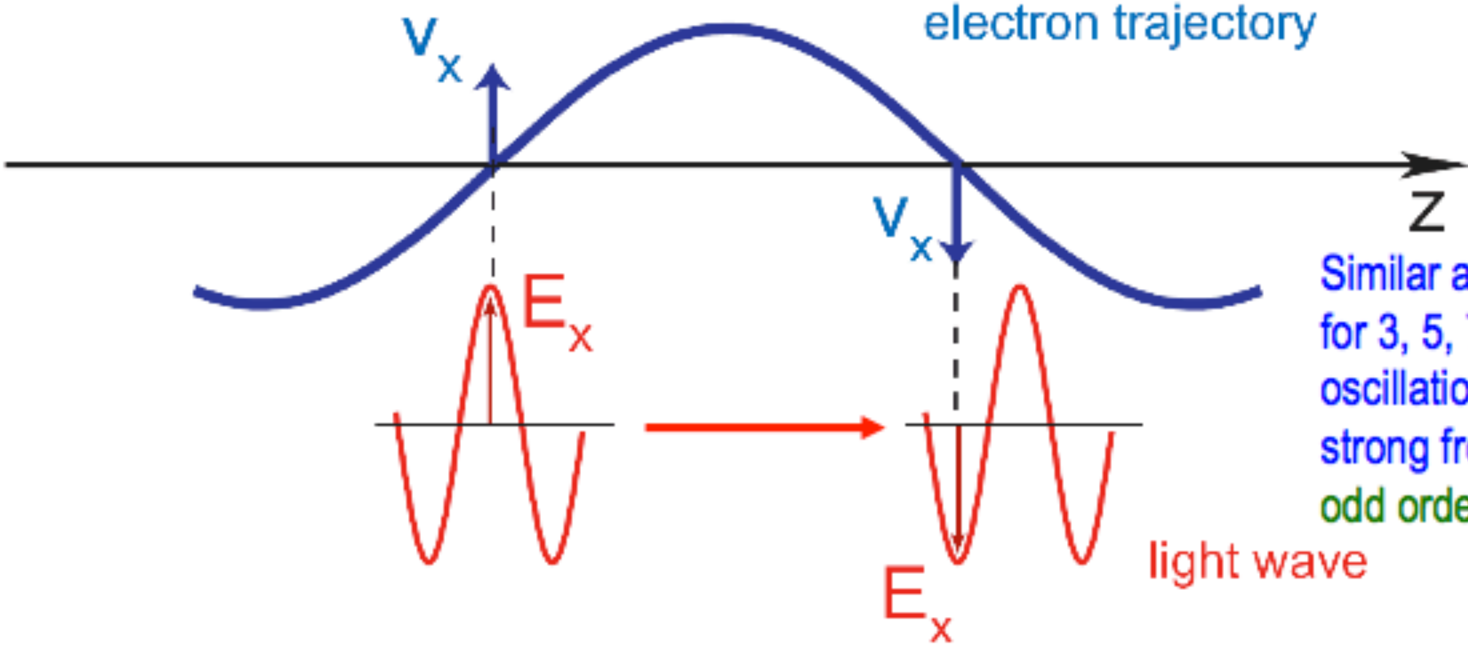




Radiation production

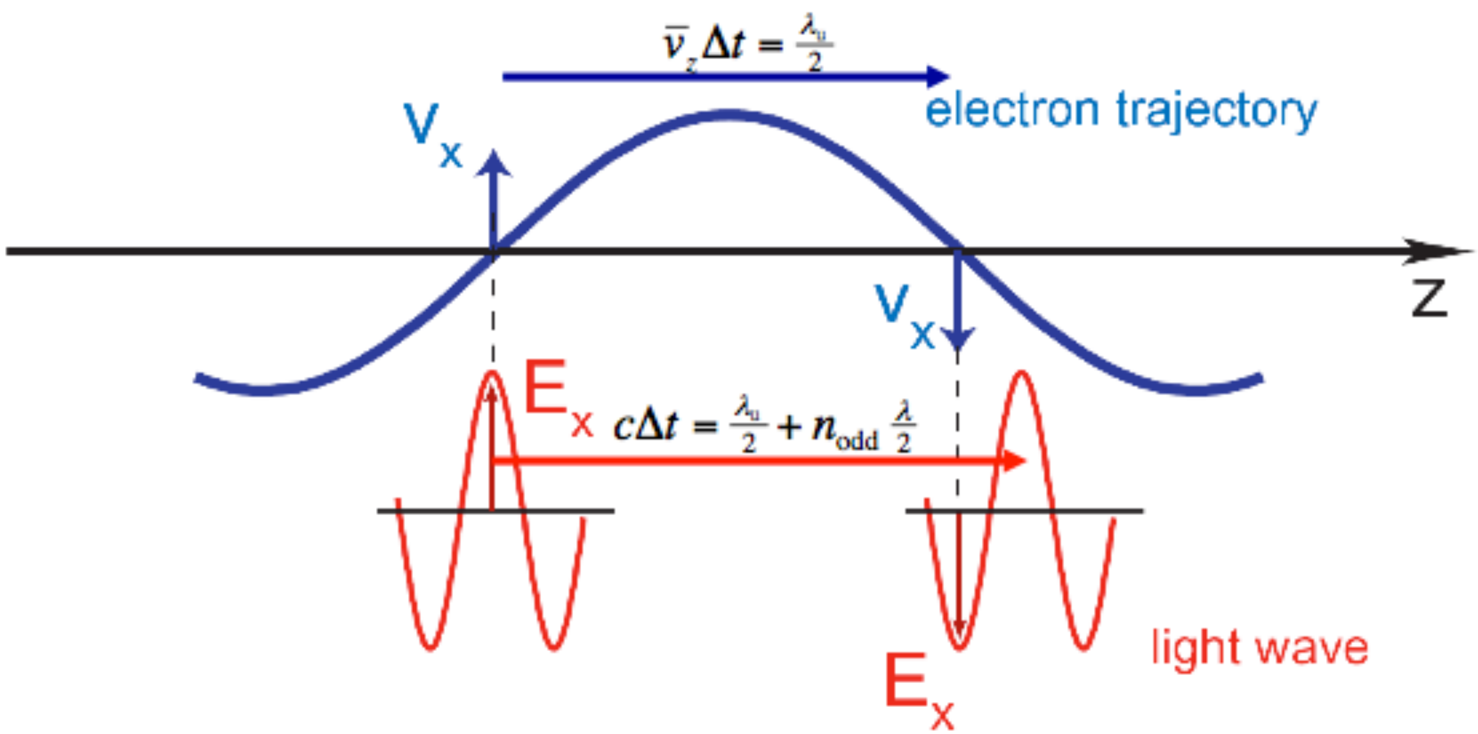


When the light wave passes the electron beam by half a wavelength per half beam oscillation, the radiation from each beam oscillation adds.



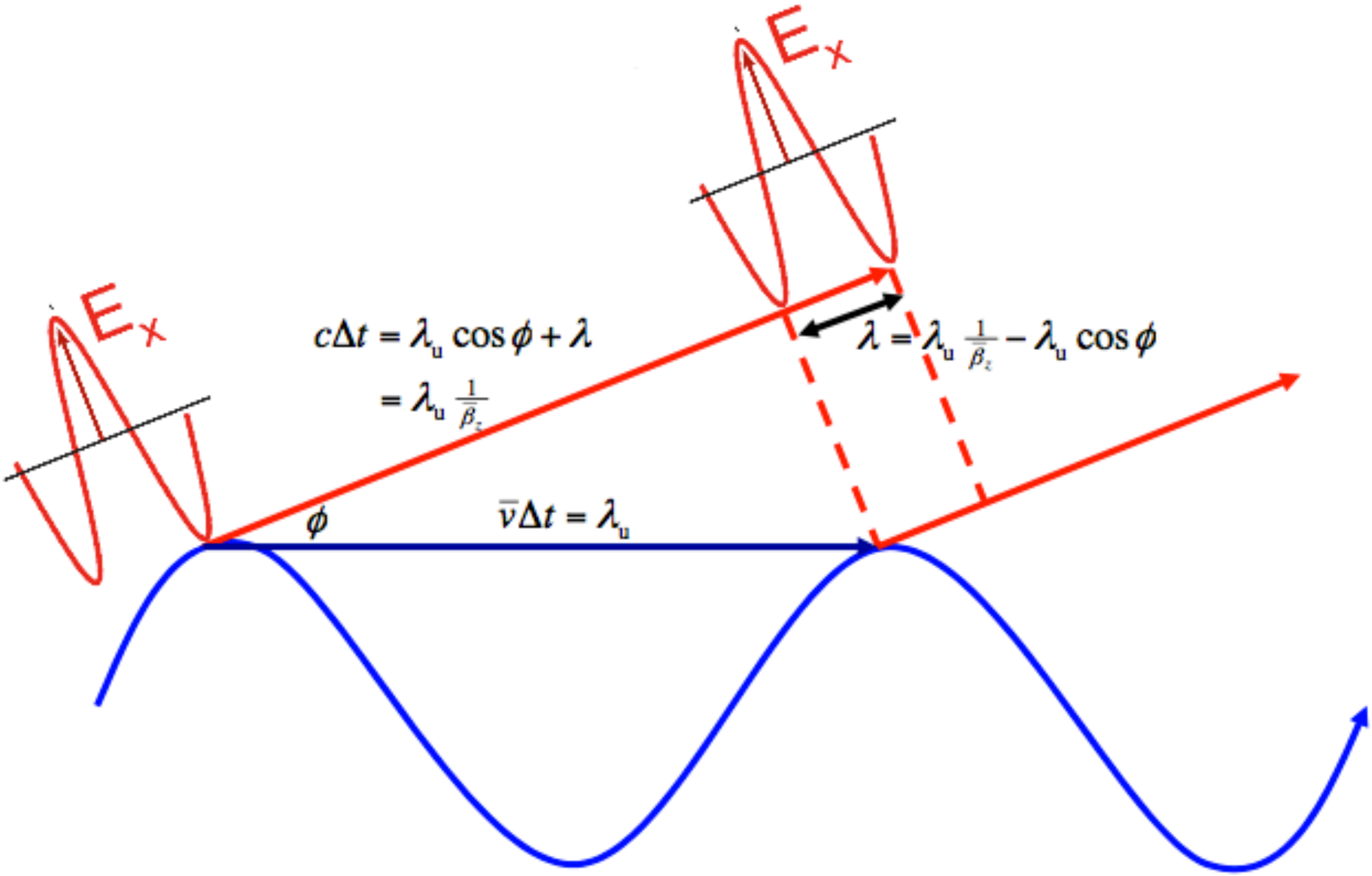
Similar adding can happen for 3, 5, 7, etc. half beam oscillations. For every strong frequency there are odd order harmonics.

Radiation production



$$\frac{c}{\bar{v}_z}\frac{\lambda_u}{2} = \frac{\lambda_u}{2} + n_{odd}\frac{\lambda}{2} \quad \Rightarrow \quad \boxed{\lambda = \frac{1}{n_{odd}}\lambda_u\left(\frac{1}{\bar{\beta}_z} - 1\right)}$$

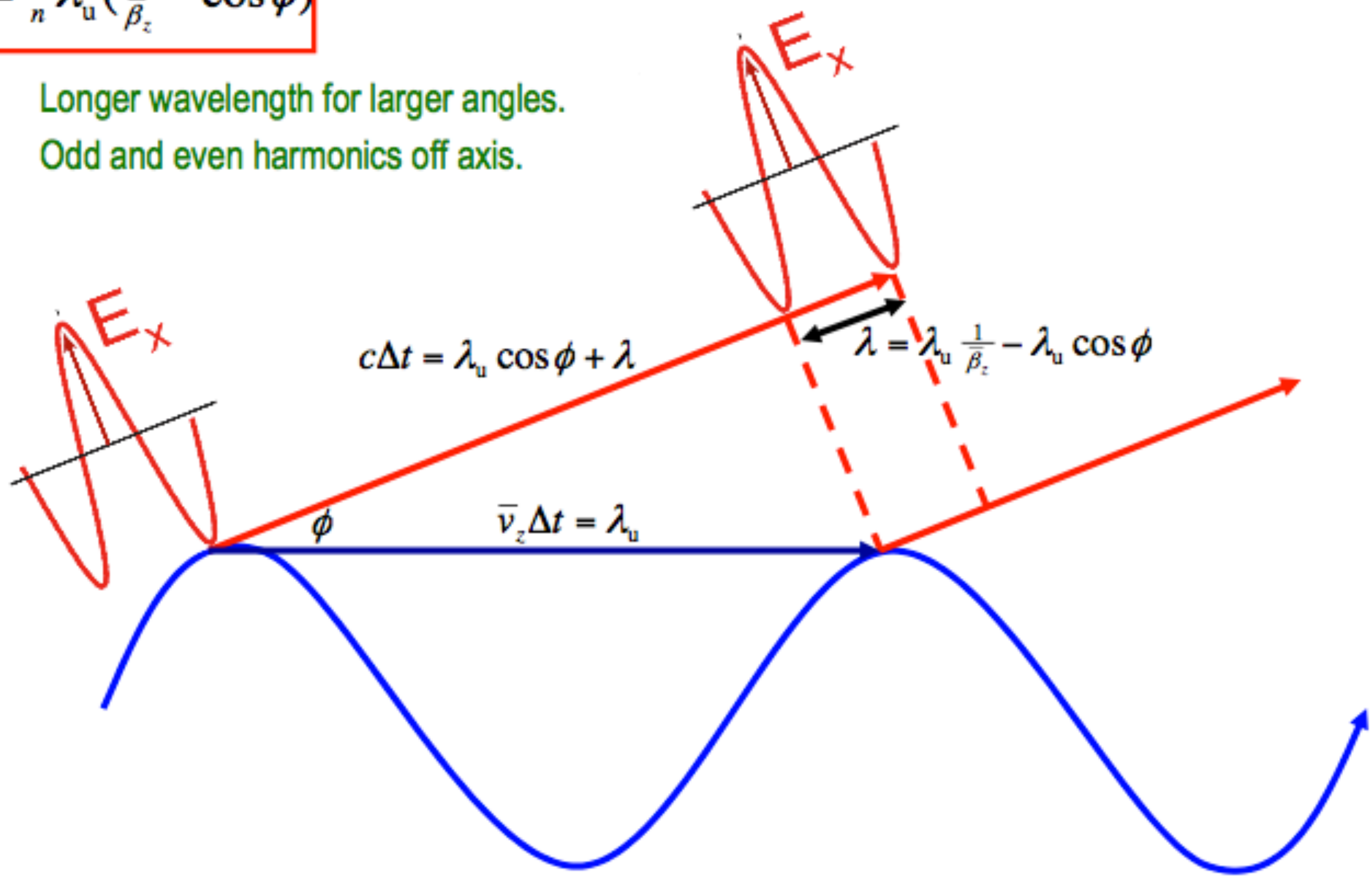
Radiation production



Radiation production

$$\lambda = \frac{1}{n}\lambda_u\left(\frac{1}{\beta_z} - \cos\phi\right)$$

1) Longer wavelength for larger angles.
2) Odd and even harmonics off axis.



Lasing at JLAB with the ERL

Wiggler gap

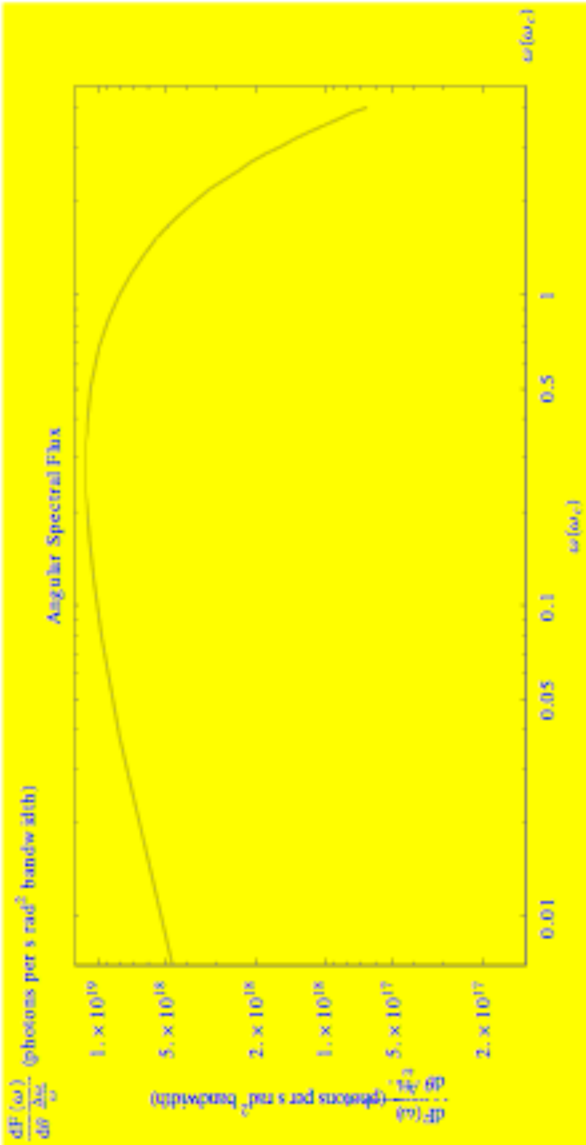
Hole Outcoupler

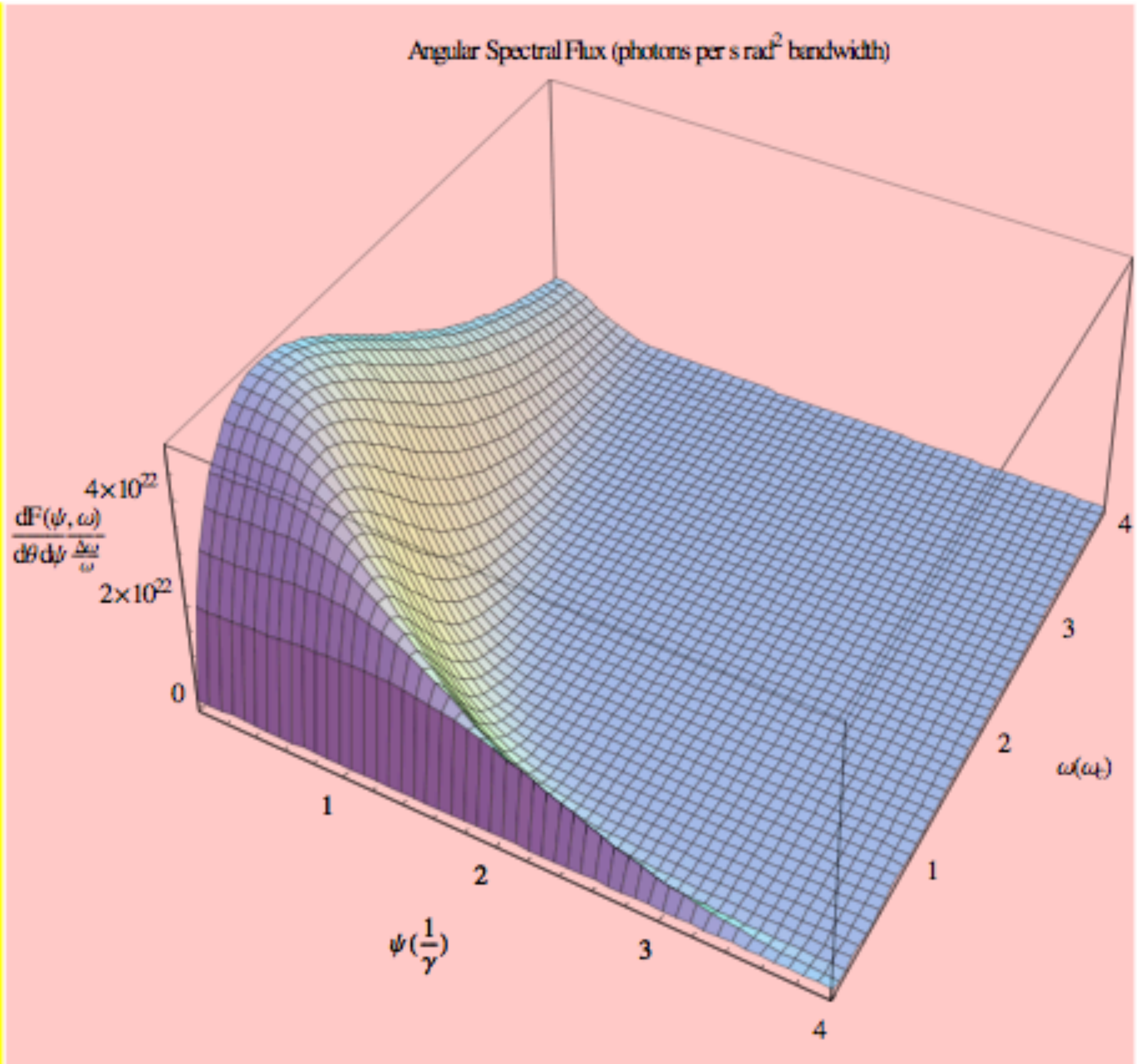
High Reflector

Beam in control room

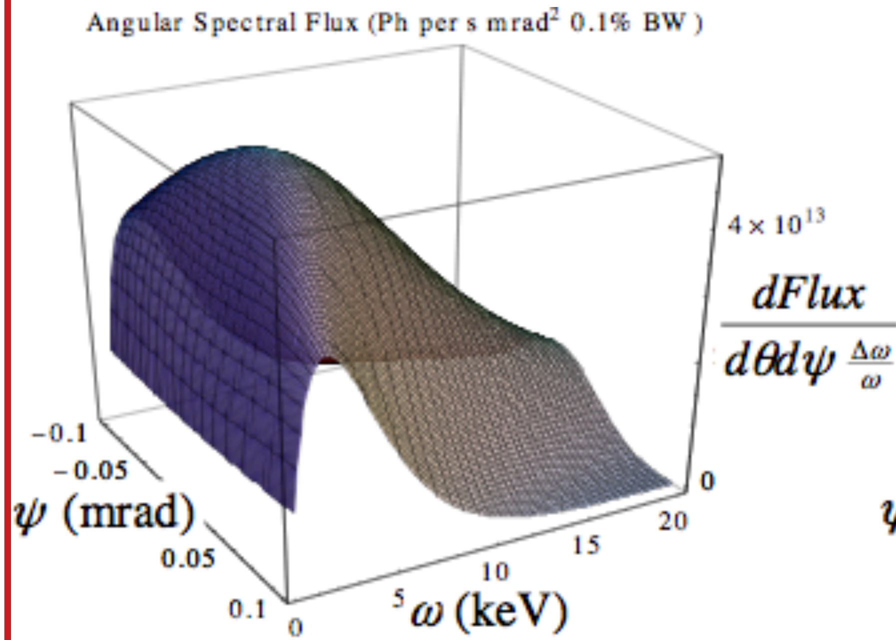
Radiation from bending magnets

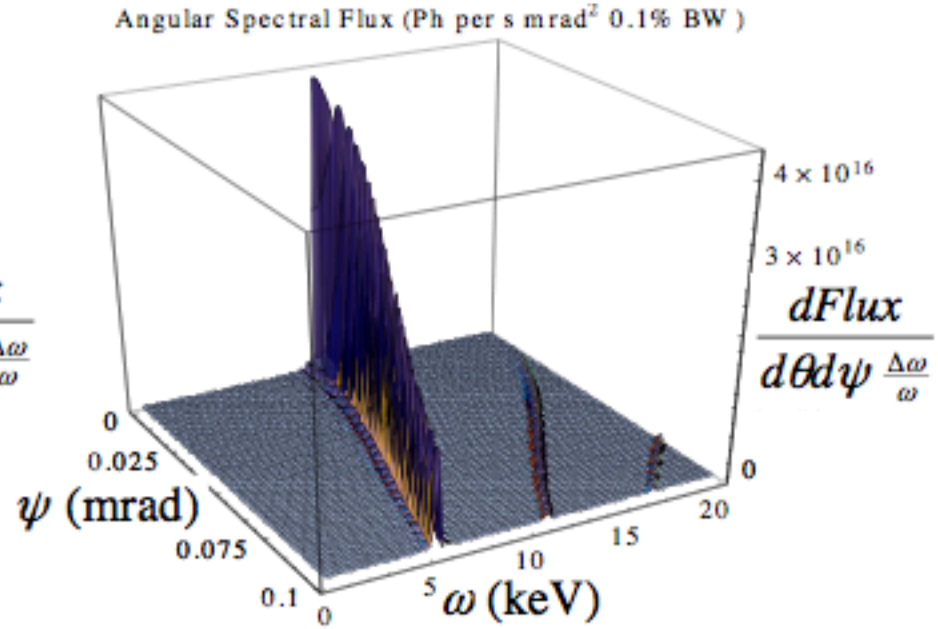
CHESS & LEPP

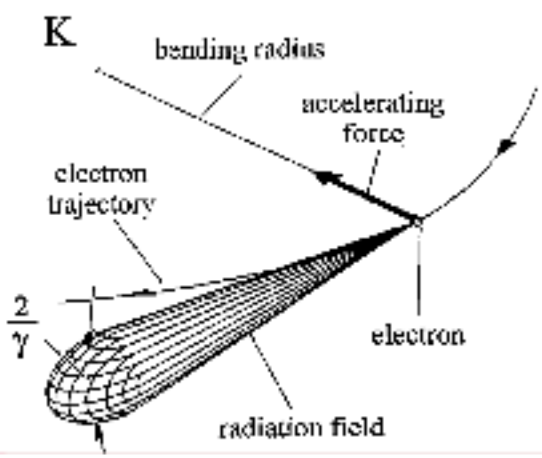


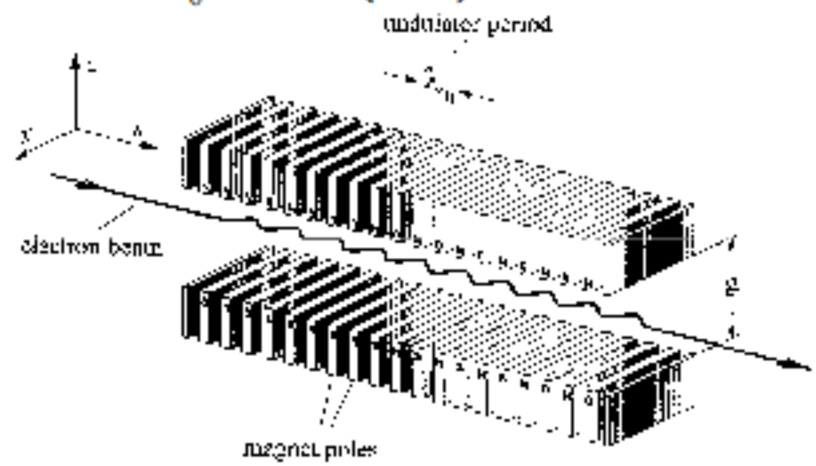


Photon flux in Bends and Undulator

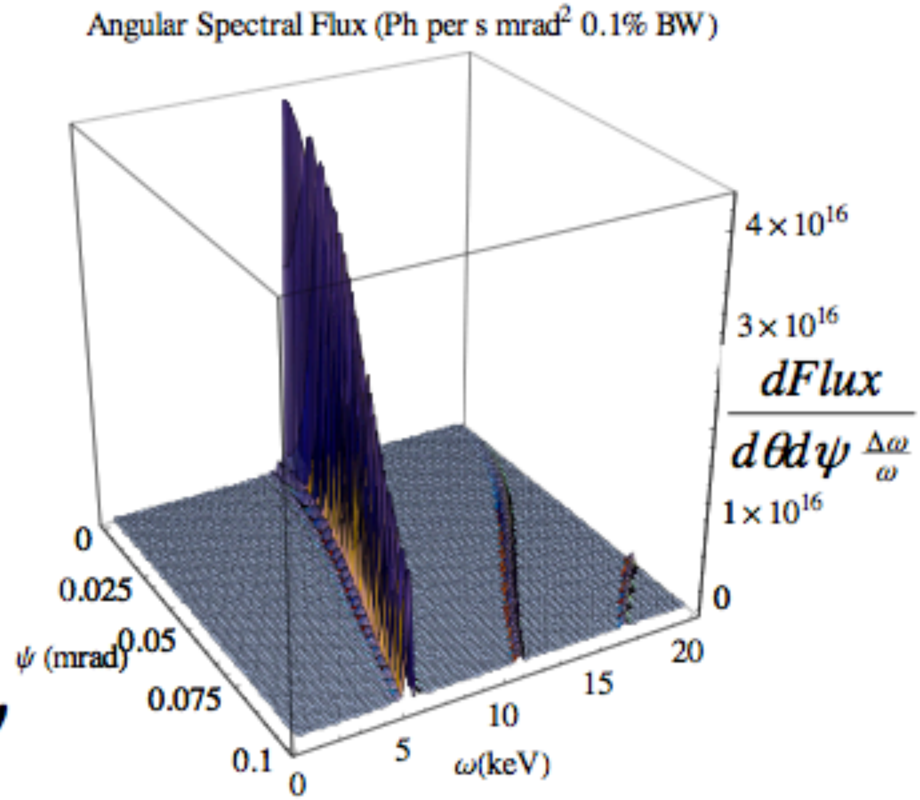








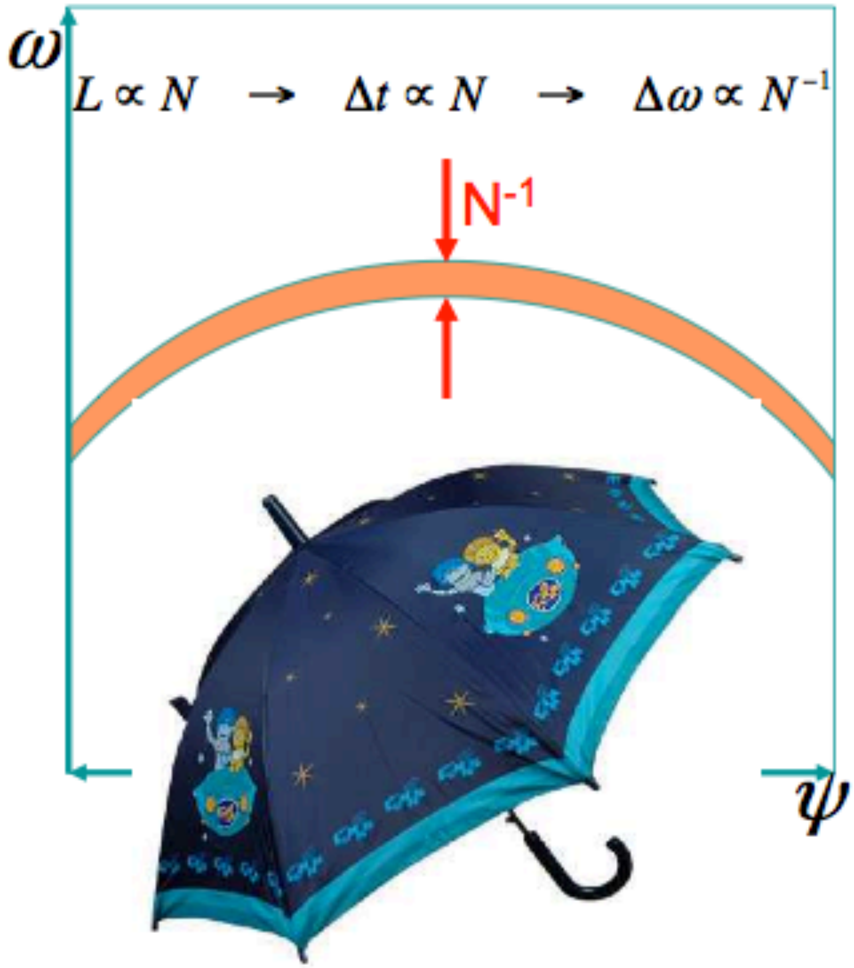
The umbrella of N-pole undulator radiation

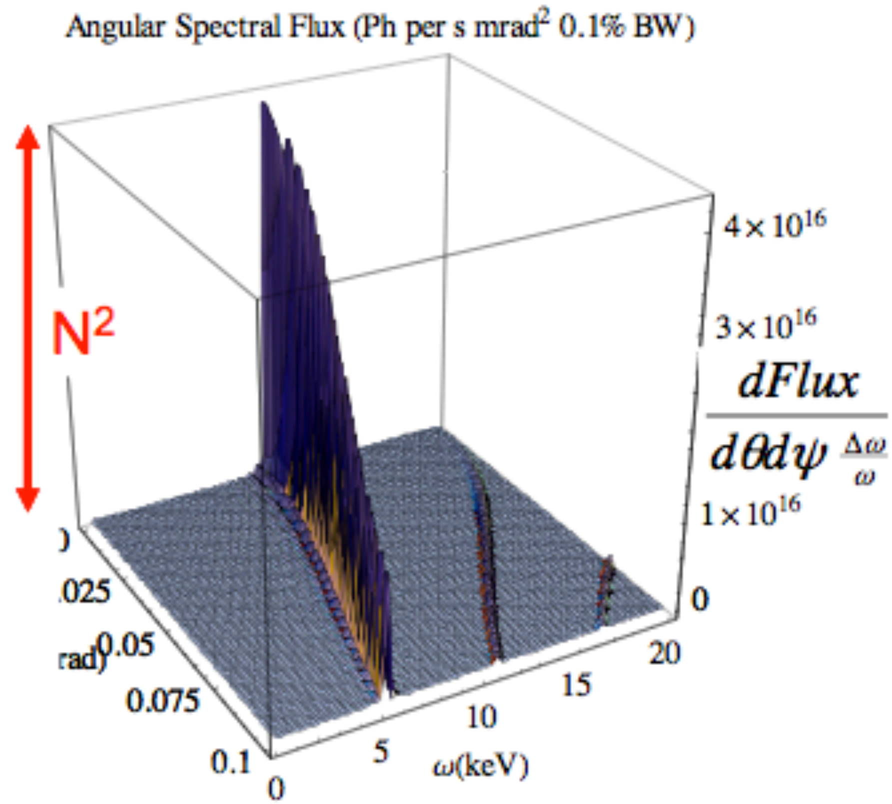


$$F_{total} \propto N$$

Flux from N poles is N times the flux from one pole

The umbrella of N-pole undulator radiation





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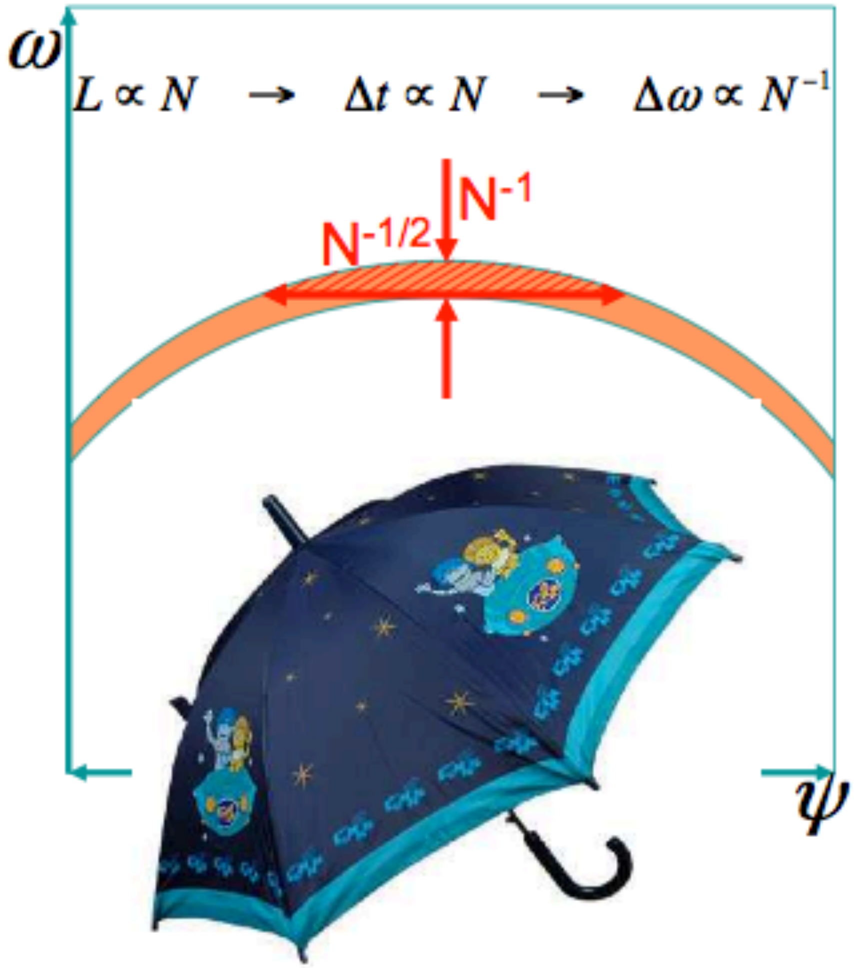
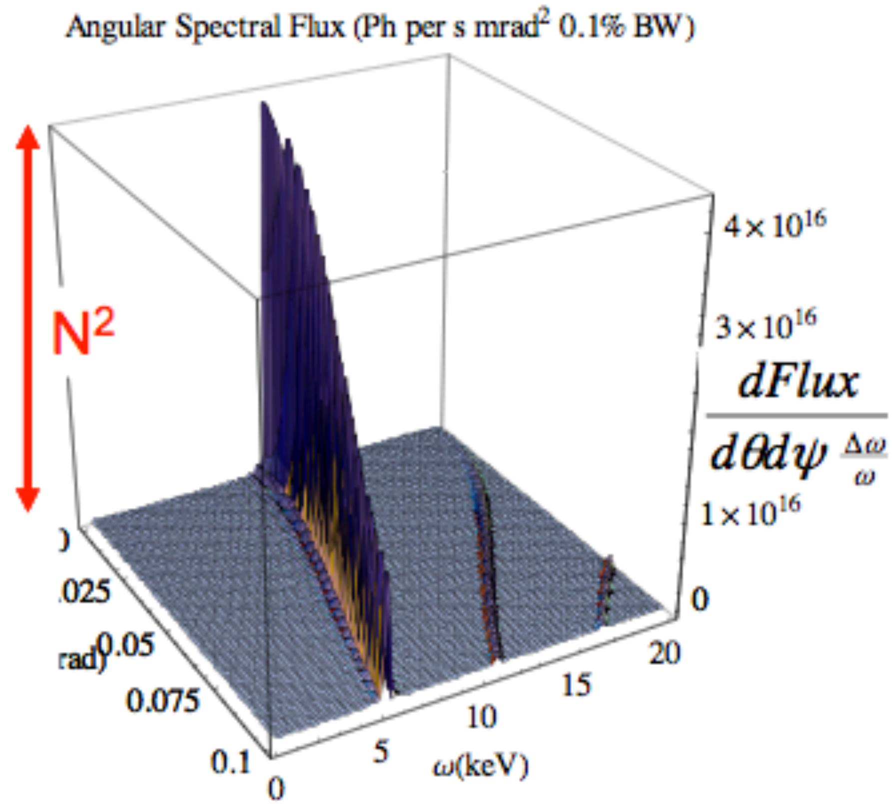
CHESS & LEPP



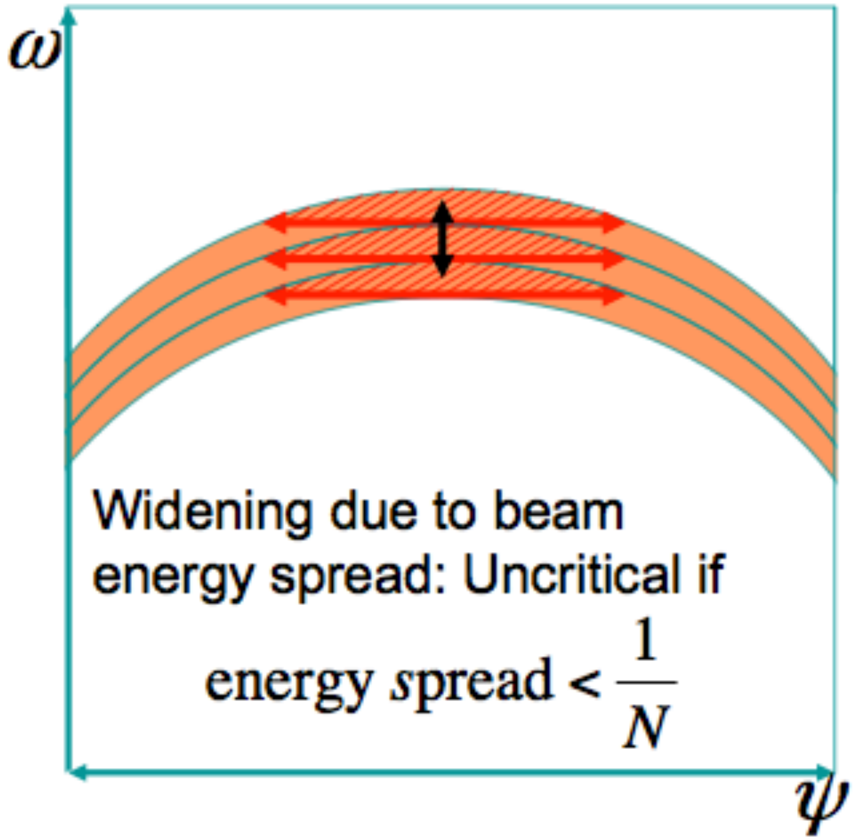


$$F_{total} \propto N$$

The power in the central cone is in dependent of N

Flux from N poles is N times the flux from one pole

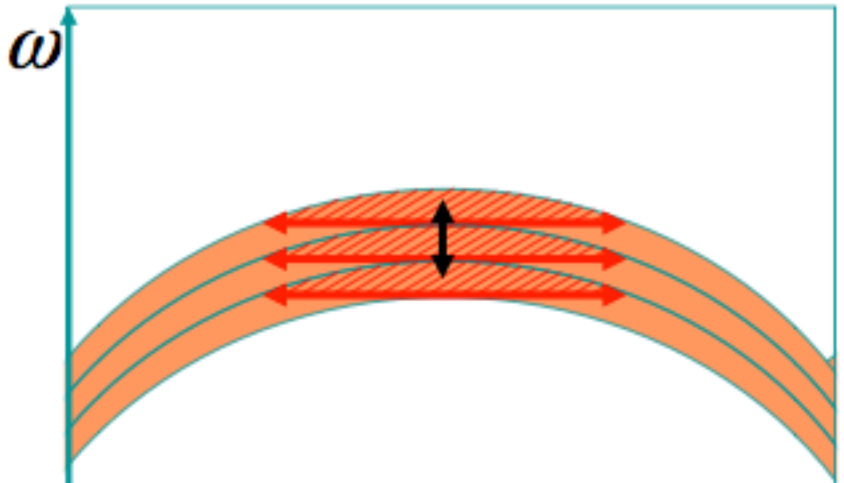
Brightness reduction by beam properties



Widening due to beam energy spread: Uncritical if

$$\text{energy spread} < \frac{1}{N}$$

Brightness reduction by beam properties



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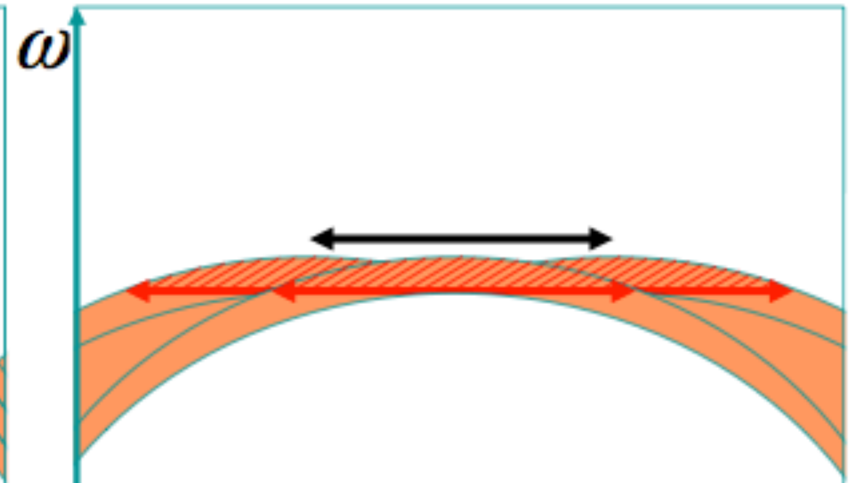
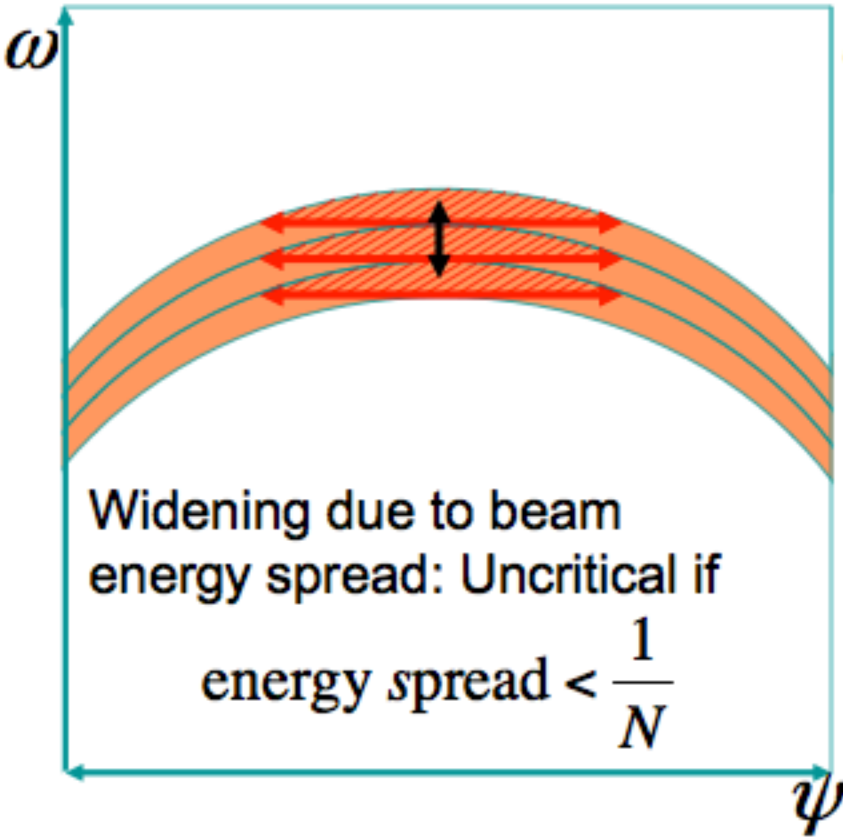


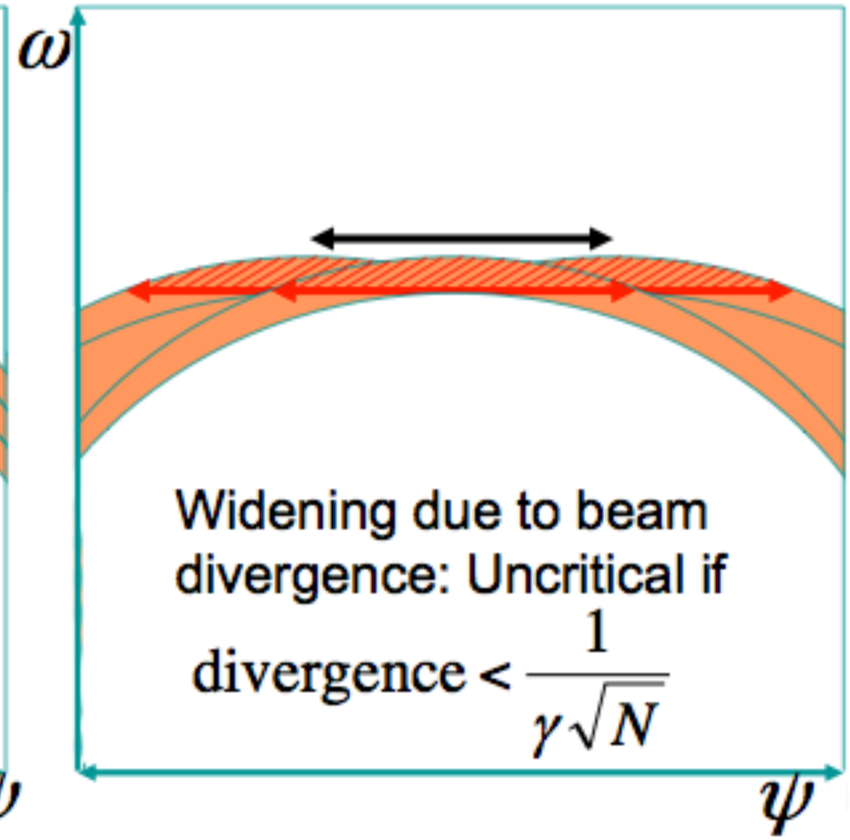
Widening due to beam divergence: Uncritical if

$$\text{divergence} < \frac{1}{\gamma\sqrt{N}}$$

ψ ψ

Brightness reduction by beam properties



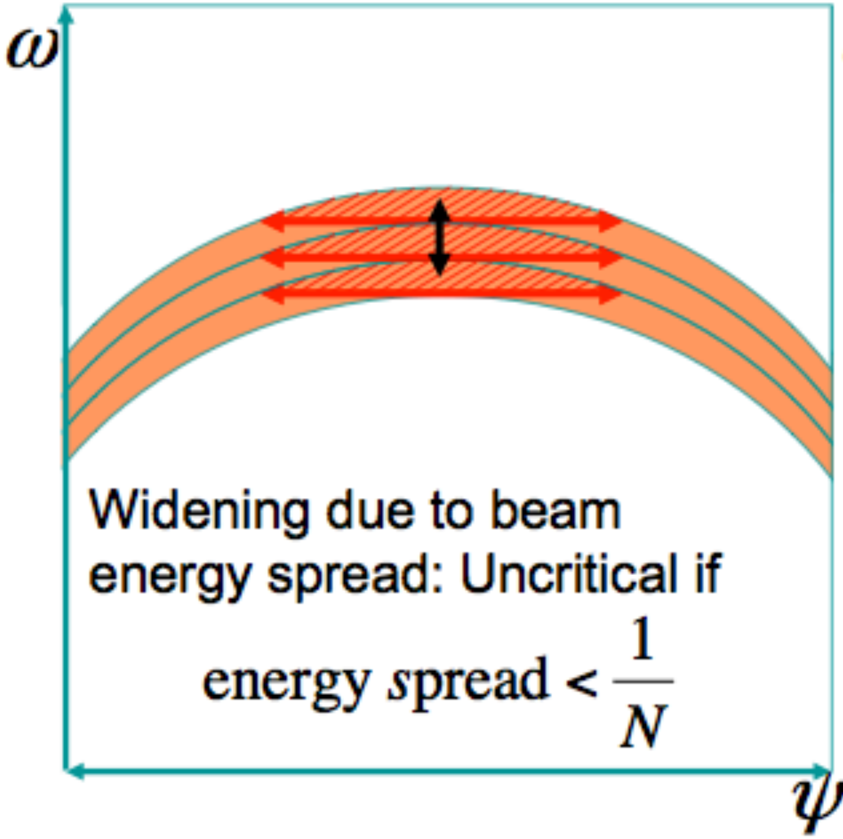


Field from a single electron cannot be distinguished from field from a spot with:

$$\text{spot size} < \frac{\lambda}{\text{divergence}}$$

Brightness reduction by beam properties

CHESS & LEPP



Widening due to beam energy spread: Uncritical if

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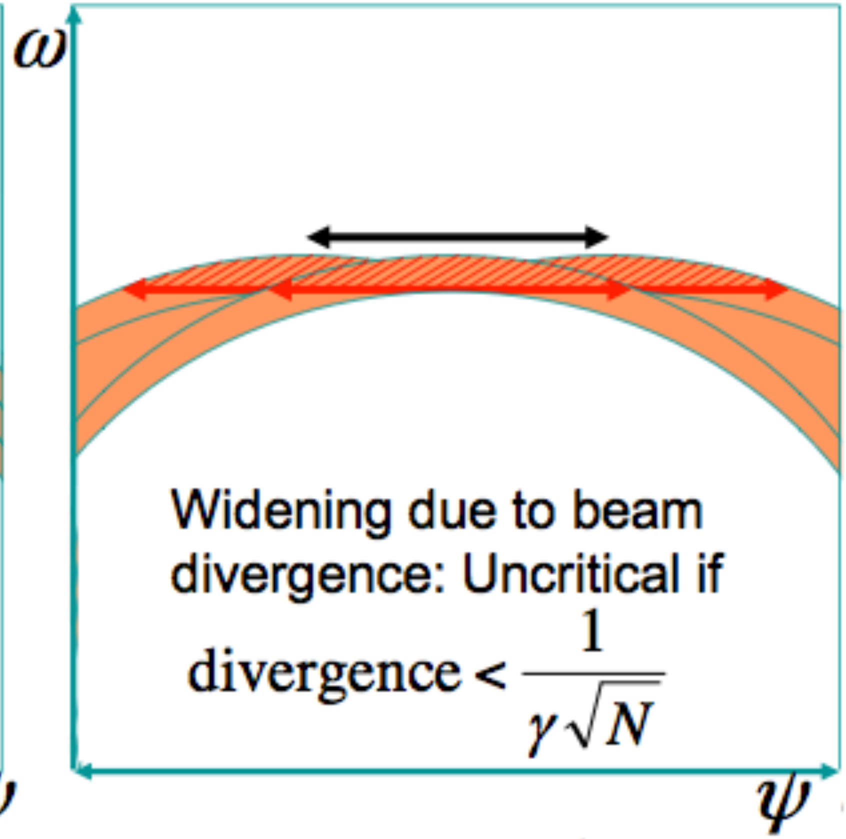

Widening due to beam divergence: Uncritical if

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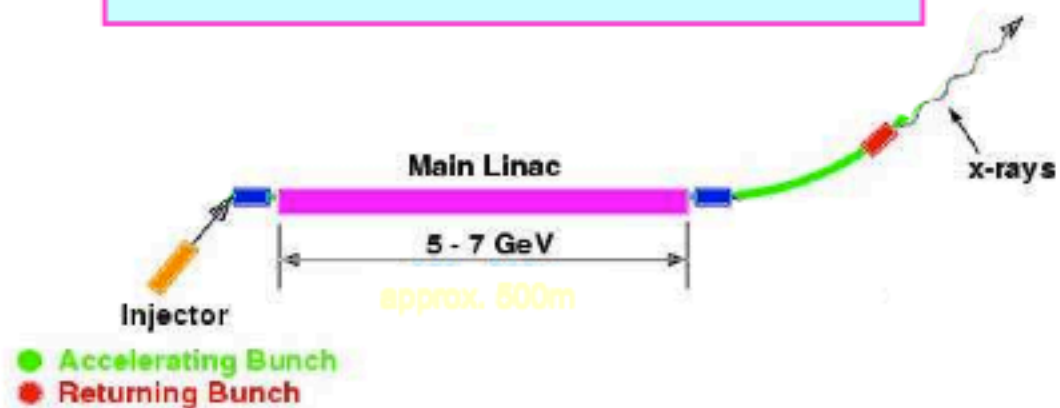
To take advantage of many undulator poles, the electron beam needs to have little energy spread, little divergence, and small beam size.

Principle of an X-ray ERL

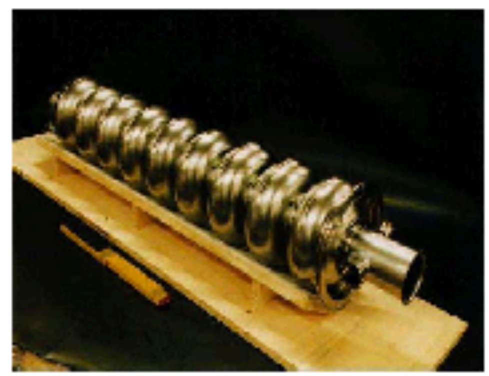
X-ray analysis with highest resolution in space and time:

5GV*100mA = 0.5GW
(good size power plant)



Challenges:

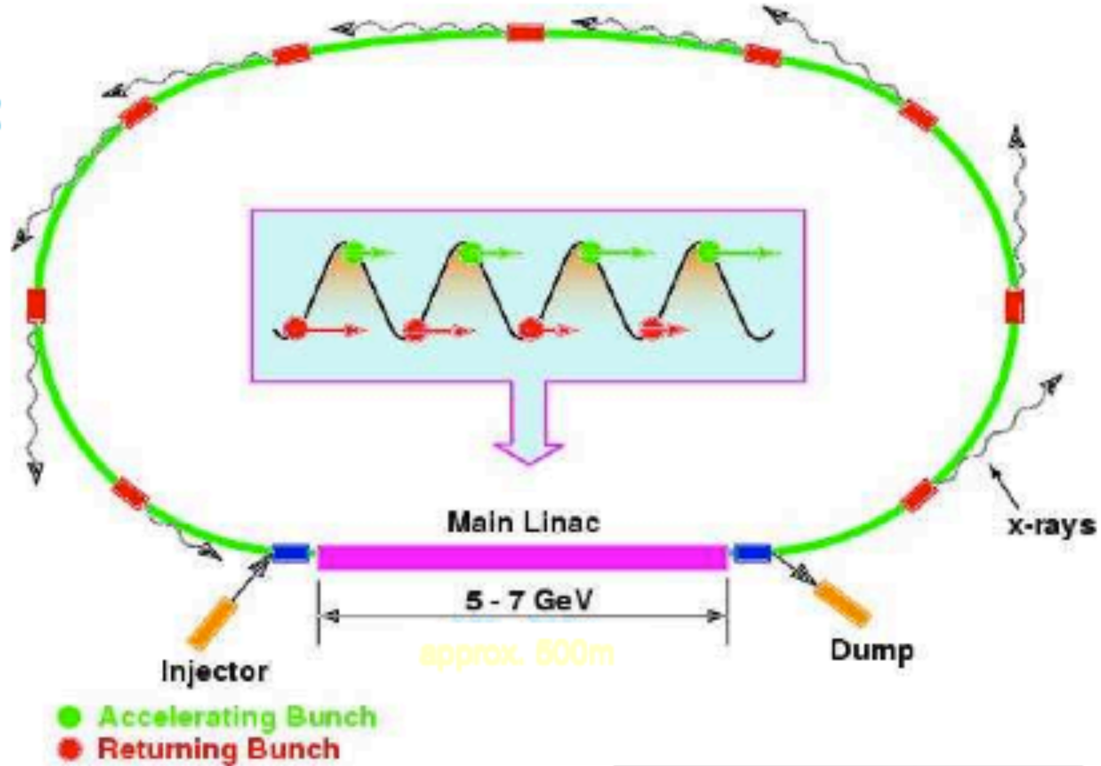
- Low emittance, high current creation
- Emittance preservation
- Beam stability at insertion devices
- Accelerator design
- Component properties, e.g. SRF



Principle of an X-ray ERL

Principle of an X-ray ERL

X-ray analysis with highest resolution in space and time:



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