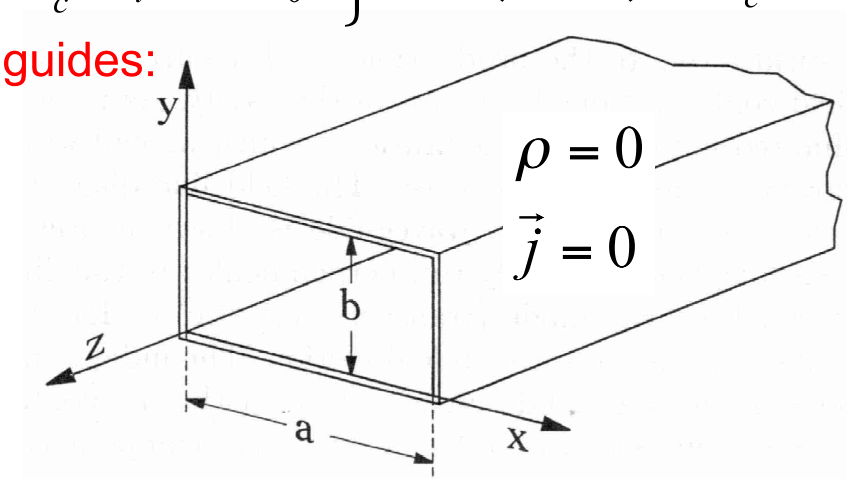
RF in Accelerators

$$\left.\begin{aligned}\vec\nabla\times\vec E &= -\partial_t\vec B\\ \vec\nabla\times\vec B &= \tfrac{1}{c^2}\partial_t\vec E+\mu_0\vec j\end{aligned}\right\}\begin{aligned}\vec\nabla\times\left(\vec\nabla\times\vec E\right) &= -\tfrac{1}{c^2}\partial_t^2\vec E-\mu_0\partial_t\vec j\\ \vec\nabla\times\left(\vec\nabla\times\vec B\right) &= -\tfrac{1}{c^2}\partial_t^2\vec B-\mu_0\vec\nabla\times\vec j\end{aligned}$$

Wave guides:



Wave equation for all components

$$\vec\nabla^2\vec E = \tfrac{1}{c^2}\partial_t^2\vec E$$
$$\vec\nabla^2\vec B = \tfrac{1}{c^2}\partial_t^2\vec B$$

$$\left.\begin{aligned}\vec\nabla\times\vec E &= -\partial_t\vec B\\ \vec\nabla\times\vec B &= \tfrac{1}{c^2}\partial_t\vec E\end{aligned}\right\}\begin{aligned}\vec\nabla_\perp\times\vec E_\perp &= -\partial_t\vec B_z\\ \vec\nabla_\perp\times\vec B_\perp &= \tfrac{1}{c^2}\partial_t\vec E_z\end{aligned}$$

$$\left.\begin{aligned}\vec\nabla\cdot\vec E &= 0\\ \vec\nabla\cdot\vec B &= 0\end{aligned}\right\}\begin{aligned}\vec\nabla_\perp\cdot\vec E_\perp+\partial_z E_z &= 0\\ \vec\nabla_\perp\cdot\vec B_\perp+\partial_z B_z &= 0\end{aligned}$$

Search for simple modes:

Transverse electric and magnetic (TEM) waves cannot exists, since:

$$E_z = 0 \text{ and } B_z = 0 \;\Rightarrow\; \vec E_\perp = const \;\text{ and } \vec B_\perp = const$$

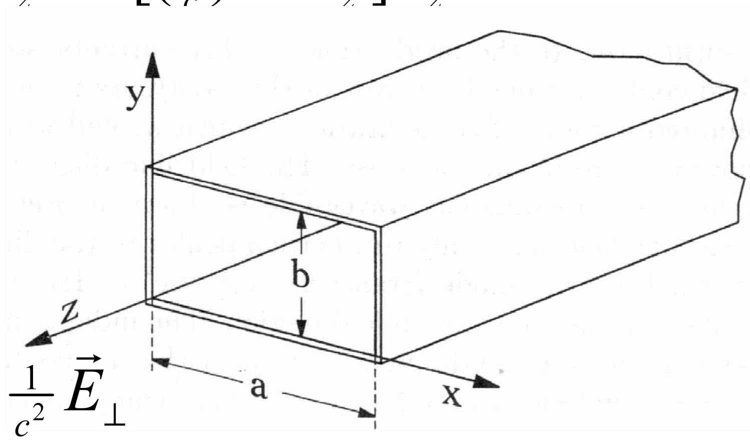
TE and TM Modes

Fourier expansion of the z-dependence: $\vec{E}(x,y,z,t) = \int \vec{E}_{k_z\omega}(x,y)e^{ik_z z - i\omega t}dk_z\, d\omega$

$$\vec{\nabla}^2\vec{E} = \tfrac{1}{c^2}\partial_t^2\vec{E} \qquad \Rightarrow \qquad \vec{\nabla}_\perp^2 E_z = -[(\tfrac{\omega}{c})^2 - k_z^2]E_z$$

$$\vec{\nabla}^2\vec{B} = \tfrac{1}{c^2}\partial_t^2\vec{B} \qquad \qquad \vec{\nabla}_\perp^2 B_z = -[(\tfrac{\omega}{c})^2 - k_z^2]B_z$$

Eigenvalue equation with boundary conditions:



Walls:

$$\vec{E}_{//} = 0 \quad \vec{B}_r = 0$$

$$E_z = 0 \quad \partial_r B_z = 0$$

$$\vec{\nabla}\times\vec{B} = \tfrac{1}{c^2}\partial_t\vec{E}$$

$$\vec{\nabla}_\perp \times \vec{B}_z + ik_z\vec{e}_z \times \vec{B}_\perp = -i\omega\tfrac{1}{c^2}\vec{E}_\perp$$

$$\vec{\nabla}_r \times \vec{B}_z + ik_z\vec{e}_z \times \vec{B}_r = -i\omega\tfrac{1}{c^2}\vec{E}_\varphi \Rightarrow \partial_r B_z = 0$$

Solutions for E or B only exist for a discrete set of eigenvalues: $(\tfrac{\omega}{c})^2 - k_z^2 = k_n^{(E)2}$

$$(\tfrac{\omega}{c})^2 - k_z^2 = k_n^{(B)2}$$

Due to different boundary conditions, E_z and B_z cannot simultaneously be nonzero.

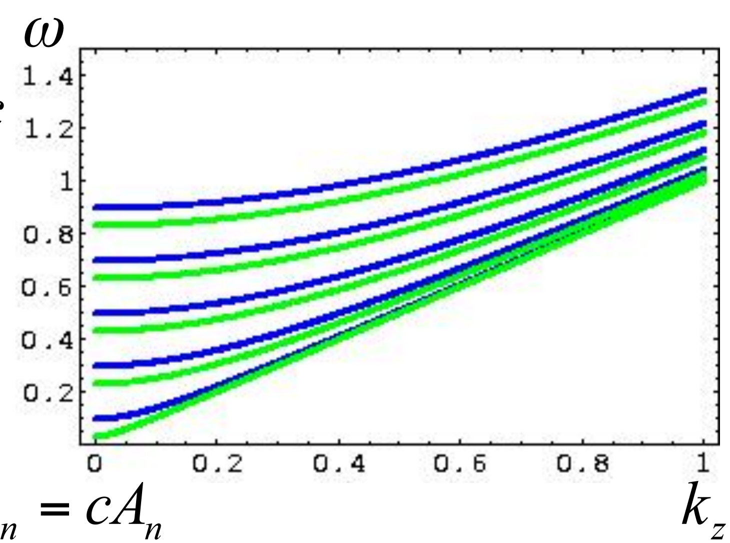
TE modes have $E_z = 0$ TM modes have $B_z = 0$

Dispersion relation

$$\omega(k_z) = c\sqrt{A_n^{\ 2} + k_z^2}$$

Phase velocity $v_{ph} = \omega / k_z = c\sqrt{1 + \left(\frac{A_n}{k_z}\right)^2} > c$

Group velocity $v_{gr} = d\omega / dk_z = c / \sqrt{1 + \left(\frac{A_n}{k_z}\right)^2} < c$



For each excitation frequency ω one obtains a propagation in the wave guide of

$$e^{ik_z z}\ , \quad k_z = \sqrt{\left(\tfrac{\omega}{c}\right)^2 - A_n^2}$$

Transport for ω above the cutoff frequency $\omega > \omega_n = cA_n$

Damping for ω below the cutoff frequency $\omega < \omega_n = cA_n$

Transverse fields

TE-modes: Once B_z is know, the full field can be found

$$\vec{\nabla}_\perp \times \vec{E}_z + ik_z\vec{e}_z \times \vec{E}_\perp = i\omega\vec{B}_\perp$$

$$\vec{\nabla}_\perp \times \vec{B}_z + ik_z\vec{e}_z \times \vec{B}_\perp = -i\omega\tfrac{1}{c^2}\vec{E}_\perp$$

$$k_z\vec{E}_\perp + \omega\vec{e}_z \times \vec{B}_\perp = \vec{e}_z \times (\vec{\nabla}_\perp \times \vec{E}_z) = -i\vec{\nabla}_\perp E_z$$

$$\omega\tfrac{1}{c^2}\vec{E}_\perp + k_z\vec{e}_z \times \vec{B}_\perp = i\vec{\nabla}_\perp \times \vec{B}_z$$

$$(\tfrac{\omega^2}{c^2} - k_z^2)\vec{E}_\perp = i(k_z\vec{\nabla}_\perp E_z + \omega\vec{\nabla}_\perp \times \vec{B}_z)$$

$$(\tfrac{\omega^2}{c^2} - k_z^2)\vec{e}_z \times \vec{B}_\perp = -i(\tfrac{\omega}{c^2}\vec{\nabla}_\perp E_z + k_z\vec{\nabla}_\perp \times \vec{B}_z)$$

$$\boxed{\begin{aligned}\vec{E}_\perp &= \frac{i}{\frac{\omega^2}{c^2} - k_z^2}(k_z\vec{\nabla}_\perp E_z + \omega\vec{\nabla}_\perp \times \vec{B}_z)\\ \vec{B}_\perp &= \frac{i}{\frac{\omega^2}{c^2} - k_z^2}(k_z\vec{\nabla}_\perp B_z - \tfrac{\omega}{c^2}\vec{\nabla}_\perp \times E_z)\end{aligned}}$$

Rectangular Wave Guide

Boundary conditions:

$$E_z(\vec{x}_0) = 0 \quad \vec{\nabla}_\perp^2 E_z = [k_z^2 - (\tfrac{\omega}{c})^2]E_z$$

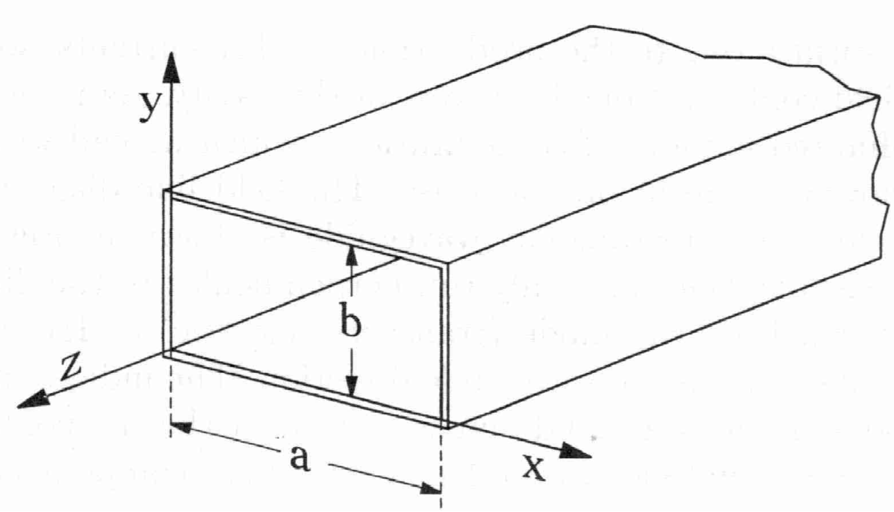
$$E_z(\vec{x}) = E_0 \sin(\tfrac{n\pi}{a}x)\sin(\tfrac{m\pi}{b}y)$$

$$(\tfrac{\omega}{c})^2 - k_z^2 = k_{nm}^{(B)2} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

$$\partial_r B_z(\vec{x}_0) = 0 \quad \vec{\nabla}_\perp^2 B_z = [k_z^2 - (\tfrac{\omega}{c})^2]B_z$$

$$B_z(\vec{x}) = B_0 \cos(\tfrac{n\pi}{a}x)\cos(\tfrac{m\pi}{b}y)$$

$$(\tfrac{\omega}{c})^2 - k_z^2 = k_{nm}^{(E)2} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$



TE and TM modes happen to have the same eigenvalues.

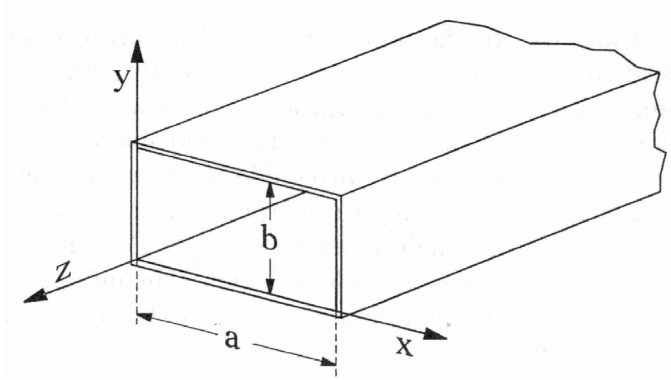
For simplicity one still looks at TE and TM modes separately.

Rectangular TE Modes

$$E_z(\vec{x}) = 0\,, \quad B_z(\vec{x}) = B_0 \cos(\tfrac{n\pi}{a}x)\cos(\tfrac{m\pi}{b}y)$$

$$\vec{E}_\perp = \frac{i}{\frac{\omega^2}{c^2}-k_z^2}(k_z\vec{\nabla}_\perp E_z + \omega\vec{\nabla}_\perp \times \vec{B}_z)$$

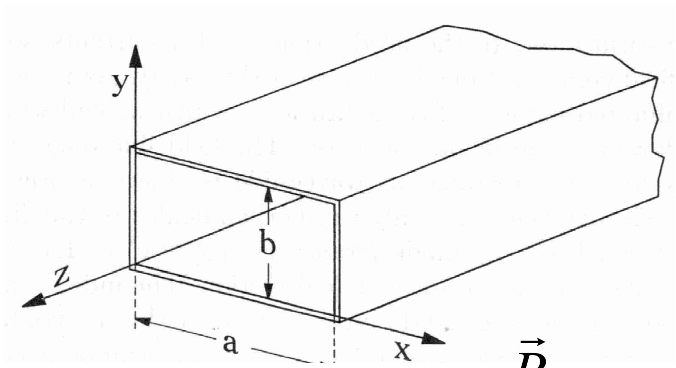
$$\vec{B}_\perp = \frac{i}{\frac{\omega^2}{c^2}-k_z^2}(k_z\vec{\nabla}_\perp B_z - \tfrac{\omega}{c^2}\vec{\nabla}_\perp \times \vec{E}_z)$$

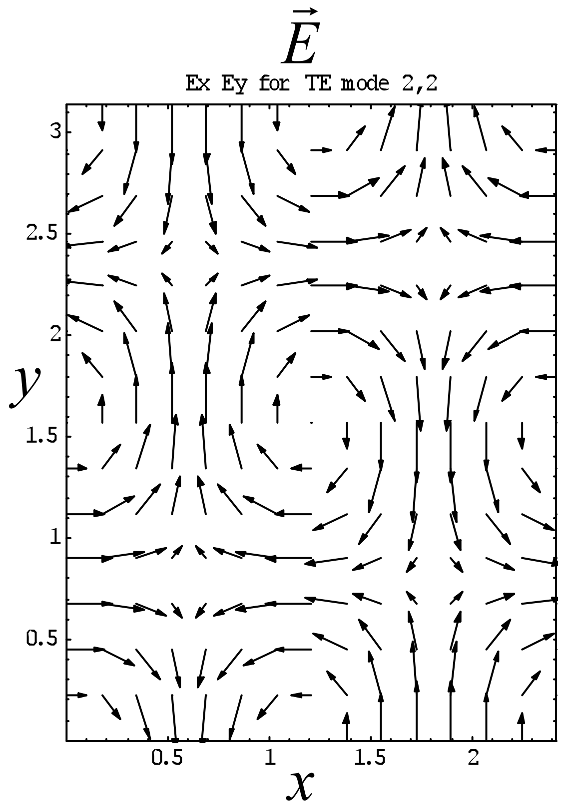


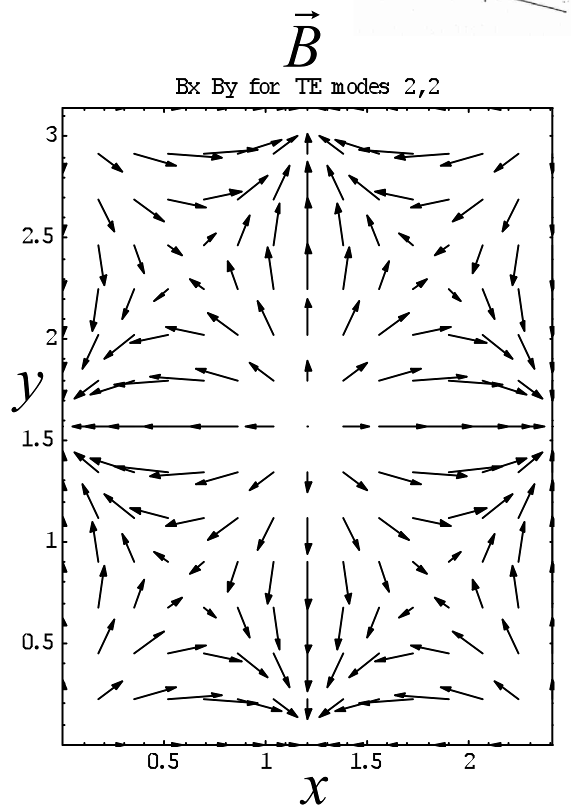
$$\vec{E}(\vec{x}) = \frac{\omega}{k_{nm}^{(E)2}} B_0 \begin{pmatrix} \frac{m\pi}{b}\cos(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ -\frac{n\pi}{a}\sin(\frac{n\pi}{a}x)\cos(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ 0 \end{pmatrix}$$

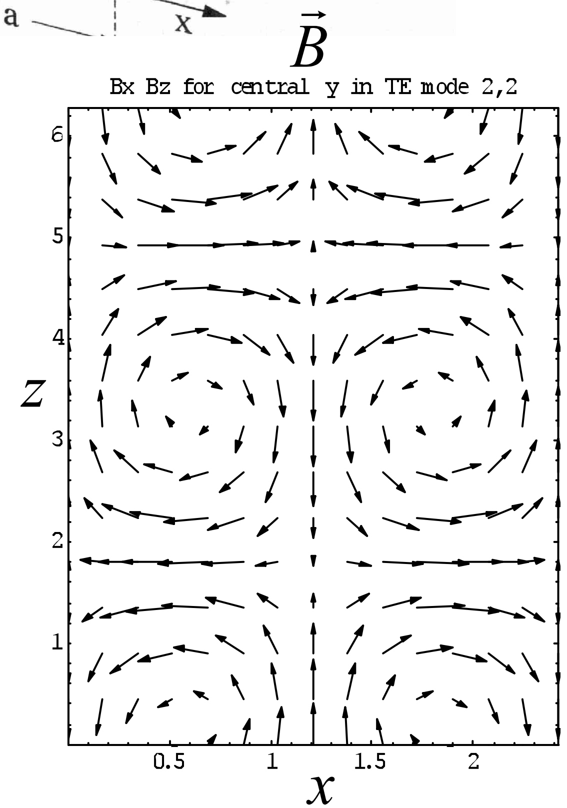
$$\vec{B}(\vec{x}) = \frac{k_z}{k_{nm}^{(E)2}} B_0 \begin{pmatrix} \frac{n\pi}{a}\sin(\frac{n\pi}{a}x)\cos(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ \frac{m\pi}{b}\cos(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ \frac{k_{nm}^{(E)2}}{k_z}\cos(\frac{n\pi}{a}x)\cos(\frac{m\pi}{b}y)\cos(k_z z-\omega t) \end{pmatrix}$$

Rectangular TE_{22} Mode







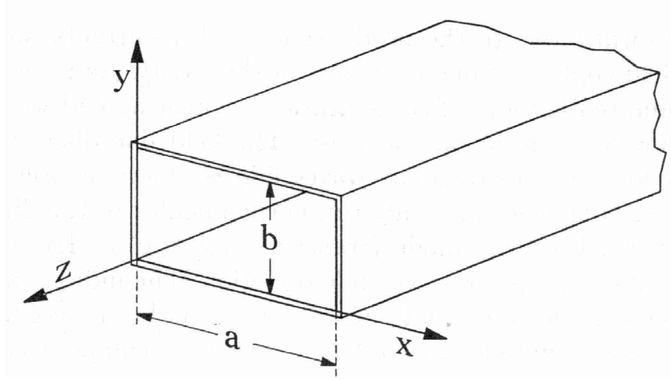


Rectangular TM Modes

$$E_z(\vec{x}) = E_0 \sin(\tfrac{n\pi}{a}x)\sin(\tfrac{m\pi}{b}y)\,, \quad B_z = 0$$

$$\vec{E}_\perp = \frac{i}{\frac{\omega^2}{c^2}-k_z^2}(k_z\vec{\nabla}_\perp E_z + \omega\vec{\nabla}_\perp \times \vec{B}_z)$$

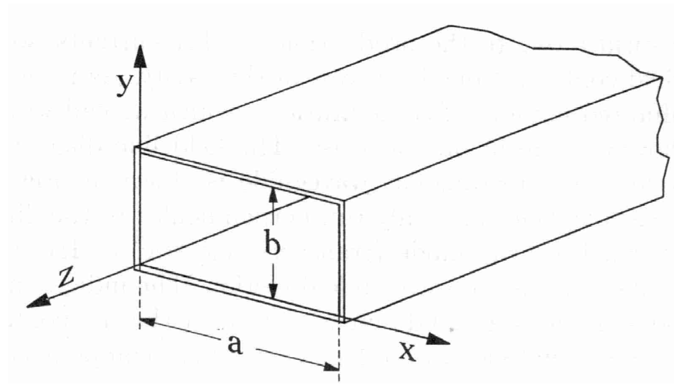
$$\vec{B}_\perp = \frac{i}{\frac{\omega^2}{c^2}-k_z^2}(k_z\vec{\nabla}_\perp B_z - \tfrac{\omega}{c^2}\vec{\nabla}_\perp \times E_z)$$

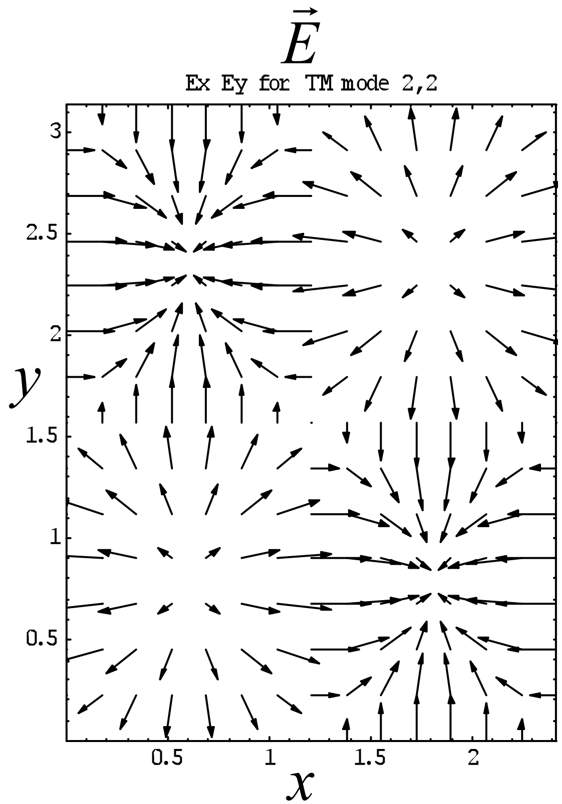


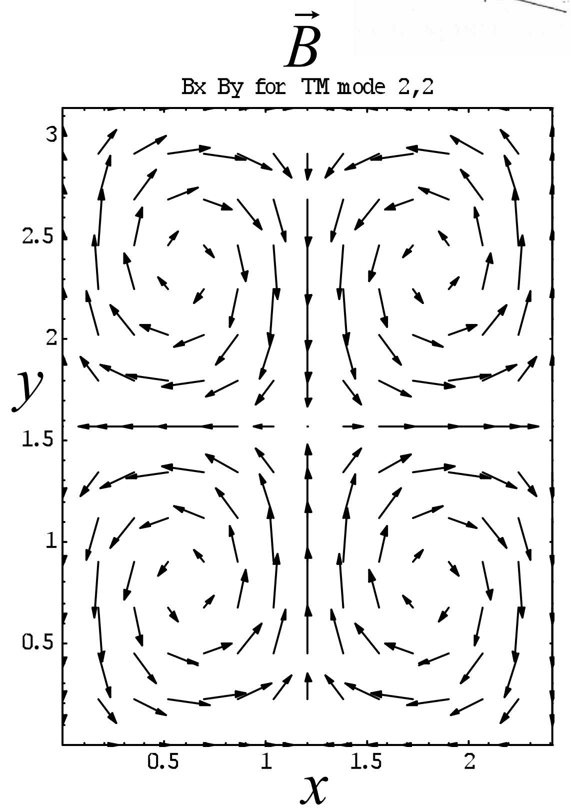
$$\vec{E}(\vec{x}) = \frac{k_z}{k_{nm}^{(E)2}} E_0 \begin{pmatrix} -\frac{n\pi}{a}\cos(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ -\frac{m\pi}{b}\sin(\frac{n\pi}{a}x)\cos(\frac{m\pi}{b}y)\sin(k_z z-\omega t) \\ \frac{k_{nm}^{(E)2}}{k_z}\sin(\frac{n\pi}{a}x)\sin(\frac{m\pi}{b}y)\cos(k_z z-\omega t) \end{pmatrix}$$

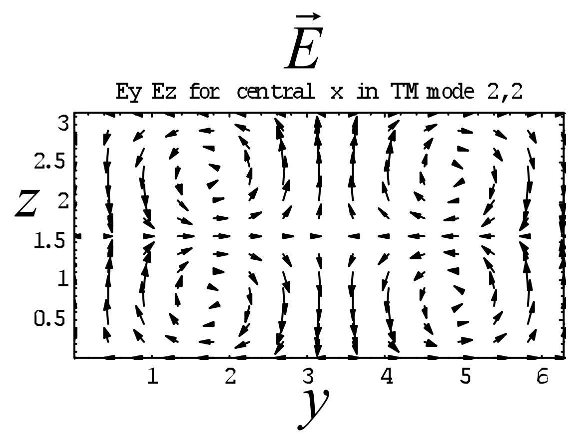
$$\vec{B}(\vec{x}) = \frac{\omega}{c^2 k_{nm}^{(E)2}} E_0 \begin{pmatrix} \frac{m\pi}{b}\sin(\frac{n\pi}{a}x)\cos(\frac{m\pi}{b}y)\ \sin(k_z z-\omega t) \\ -\frac{n\pi}{a}\cos(\frac{n\pi}{a}x)\ \sin(\frac{m\pi}{b}y)\ \sin(k_z z-\omega t) \\ 0 \end{pmatrix}$$

Rectangular TM_{22} Mode

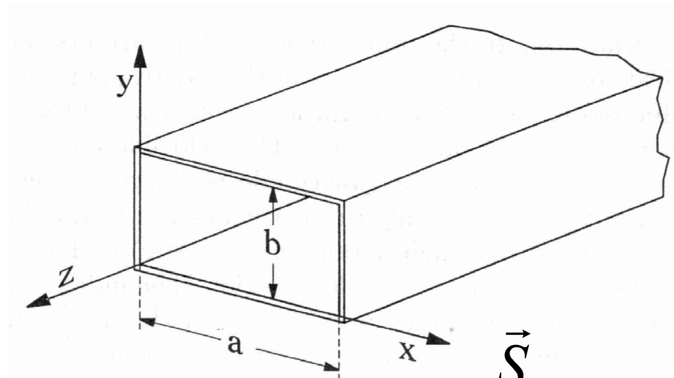




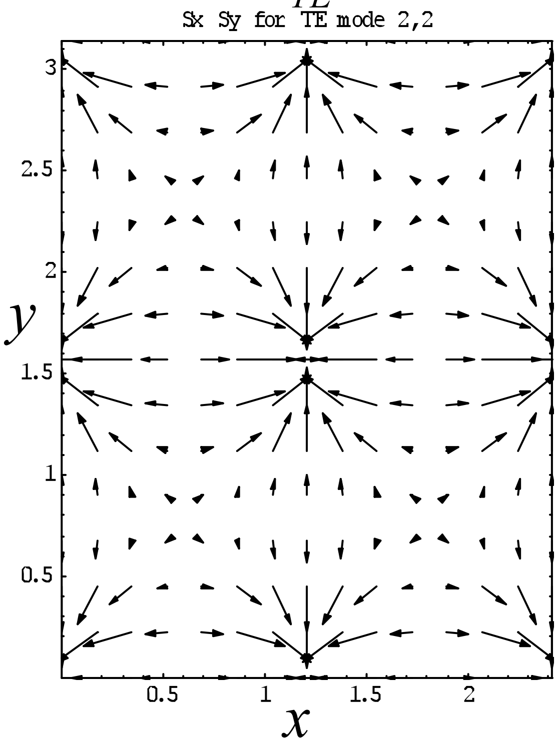




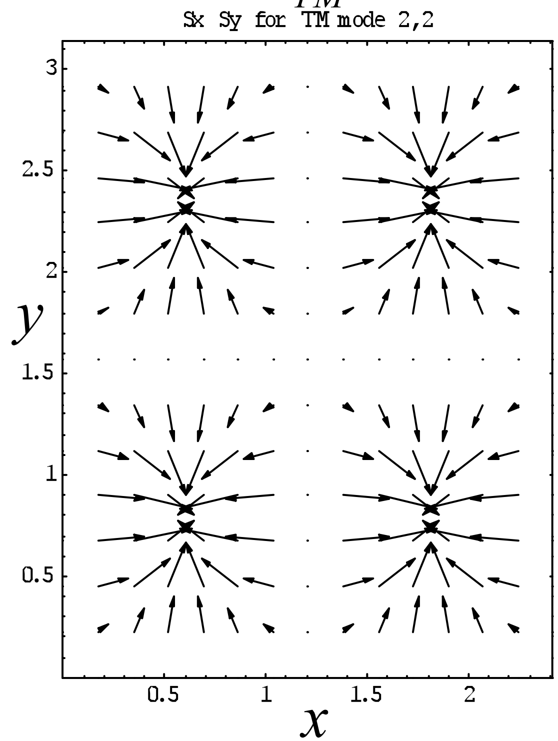
Pointing Vector



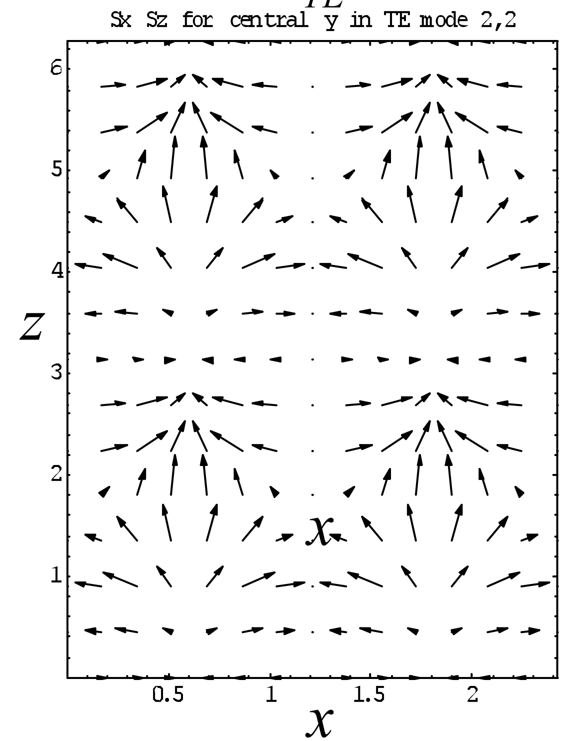
$\vec{S}_{TE}$



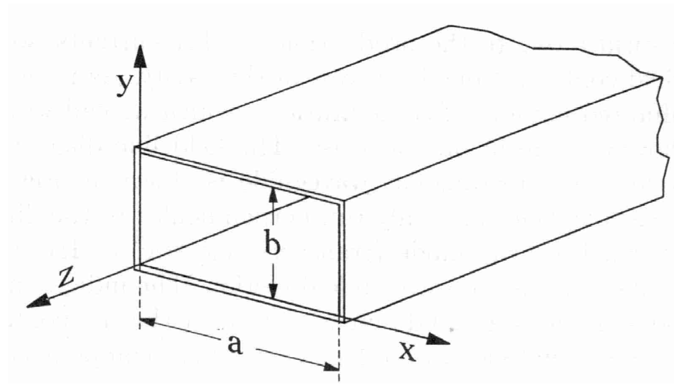
$\vec{S}_{TM}$



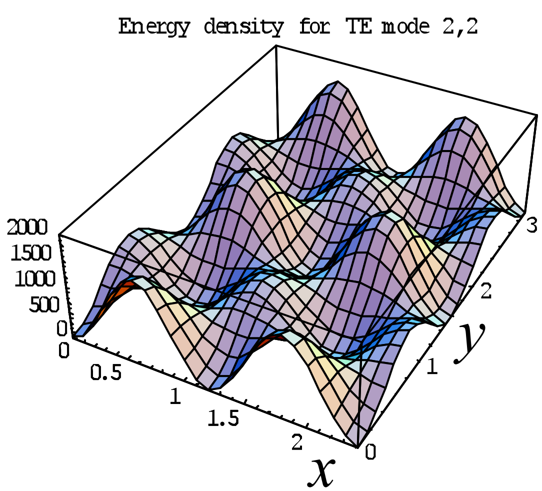
$\vec{S}_{TE}$



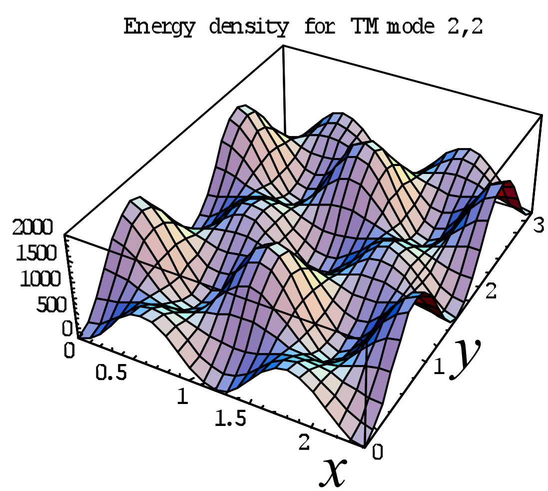
Energy Density



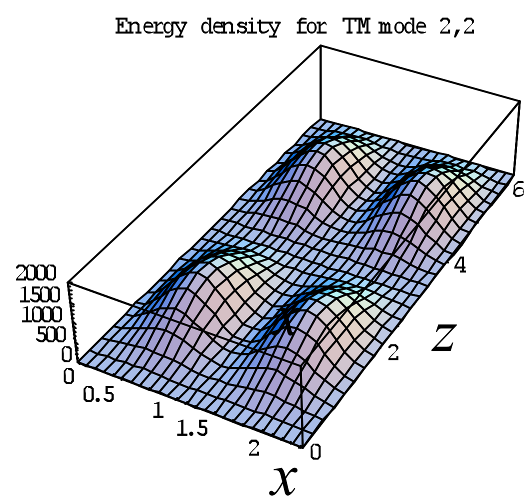
U_{TE}



U_{TM}



U_{TM}



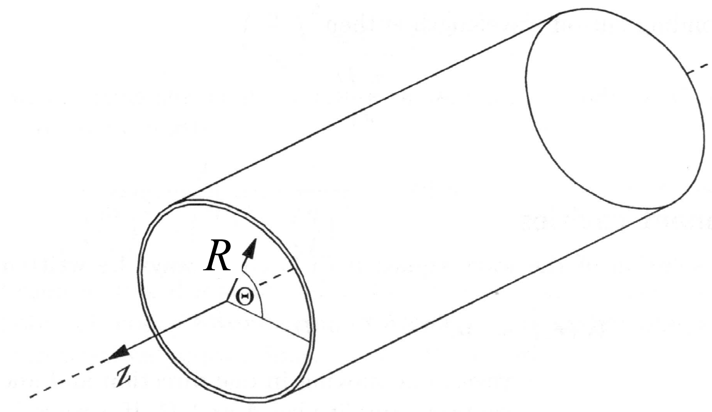
Cylindrical Wave Guides

TM Modes:

$$E_z(\vec{x}_0)=0 \qquad \vec{\nabla}_\perp^2 E_z=[k_z^2-(\tfrac{\omega}{c})^2]E_z$$

$$(\partial_r^2+\tfrac{1}{r}\partial_r+\tfrac{1}{r^2}\partial_\varphi^2)E_z=[k_z^2-(\tfrac{\omega}{c})^2]E_z$$

$$(\xi^2\partial_\xi^2+\xi\partial_\xi+\xi^2-n^2)E_z=0\,,\quad \xi=k_{nm}^{(B)}\,r$$



$$E_z(\vec{x})=E_0 J_n(k_{nm}^{(B)}r)e^{in\varphi} \qquad k_{nm}^{(B)}=\frac{Z_{nm}}{R}$$ with the m^th 0 of the n^th Bessel function

Notation: TM_{nm} Mode

TE Modes:

$$\partial_r B_z(\vec{x}_0)=0 \qquad \vec{\nabla}_\perp^2 B_z=[k_z^2-(\tfrac{\omega}{c})^2]B_z$$

$$B_z(\vec{x})=B_0 J_n(k_{nm}^{(E)}r)e^{in\varphi} \qquad k_{nm}^{(E)}=\frac{S_{nm}}{R}$$ with the m^th extremum of J_n

Notation: TE_{nm} Mode

Fundamental Mode

Mode for particle acceleration: TM_{01}

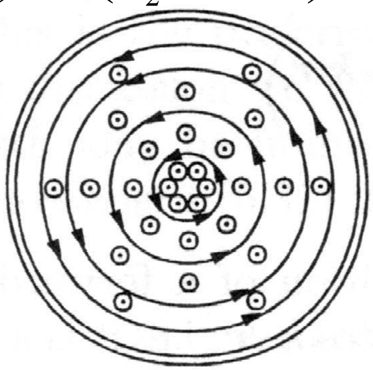
$$E_z(\vec{x}) = E_z J_0(\tfrac{r}{r_0})\cos(k_z z - \omega t)$$

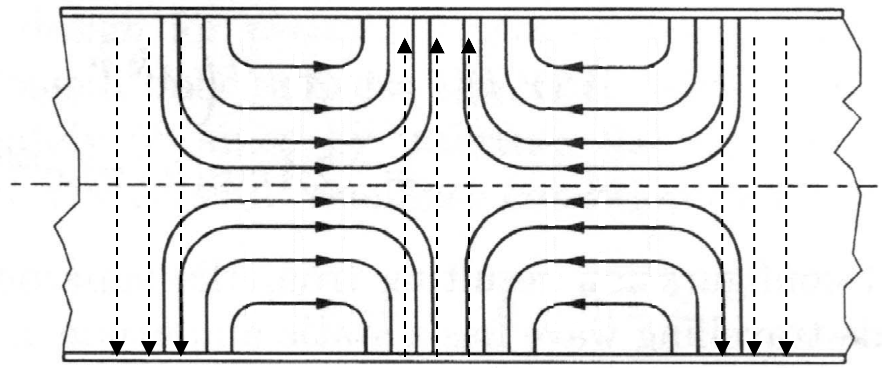
$$E_r(\vec{x}) = -E_z r_1 k_z J_0'(\tfrac{r}{r_1})\sin(k_z z - \omega t)$$

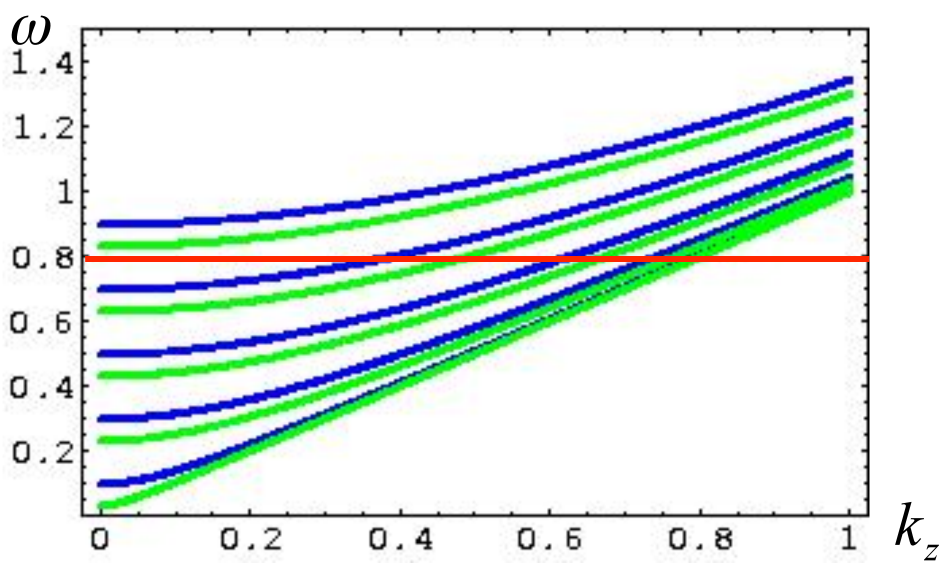
$$E_\varphi(\vec{x}) = 0$$

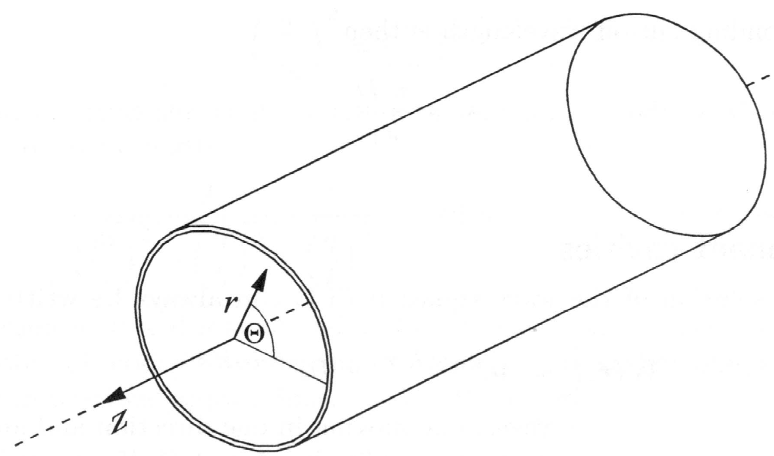
$$B_r(\vec{x}) = 0$$

$$B_\varphi(\vec{x}) = -E_z r_1 \tfrac{\omega}{c^2} J_0'(\tfrac{r}{r_1})\sin(k_z z - \omega t)$$







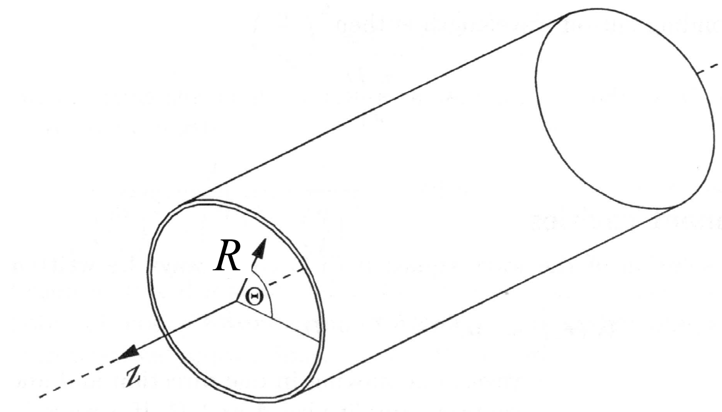


Cylindrical Wave TE Modes

$$E_z(\vec{x}) = 0\,, \quad B_z(\vec{x}) = B_0 J_n(\tfrac{S_{nm}}{R} r) e^{in\varphi}$$

$$\vec{E}_\perp = \frac{i}{\frac{\omega^2}{c^2} - k_z^2}(k_z \vec{\nabla}_\perp E_z + \omega \vec{\nabla}_\perp \times \vec{B}_z)$$

$$\vec{B}_\perp = \frac{i}{\frac{\omega^2}{c^2} - k_z^2}(k_z \vec{\nabla}_\perp B_z - \tfrac{\omega}{c^2} \vec{\nabla}_\perp \times E_z)$$



$$E_r = i\omega\left(\frac{R}{S_{nm}}\right)^2 \frac{1}{r}\partial_\varphi B_z = -B_0 n\omega R \frac{1}{S_{nm}^2}\frac{R}{r} J_n(\tfrac{S_{nm}}{R} r)\cos(n\varphi + k_z z - \omega t)$$

$$E_\varphi = -i\omega\left(\frac{R}{S_{nm}}\right)^2 \partial_r B_z = B_0 \omega R \frac{1}{S_{nm}} J'_n(\tfrac{S_{nm}}{R} r)\sin(n\varphi + k_z z - \omega t)$$

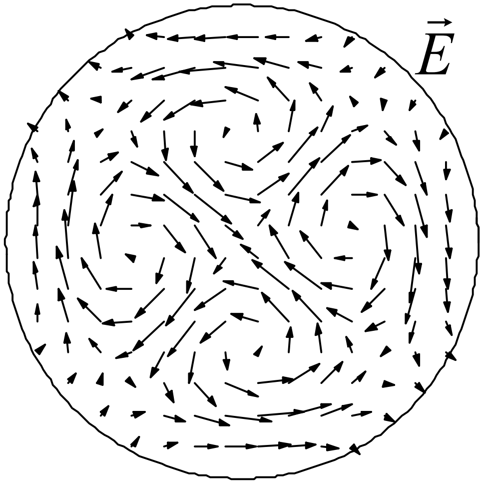
$$E_z = 0$$

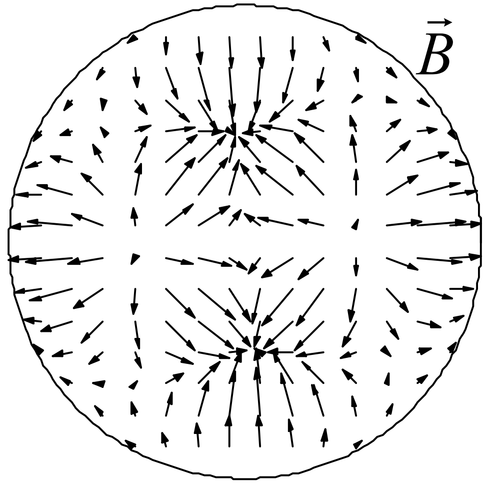
$$B_r = ik_z\left(\frac{R}{S_{nm}}\right)^2 \partial_r B_z = -B_0 k_z R \frac{1}{S_{nm}} J'_n(\tfrac{S_{nm}}{R} r)\sin(n\varphi + k_z z - \omega t)$$

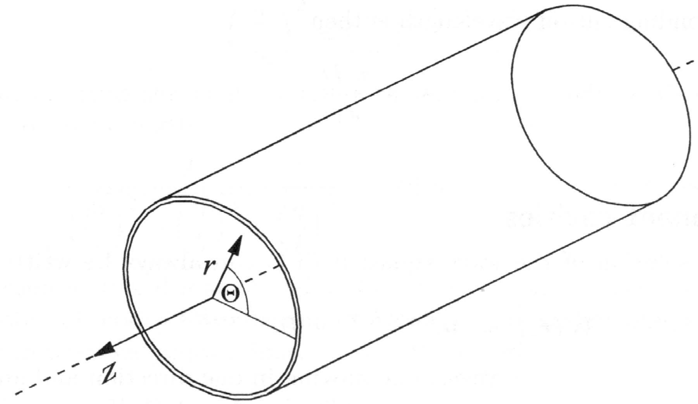
$$B_\varphi = ik_z\left(\frac{R}{S_{nm}}\right)^2 \frac{1}{r}\partial_\varphi B_z = -B_0 n k_z R \frac{1}{S_{nm}^2}\frac{R}{r} J_n(\tfrac{S_{nm}}{R} r)\cos(n\varphi + k_z z - \omega t)$$

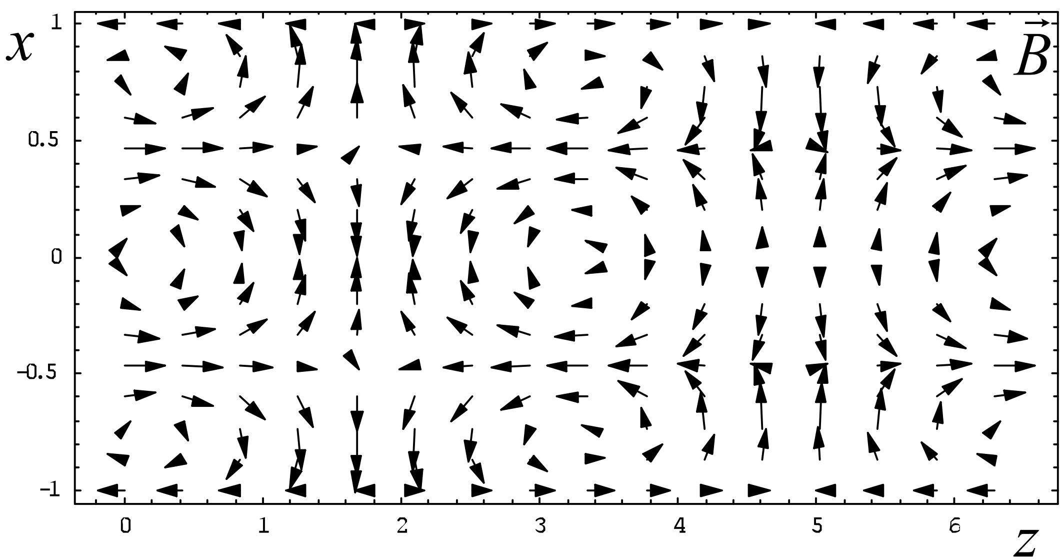
$$B_z = B_0 J_n(\tfrac{S_{nm}}{R} r)\cos(n\varphi + k_z z - \omega t)$$

Rectangular TE_{22} Mode









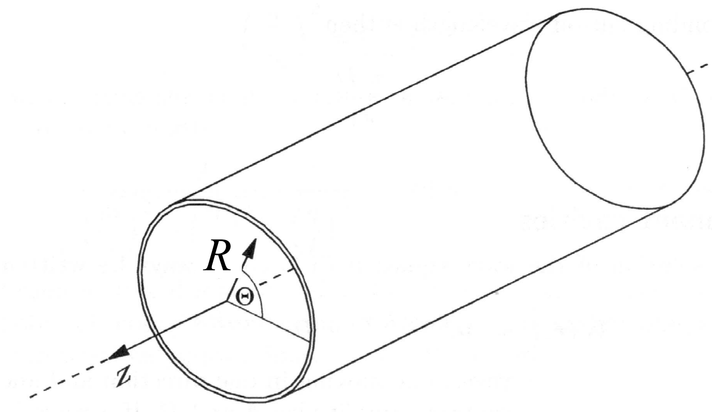
Cylindrical Wave TM Modes

CHESS & LEPP

$$E_z(\vec{x}) = E_0 J_n(\tfrac{Z_{nm}}{R} r) e^{in\varphi} \ , \quad B_z(\vec{x}) = 0$$

$$\vec{E}_\perp = \frac{i}{\frac{\omega^2}{c^2} - k_z^2}(k_z \vec{\nabla}_\perp E_z + \omega \vec{\nabla}_\perp \times \vec{B}_z)$$

$$\vec{B}_\perp = \frac{i}{\frac{\omega^2}{c^2} - k_z^2}(k_z \vec{\nabla}_\perp B_z - \tfrac{\omega}{c^2} \vec{\nabla}_\perp \times \vec{E}_z)$$



$$E_r = i \frac{Rk_z}{Z_{nm}} \partial_r E_z = -E_0 k_z J'_n(\tfrac{Z_{nm}}{R} r) \sin(n\varphi + k_z z - \omega t)$$

$$E_\varphi = i \frac{Rk_z}{Z_{nm}} \frac{1}{r} \partial_\varphi E_z = -E_0 n k_z \frac{1}{Z_{nm}} \frac{R}{r} J_n(\tfrac{Z_{nm}}{R} r) \cos(n\varphi + k_z z - \omega t)$$

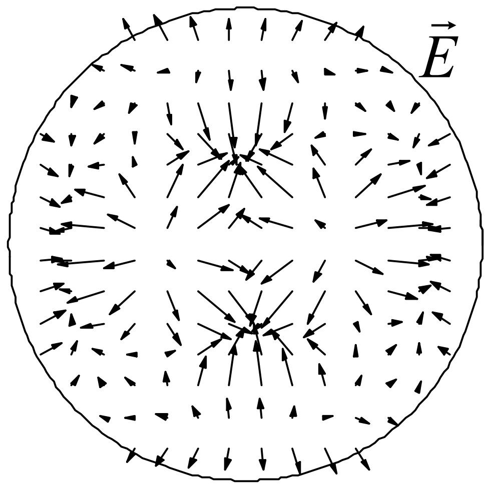
$$E_z = E_0 J_n(\tfrac{Z_{nm}}{R} r) \cos(n\varphi + k_z z - \omega t)$$

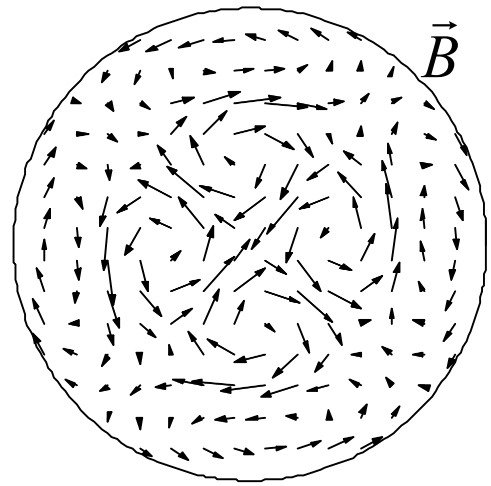
$$B_r = -i \frac{R\omega}{Z_{nm}} \frac{1}{r} \partial_\varphi E_z = E_0 n \frac{\omega}{c^2} \frac{1}{Z_{nm}} \frac{R}{r} J_n(\tfrac{Z_{nm}}{R} r) \cos(n\varphi + k_z z - \omega t)$$

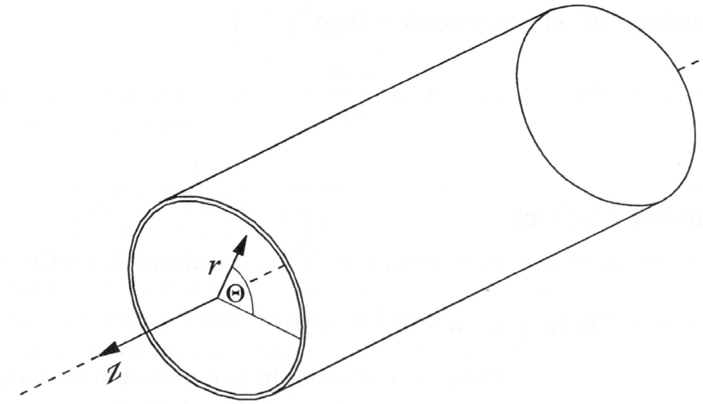
$$B_\varphi = i \frac{R\omega}{Z_{nm}} \partial_r E_z = -E_0 \frac{\omega}{c^2} J'_n(\tfrac{Z_{nm}}{R} r) \sin(n\varphi + k_z z - \omega t)$$

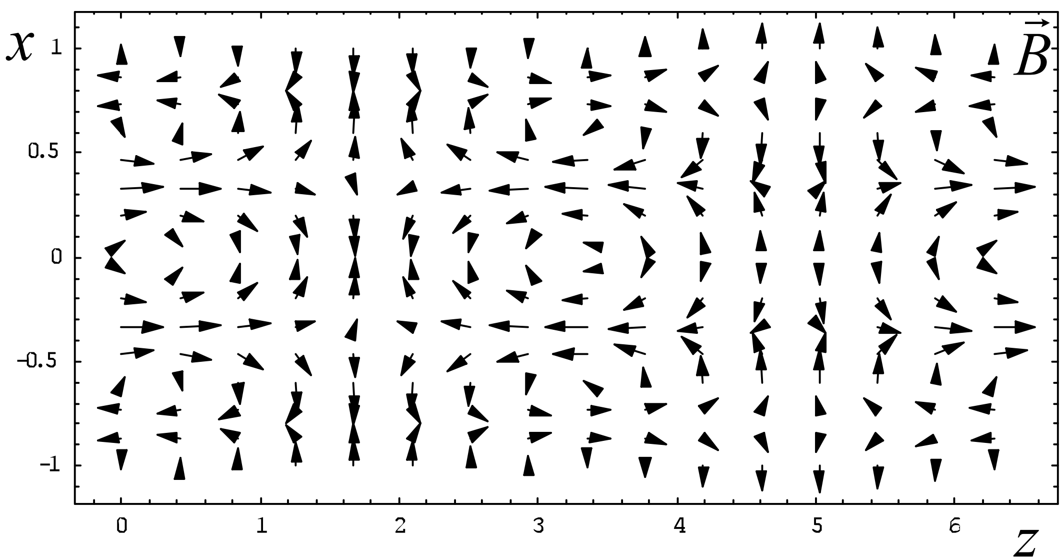
$$B_z = 0$$

Rectangular TM_{22} Mode

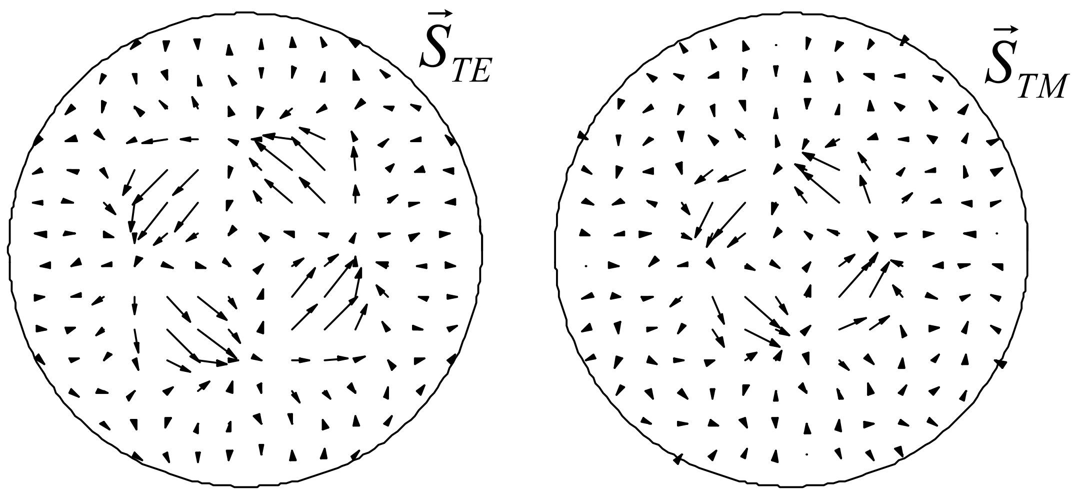


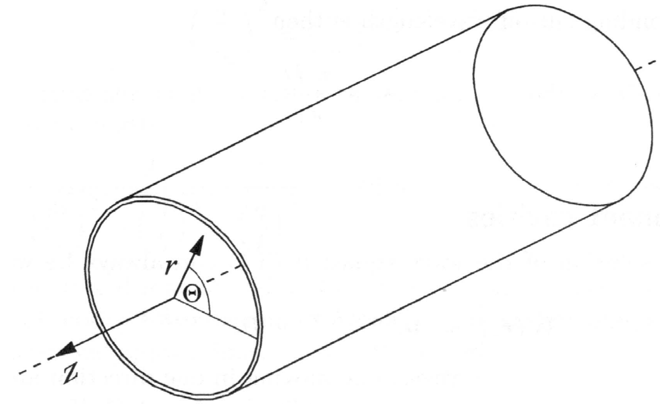


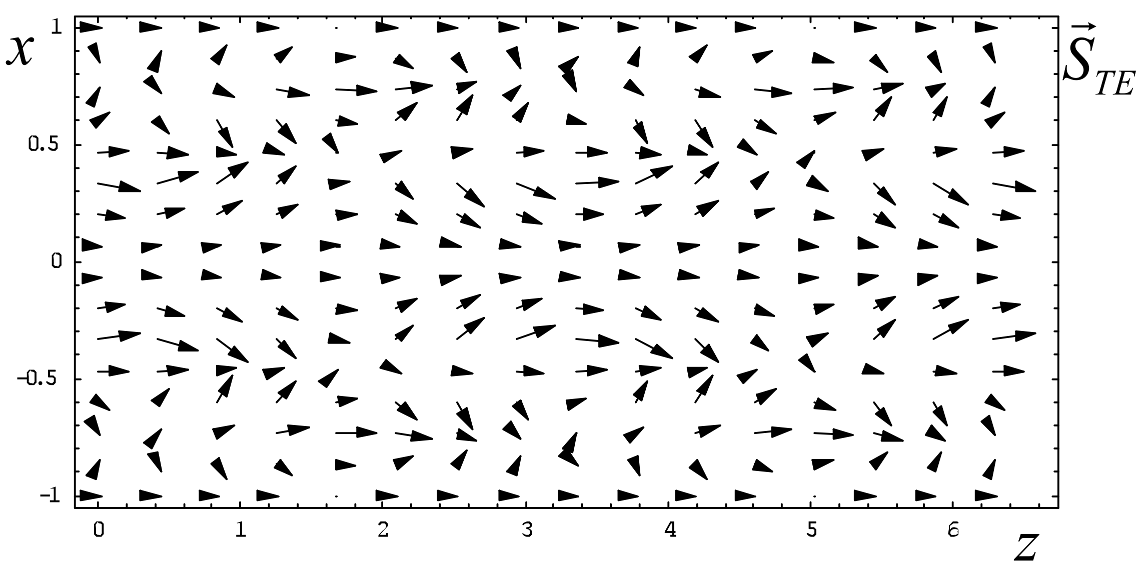




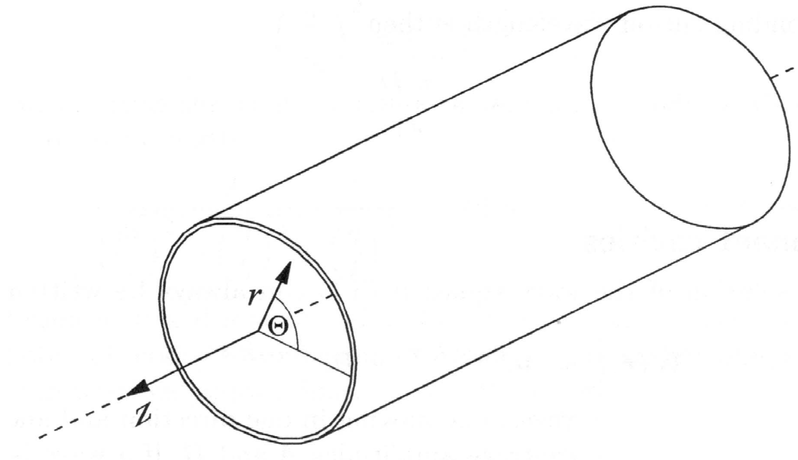
Pointing Vector



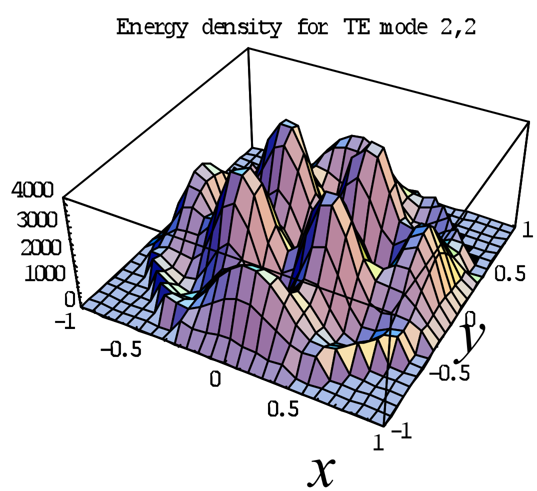




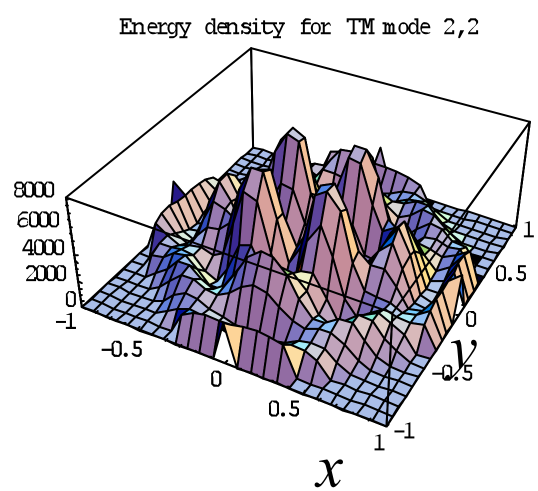
Energy Density



U_{TE}



U_{TM}



U_{TM}