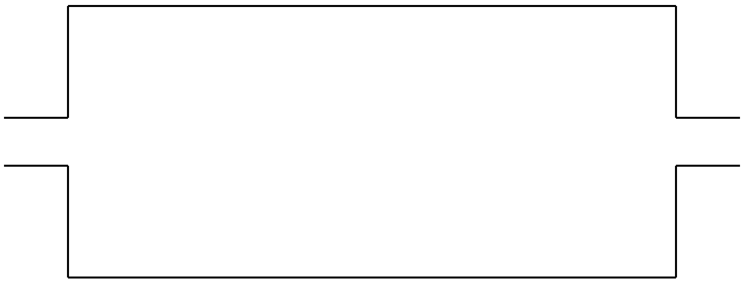
Resonant Cavities

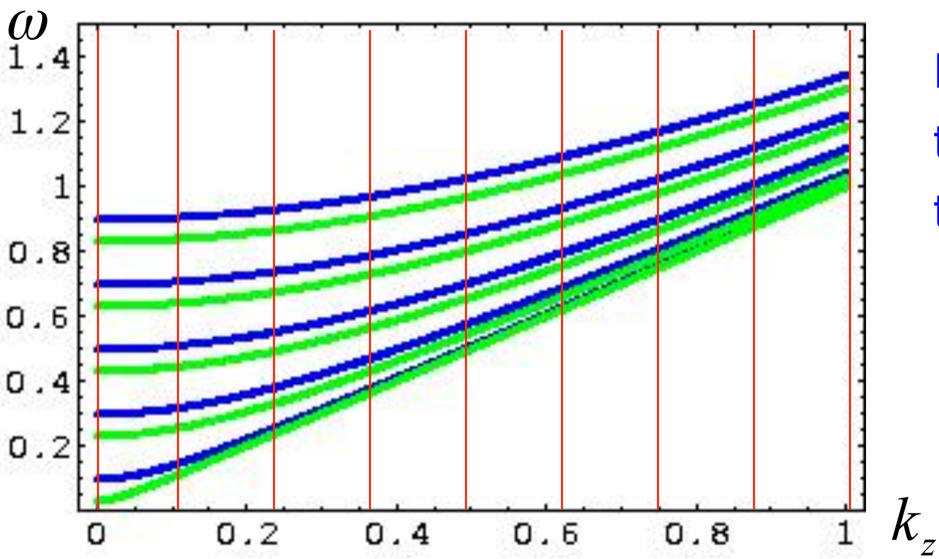


TE Modes: Standing waves with nodes

$$B_z(\vec{x}) \propto \sin(k_z z)\sin(\omega t)\,, \quad k_z = \frac{l\pi}{L}$$
$$l > 0$$

TM Modes: Standing waves with nodes

$$E_z(\vec{x}) \propto \cos(k_z z)\cos(\omega t)\,, \quad k_z = \frac{l\pi}{L}$$
$$l \geq 0$$

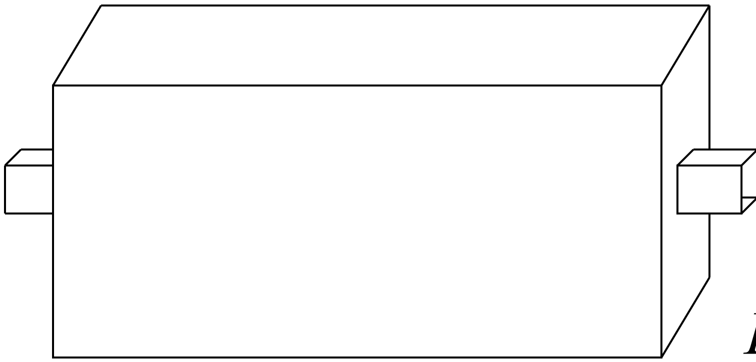


For each mode TE_{nm} or TM_{nm} there is a discrete set of frequencies that can be excited.

$$\omega_{nm}^{(E/B)} = c\sqrt{k_{nm}^{(E/B)2} + \left(\frac{l\pi}{L}\right)^2}$$

Resonant Cavities Examples

Rectangular cavity:

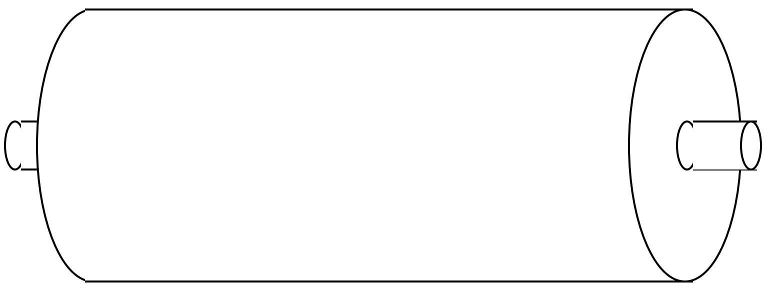


$$\omega_{nml}^{(E/B)} = c\sqrt{\left(\frac{n\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{l\pi}{L_z}\right)^2}$$

Fundamental acceleration mode: $\omega_{110}^{(B)} = c\frac{\pi}{L}\sqrt{2}$

$$L_x = L_y = 21.2cm \quad \Rightarrow \quad f_{110}^{(B)} = 1.0\text{GHz}$$

Pill Box cavity:



$$\omega_{nm}^{(E/B)} = c\sqrt{k_{nm}^{(E/B)2} + \left(\frac{l\pi}{L}\right)^2}$$

$k_{nm}^{(B)}r$ is the mth 0 of the nth Bessel function.

$k_{nm}^{(E)}r$ is the mth extremeum of J_n

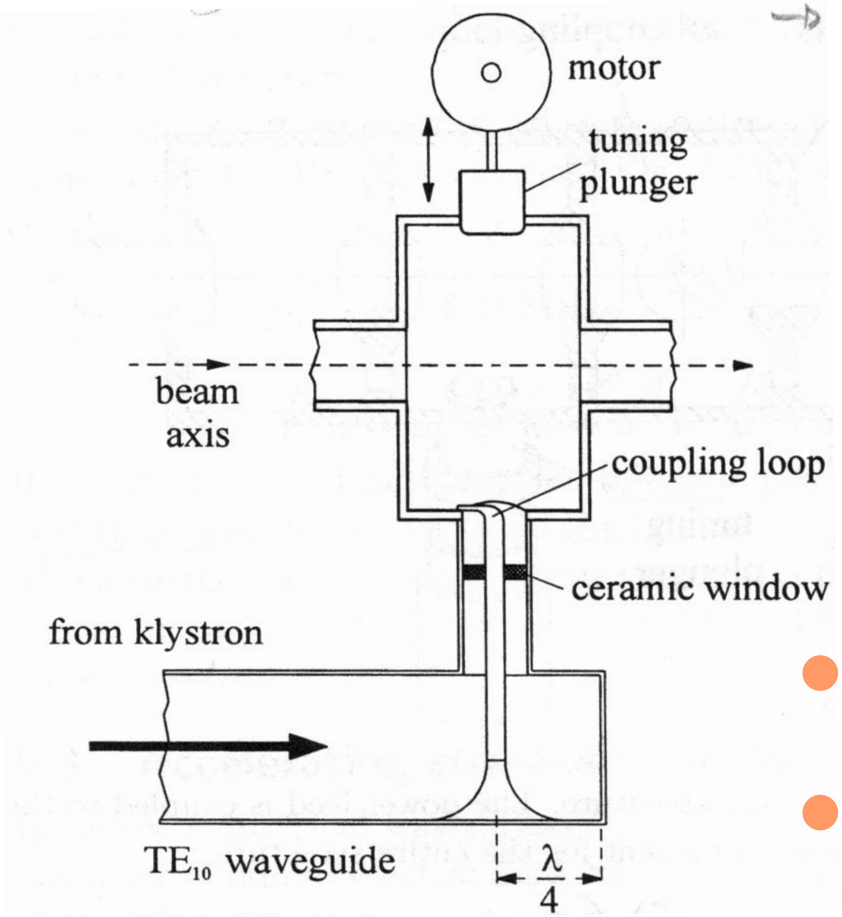
Fundamental acceleration mode: $\omega_{010}^{(B)} = c\frac{2.40}{r}$

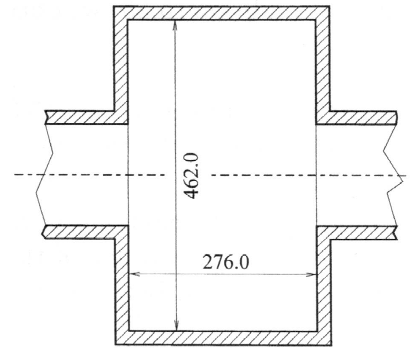
$$2r = 22.9cm \quad \Rightarrow \quad f_{010}^{(B)} = 1.0\text{GHz}$$

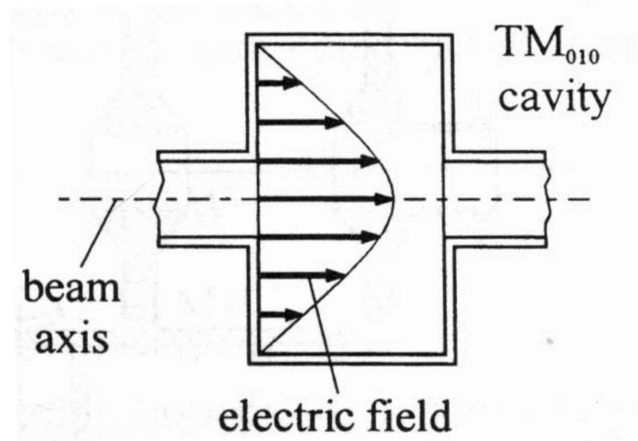
Cavity Operation

500MHz Cavity of DORIS:

$$r = 23.1cm \quad \Rightarrow \quad f_{010}^{(M)} = 0.4967\text{GHz}$$

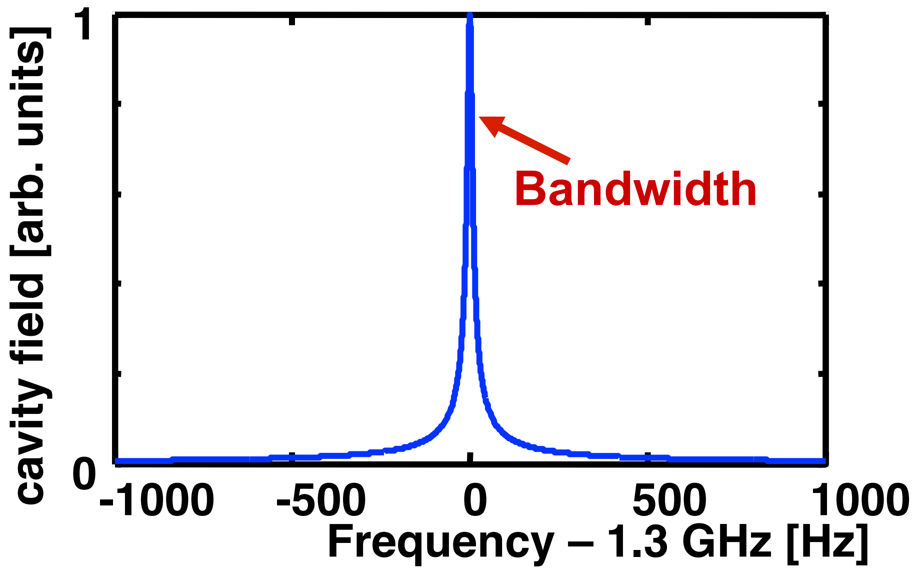






- The frequency is increased and tuned by a tuning plunger.
- An inductive coupling loop excites the magnetic field at the equator of the cavity.

3 dominant features of RF systems



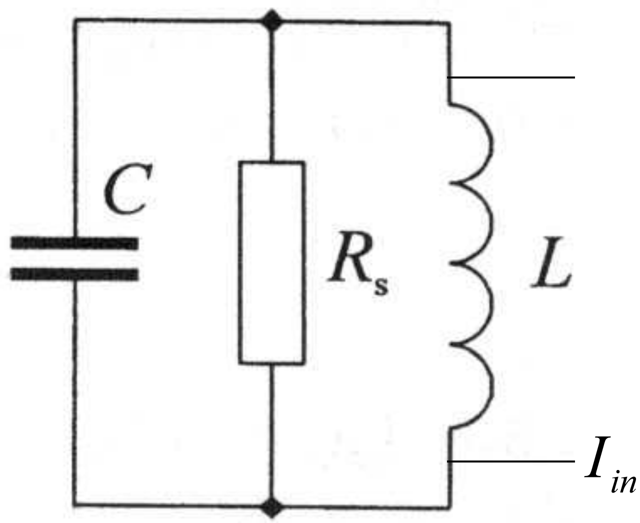
$$I_{in} \Rightarrow V(\omega)$$

(1) The RF system has a resonant frequency ω_0

(2) The resonance curve has a characteristic width $\Delta\omega = \frac{\omega_0}{2Q}$

A resonant L/C/R circuit also has such characteristics

RF systems for accelerators



L and C: determined by the cavity geometry

R_s : shunt impedance, related to surface res. R

$$I_{in} = \left(\frac{1}{R_s} + iC\omega + \frac{1}{iL\omega}\right) U_C$$

$$\hat{U}_C = \frac{1}{\sqrt{\frac{1}{R_s^2} + \left(\frac{1}{L\omega} - C\omega\right)^2}} \hat{I}_{in} \rightarrow \hat{U}_{Cres} = R_s \hat{I}_{in}$$

$$P_{RF} = \langle U_C I_{in} \rangle_t = \tfrac{1}{T}\int_0^T \mathrm{Re}[\left(\frac{1}{R_s} + iC\omega + \frac{1}{iL\omega}\right) U_C]\,\mathrm{Re}[U_C]dt = \tfrac{1}{2}\frac{1}{R_s}\hat{U}_C^2$$

Quallity factor: $$Q = 2\pi\frac{E}{\Delta E} = 2\pi\frac{\frac{1}{2}CU_C^2}{TP_{RF}} = \omega R_s C = R_s\sqrt{\tfrac{C}{L}}$$

Geometry factor: $$\frac{R_s}{Q} = \sqrt{\frac{L}{C}}$$

Geometry factor

$$\vec\nabla\cdot\vec E = \tfrac{1}{\varepsilon_0}\rho \qquad \vec E' = \tfrac{1}{\alpha}\vec E(\alpha\vec r',\alpha t') \qquad \vec\nabla'\cdot\vec E' = \tfrac{1}{\varepsilon_0}\rho'$$

$$\vec\nabla\times\vec E = -\partial_t\vec B \quad\Rightarrow\quad \vec B' = \tfrac{1}{\alpha}\vec B(\alpha\vec r',\alpha t') \quad\Rightarrow\quad \vec\nabla'\times\vec E' = -\partial_{t'}\vec B'$$

$$\vec\nabla\cdot\vec B = 0 \qquad \rho' = \rho(\alpha\vec r',\alpha t') \qquad \vec\nabla'\cdot\vec B' = 0$$

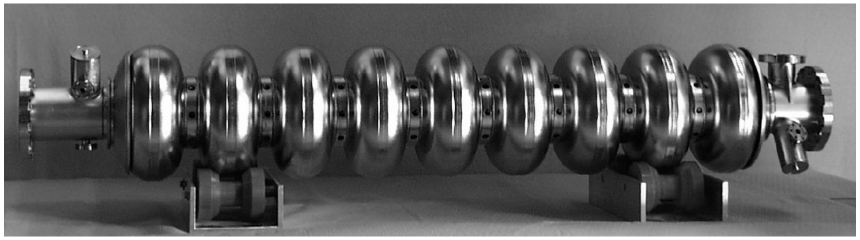
$$\vec\nabla\times\vec B = \tfrac{1}{c^2}[\tfrac{1}{\varepsilon_0}\vec j + \partial_t\vec E] \qquad \vec j' = \vec j(\alpha\vec r',\alpha t') \qquad \vec\nabla'\times\vec B' = \tfrac{1}{c^2}[\tfrac{1}{\varepsilon_0}\vec j' + \partial_{t'}\vec E']$$

Reducing all sizes by **a**, letting the time pass **a** times faster, reducing all charges by **a**^{3} and all currents by **a**^{2} leads to fields that are **a** times smaller !

$$L = \frac{V}{\dot I} = \frac{\alpha^2 V'}{\alpha \dot I'} = \alpha L' \qquad\qquad C = \frac{Q}{V} = \frac{\alpha^3 Q'}{\alpha^2 V'} = \alpha C'$$

For any oscillating circuit $\sqrt{\frac{L}{C}}$ is a size independent geometry factor !

Superconducting Cavities



$Q = 10^{10}$

$E = 20MV/m$

A bell with this Q would ring for a year.

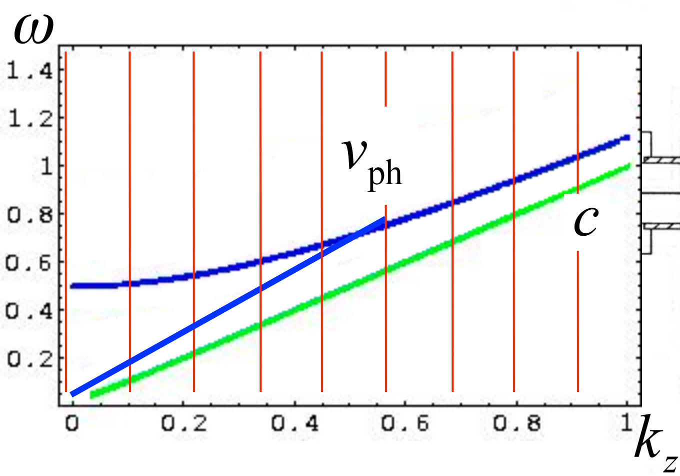
- Very low wall losses.
- Therefore continuous operation is possible.
- Energy recovery becomes possible.

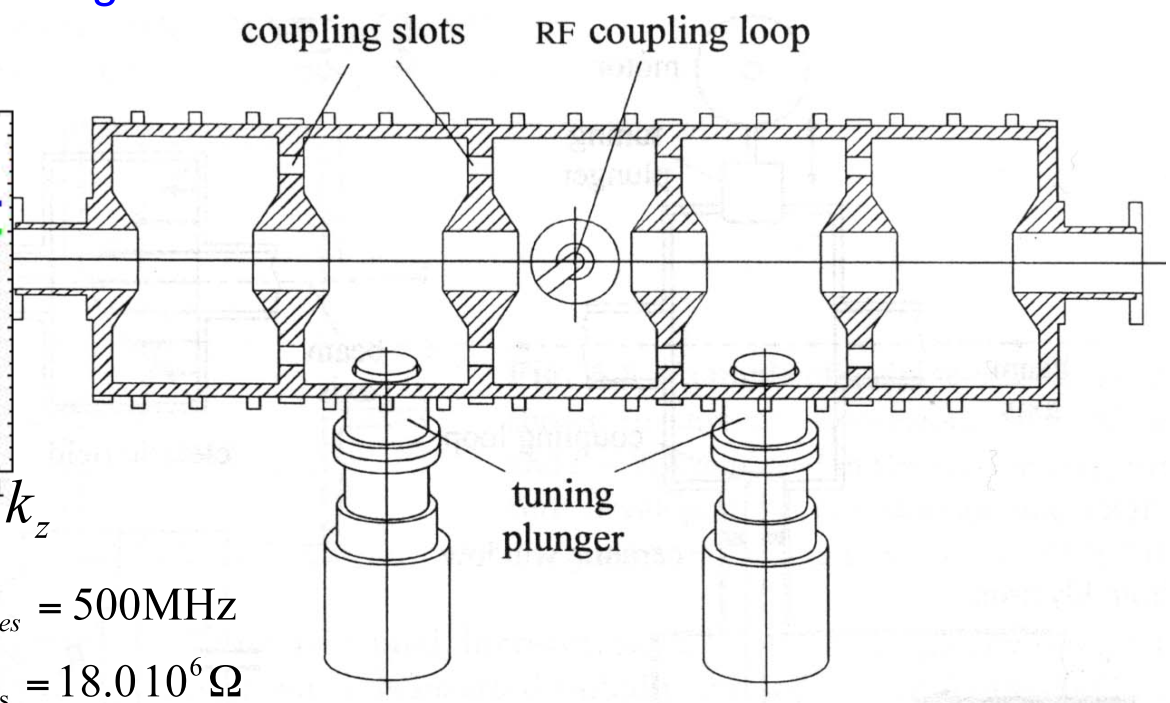
Normal conducting cavities

- Significant wall losses.
- Cannot operate continuously with appreciable fields.
- Energy recovery was therefore not possible.

Multicell standing-wave cavities

The filed in many cells can be excited by a single power source and a single input coupler in order to have the voltage of several cavities available.





Example: PETRA Cavity $f_{res} = 500\text{MHz}$

$R_s = 18.0\,10^6\,\Omega$

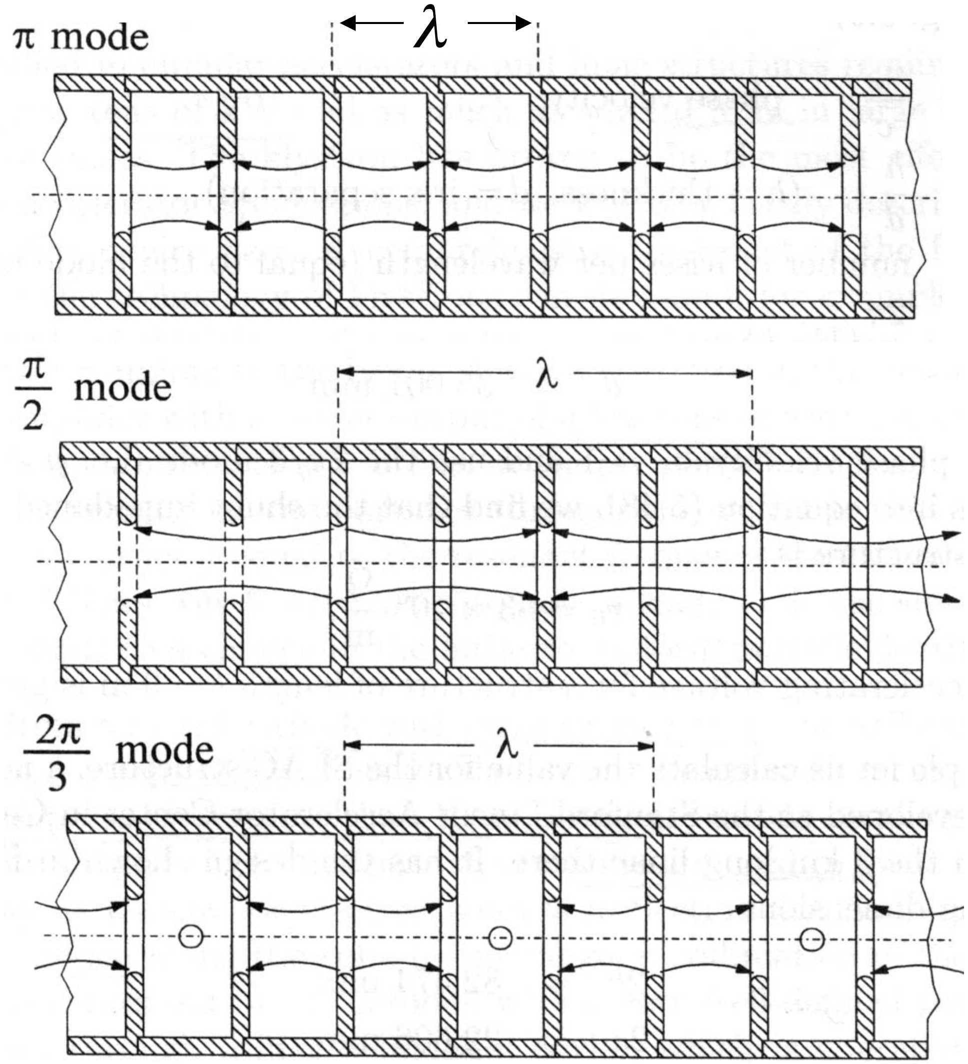
$125\text{kW} \rightarrow 2.12\text{MV}$

Without the walls: Long single cavity with too large wave velocity. $v_{ph} = \frac{\omega}{k}$

Thick walls: shield the particles from regions with decelerating phase.

Modes in Waveguides

The iris size is chosen to let the phase velocity equal the particle velocity.



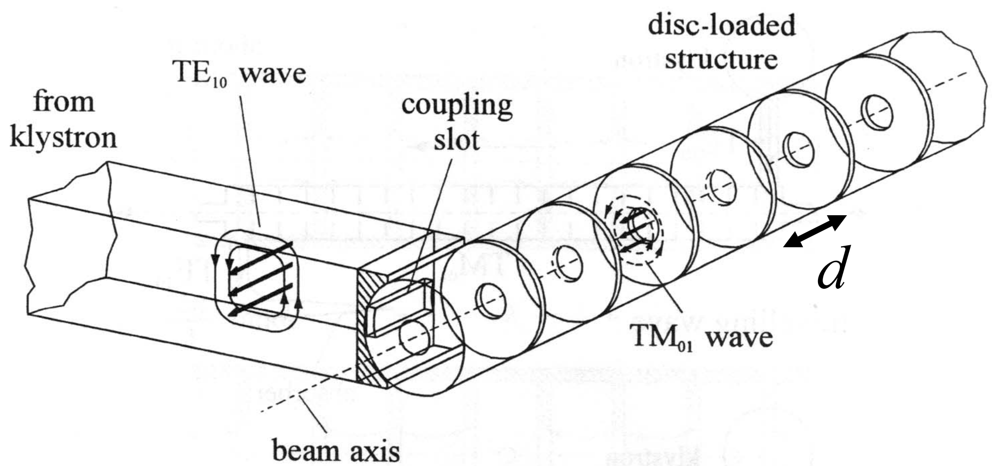
Long initial settling or filling time, not good for pulsed operation.

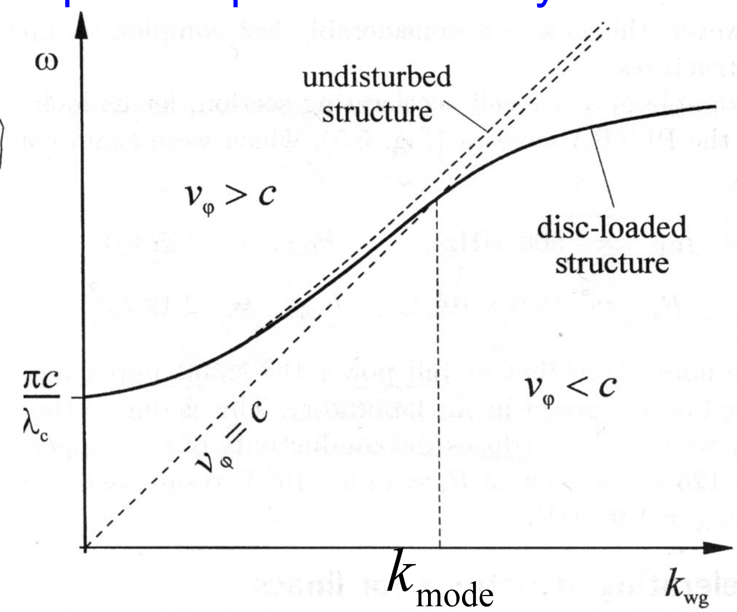
Small shunt impedance per length.

Common compromise.

Disc Loaded Waveguides

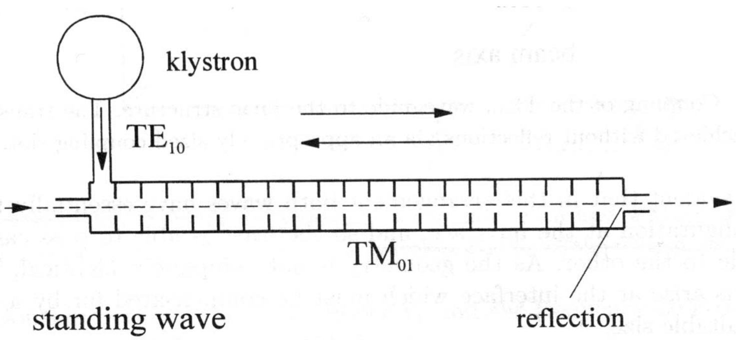
The iris size is chosen to let the phase velocity equal the particle velocity.



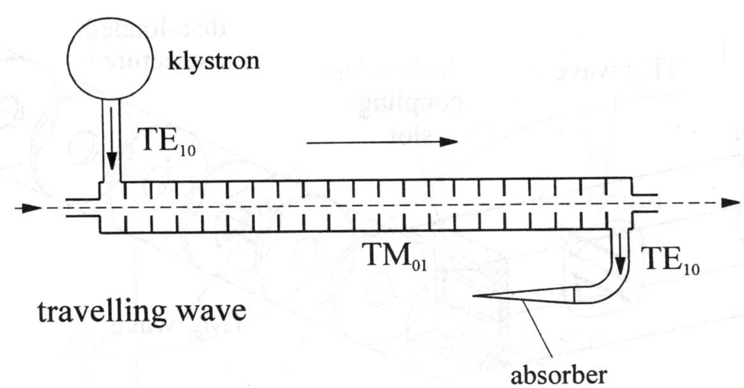


Loss free propagation: $k = \frac{2\pi}{nd}$

Standing wave cavity.



Traveling wave cavity (wave guide).



Transport maps of cavities

(1) Linearization: $E_r(r,z,t) =_1 -\frac{r}{2}\partial_z E_z(0,z,t) \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{E} = 0$

$$B_\varphi(r,z,t) =_1 \frac{1}{c^2}\frac{r}{2}\partial_t E_z(0,z,t) \quad \Rightarrow \quad \vec{\nabla}\times\vec{B} = \frac{1}{c^2}\partial_t\vec{E}$$

(2) Equation of motion:

$$a = \frac{p_x}{p_0}$$

Exact for a traveling wave with w/k = v
because of: $\frac{d}{dz}\cos(\omega t - kz + \varphi_0) f(z) = \frac{d}{dz} f(z)$

$$a' = \frac{1}{p_0 v}(F_x - aF_z) = -\frac{q}{p_0 v}[\frac{r}{2}(\partial_z + \frac{v}{c^2}\partial_t)E_z + aE_z]$$

$$= -\frac{q}{p_0 v}[\frac{r}{2}(\frac{d}{dz} - \frac{1}{v}(1-\frac{v^2}{c^2})\partial_t)E_z + aE_z] \approx -\frac{1}{p_0}[r\frac{1}{2}p_0'' + ap_0']$$

$$u = r\sqrt{p}$$

p denotes p_0 for simplicity

$$u' = a\sqrt{p} + r\sqrt{p}\frac{p'}{2p}$$

Focusing !

$$u'' \approx -\frac{1}{\sqrt{p}}(r\frac{1}{2}p'' + ap') + a\sqrt{p}\frac{p'}{p} + r(\frac{p''}{2\sqrt{p}} - \frac{p'}{4\sqrt{p}^3}) = -u\left(\frac{p'}{2p}\right)^2$$

Transport maps of cavities

(3) Average focusing over one period with relatively little energy change:

$$u'' \approx -u\frac{\Delta^2/4}{p^2}\,, \quad \Delta = \sqrt{\langle p'^2 \rangle}$$

(4) Continuous energy change:

$$p' \approx \Omega\,, \quad \Omega = \langle p' \rangle$$

$$\frac{d^2}{dp^2}u \approx \frac{1}{\Omega^2}u'' \approx -u\frac{(\Delta/\Omega)^2}{4p^2}$$

$$\frac{d^2}{dp^2}(r\sqrt{p}) = \frac{d^2}{dp^2}r\sqrt{p} + \frac{d}{dp}r\frac{1}{\sqrt{p}} - r\frac{1}{4\sqrt{p}^3} \approx -r\frac{(\Delta/\Omega)^2}{4\sqrt{p}^3}$$

$$\frac{d^2}{dp^2}r + \frac{d}{dp}r\frac{1}{p} \approx -r\frac{(\Delta/\Omega)^2-1}{4p^2} = -r\frac{\varepsilon^2}{p^2}$$

$$r(p) = \eta(-\ln(p)) \quad \Rightarrow \quad \frac{d^2}{dp^2}r = \frac{1}{p^2}\eta' + \frac{1}{p^2}\eta'' = \frac{1}{p^2}\eta' - \eta\frac{\varepsilon^2}{p^2}$$

Transport maps of traveling wave cavities

$$\eta'' = -\varepsilon^2 \eta \,, \quad \eta(\xi) = A\cos(\varepsilon\xi) - B\sin(\varepsilon\xi)$$

$$\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \frac{p'}{p} \end{pmatrix} \begin{pmatrix} \cos(\varepsilon \ln(p)) & \sin(\varepsilon \ln(p)) \\ -\sin(\varepsilon \ln(p)) & \cos(\varepsilon \ln(p)) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \frac{p'}{p} \end{pmatrix} \begin{pmatrix} \cos(\varepsilon \ln(\frac{p}{p_i})) & \sin(\varepsilon \ln(\frac{p}{p_i})) \\ -\sin(\varepsilon \ln(\frac{p}{p_i})) & \cos(\varepsilon \ln(\frac{p}{p_i})) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\varepsilon}\frac{p_i}{p'} \end{pmatrix} \begin{pmatrix} r_i \\ a_i \end{pmatrix}$$

$$E_z = \sum_{n=0}^{\infty} g_n \cos(n \tfrac{2\pi}{L} z + \alpha_n) \cos(\omega t - kz + \varphi_0)$$

$$= \sum_{n=0}^{\infty} g_n \cos(n \tfrac{2\pi}{L} z + \alpha_n) \cos(\varphi_0) \,, \quad \alpha_0 = 0$$

$$\left.\langle p' \rangle = g_0 \cos(\varphi_0) \,, \langle p'^2 \rangle = \tfrac{1}{2} \sum_{n=0}^{\infty} g_n^2 \cos^2(\varphi_0) \right\} \varepsilon = \tfrac{1}{2} \sqrt{\tfrac{1}{2} \sum_{n=0}^{\infty} \left(\tfrac{g_n}{g_0}\right)^2 - 1}$$

Transport maps of standing wave cavities

$$\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \frac{p'}{p} \end{pmatrix} \begin{pmatrix} \cos(\varepsilon \ln(\frac{p}{p_i})) & \sin(\varepsilon \ln(\frac{p}{p_i})) \\ -\sin(\varepsilon \ln(\frac{p}{p_i})) & \cos(\varepsilon \ln(\frac{p}{p_i})) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\varepsilon} \frac{p_i}{p'} \end{pmatrix} \begin{pmatrix} r_i \\ a_i \end{pmatrix}$$

$$E_z = \sum_{n=-\infty}^{\infty} g_n e^{in\frac{2\pi}{L}z} \cos(kz)\cos(\omega t + \varphi_0) \ , \quad \omega t = kz \, , k = m\tfrac{\pi}{L} \, , \quad g_{-n} = g_n^*$$

$$= \tfrac{1}{4} \sum_{n=-\infty}^{\infty} g_n [e^{i(n+m)\frac{2\pi}{L}z+\varphi_0} + e^{i(n-m)\frac{2\pi}{L}z-\varphi_0} + e^{in\frac{2\pi}{L}z+\varphi_0} + e^{in\frac{2\pi}{L}z-\varphi_0}]$$

$$= \tfrac{1}{4} \sum_{n=-\infty}^{\infty} [g_{n-m} e^{i\varphi_0} + g_{n+m} e^{-i\varphi_0} + 2 g_n \cos(\varphi_0)] e^{in\frac{2\pi}{L}z} = \sum_{n=-\infty}^{\infty} f_n e^{in\frac{2\pi}{L}z}$$

$$\left.\begin{array}{l} \langle p' \rangle = f_0 \, , \\ \langle p'^2 \rangle = \sum_{n=0}^{\infty} |f_n|^2 \end{array}\right\} \varepsilon = \tfrac{1}{2} \sqrt{\sum_{n=0}^{\infty} \left|\frac{f_n}{f_0}\right|^2 - 1}$$

Phase space precervation in cavities

CHESS & LEPP

Average focusing over one period with relatively little energy change:

$$\begin{pmatrix} r \\ a \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \frac{p'}{p} \end{pmatrix} \begin{pmatrix} \cos(\varepsilon \ln(\frac{p}{p_i})) & \sin(\varepsilon \ln(\frac{p}{p_i})) \\ -\sin(\varepsilon \ln(\frac{p}{p_i})) & \cos(\varepsilon \ln(\frac{p}{p_i})) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\varepsilon} \frac{p_i}{p'} \end{pmatrix}}_{\underline{M}} \begin{pmatrix} r_i \\ a_i \end{pmatrix}$$

$$\det(\underline{M}) = \frac{p_i}{p}$$

Because the determinant is not 1, the phase space volume is no longer conserved but changes by p_0/p.

A new propagation and definition of Twiss parameters is therefore needed:

$$r = \sqrt{2J \frac{1}{\beta_r \gamma_r} \beta} \sin(\psi + \phi_0)$$

Twiss parameters in accelerating cavities

$$\alpha = -\tfrac{1}{2}\beta' , \quad \gamma = \tfrac{1+\alpha^2}{\beta}$$

$$a = r' = \sqrt{2J\tfrac{mc}{p}}[-\tfrac{2\alpha+\beta\frac{p'}{p}}{2\sqrt{\beta}}\sin(\psi+\phi_0) + \tfrac{\beta\psi'}{\sqrt{\beta}}\cos(\psi+\phi_0)]$$

$$a' \approx -\tfrac{1}{p}[r(pK+\tfrac{1}{2}p'') + ap']$$

$$a' = -\sqrt{2J\tfrac{mc}{\beta p}}\begin{pmatrix} \frac{(\beta\psi')^2+\alpha^2}{\beta} + \alpha' - \alpha\frac{p'}{p} + \beta\frac{p''}{2p} - \beta\frac{3p'^2}{4p^2} \\ 2\alpha\psi' + \beta\frac{p'}{p}\psi' - \beta\psi'' \end{pmatrix}\begin{pmatrix} \sin(\psi+\phi_0) \\ \cos(\psi+\phi_0) \end{pmatrix}$$

$$= -\sqrt{2J\tfrac{mc}{\beta p}}\begin{pmatrix} \beta(K+\frac{1}{2}\frac{p''}{p}) - (\alpha+\beta\frac{p'}{2p})\frac{p'}{p} \\ \beta\frac{p'}{p}\psi' \end{pmatrix}\begin{pmatrix} \sin(\psi+\phi_0) \\ \cos(\psi+\phi_0) \end{pmatrix}$$

$$\Rightarrow \quad \psi' = \frac{A}{\beta}, \text{choice} : A = 1$$

$$\alpha' + \gamma = \beta\left[K + \left(\tfrac{p'}{2p}\right)^2\right]$$

$$\begin{pmatrix} r \\ a \end{pmatrix} = \sqrt{2J\tfrac{mc}{p}}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha+\beta\frac{p'}{2p}}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi+\phi_0) \\ \cos(\psi+\phi_0) \end{pmatrix}$$

Beta functions in accelerating cavities

$$\begin{pmatrix} r \\ a \end{pmatrix} = \sqrt{2J_n \frac{mc}{p}} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\tilde{\alpha}}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos(\psi + \phi_0) \\ \sin(\psi + \phi_0) \end{pmatrix} \quad , \quad \tilde{\alpha} = \alpha + \beta \frac{p'}{2p}$$

For systems with changing energy one uses the normalized Courant-Snyder invariant $J_n = J\, b_r\, g_r$

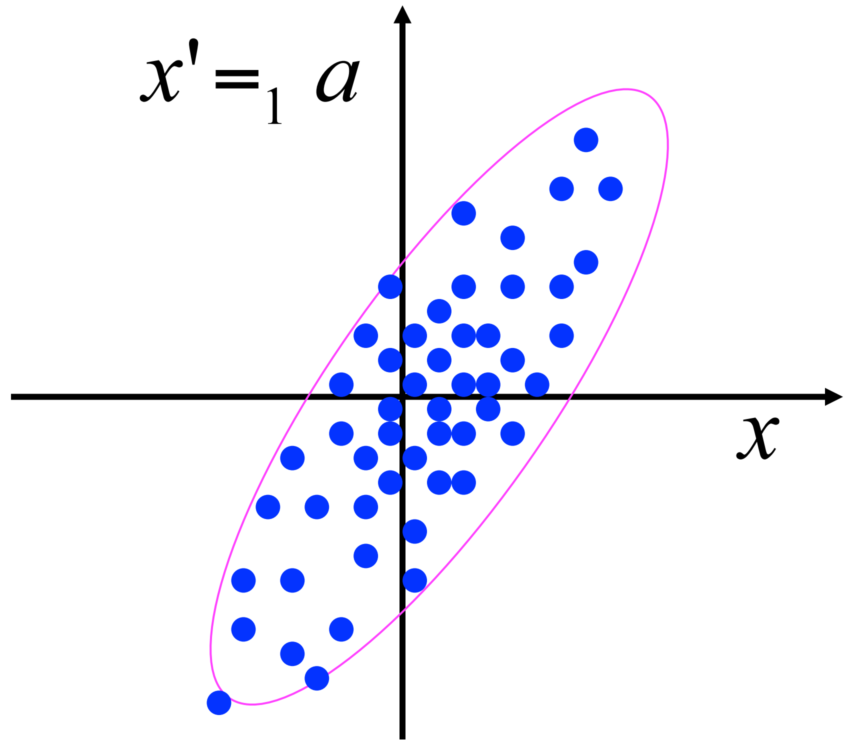
$$\begin{pmatrix} r & a \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta}} & \frac{\tilde{\alpha}}{\sqrt{\beta}} \\ 0 & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\tilde{\alpha}}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} r \\ a \end{pmatrix} \frac{p}{2mc} = \begin{pmatrix} r & a \end{pmatrix} \begin{pmatrix} \frac{1+\tilde{\alpha}^2}{\beta} & \tilde{\alpha} \\ \tilde{\alpha} & \beta \end{pmatrix} \begin{pmatrix} r \\ a \end{pmatrix} \frac{p}{2mc} = J_n$$

Reasons:

(1) **J** is the phase space amplitude of a particle in **(x , a)** phase space, which is the area in phase space (over 2p) that its coordinate would circumscribe during many turns in a ring. However, **$a=p_x/p_0$** is not conserved when p0 changes in a cavity. Therefore J is not conserved.

(2) **$J_n = J\, p_0/mc$** is therefore proportional to the corresponding area in **(x , p_x)** phase space, and is thus conserved.

The normazized emittance



Remarks:

(1) The phase space area that a beam fills in (x , a) phase space shrinks during acceleration by the factor p_i/p. This area is the emittance e.

(2) The phase space area that a beam fills in (x , p_x) phase space is conserved. This area (divided by mc) is the normalized emittance e_n.

$$\varepsilon = \varepsilon_n / \beta_r \gamma_r$$