Complex Potential of a Wire

Straight wire at the origin: $\vec{\nabla}\times\vec{B}=\mu_0\,\vec{j} \quad\Rightarrow\quad \vec{B}(r)=\frac{\mu_0\,I}{2\pi\,r}\vec{e}_\phi=\frac{\mu_0\,I}{2\pi\,r^2}\begin{pmatrix}-y\\ x\end{pmatrix}$

Wire at $\vec{a}$:

$$\vec{B}(x,y)=\frac{\mu_0\,I}{2\pi\,(\vec{r}-\vec{a})^2}\begin{pmatrix}-[y-a_y]\\ x-a_x\end{pmatrix}$$

This can be represented by complex multipole coefficients Ψ_ν

$$\vec{B}(x,y)=-\vec{\nabla}\Psi \quad\Rightarrow\quad B_x+iB_y=-\left(\partial_x+i\partial_y\right)\psi=-2\partial_{\overline{w}}\psi$$

$$B_x+iB_y=\tfrac{\mu_0 I}{2\pi}\frac{-i(w_a-w)}{(w_a-w)(\overline{w}_a-\overline{w})}=i\,\tfrac{\mu_0 I}{2\pi}\frac{-\frac{w_a}{a^2}}{1-\frac{\overline{w}w_a}{a^2}}$$

$$=i\,\tfrac{\mu_0 I}{2\pi}\,\partial_{\overline{w}}\ln(1-\tfrac{\overline{w}w_a}{a^2})=-2\partial_{\overline{w}}\,\mathrm{Im}\left\{\tfrac{\mu_0 I}{2\pi}\ln(1-\tfrac{\overline{w}w_a}{a^2})\right\}$$

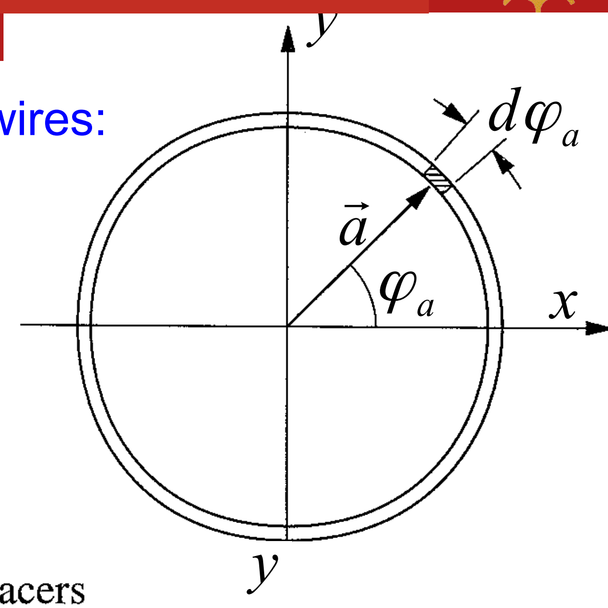
$$\psi=\mathrm{Im}\left\{\tfrac{\mu_0 I}{2\pi}\ln(1-\tfrac{\overline{w}w_a}{a^2})\right\}=-\,\mathrm{Im}\left\{\tfrac{\mu_0 I}{2\pi}\sum_{\nu=1}^{\infty}\tfrac{1}{\nu}\left(\tfrac{w_a}{a^2}\right)^\nu\overline{w}^\nu\right\} \quad\Rightarrow\quad \Psi_\nu=\tfrac{\mu_0 I}{2\pi}\tfrac{1}{\nu}\tfrac{1}{a^\nu}e^{i\nu\varphi_a}$$

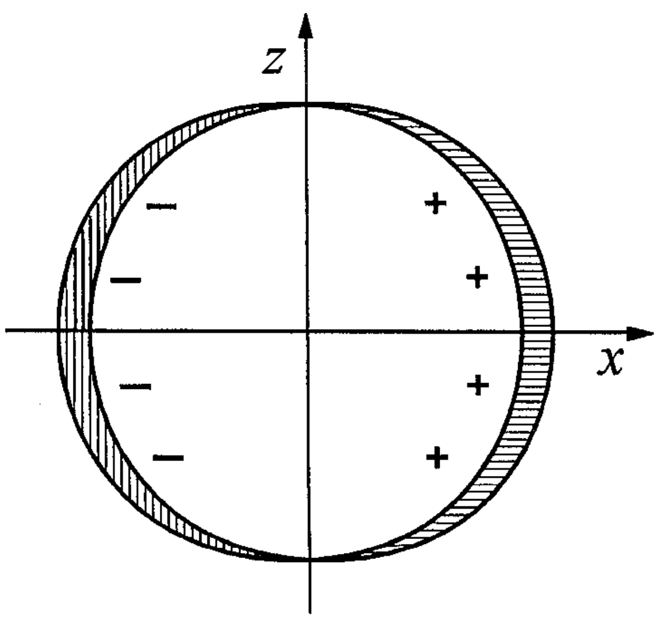
Air-coil Multipoles

Creating a multipole be created by an arrangement of wires:

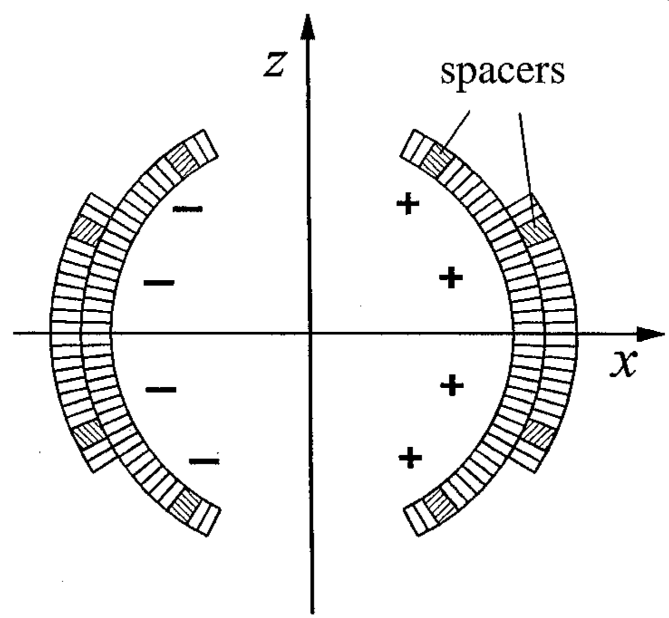
$$\Psi_\nu = \int_0^{2\pi} \frac{\mu_0}{2\pi}\frac{1}{\nu}\frac{1}{a^\nu} e^{i\nu\varphi_a} \frac{dI}{d\varphi_a} d\varphi_a$$

$$\Psi_\nu = \delta_{\nu n}\frac{\mu_0}{2}\frac{1}{n}\frac{1}{a^n}\hat{I} \quad \text{if } I(\varphi_a) = \hat{I}\cos n\varphi_a$$





Ideal multipole

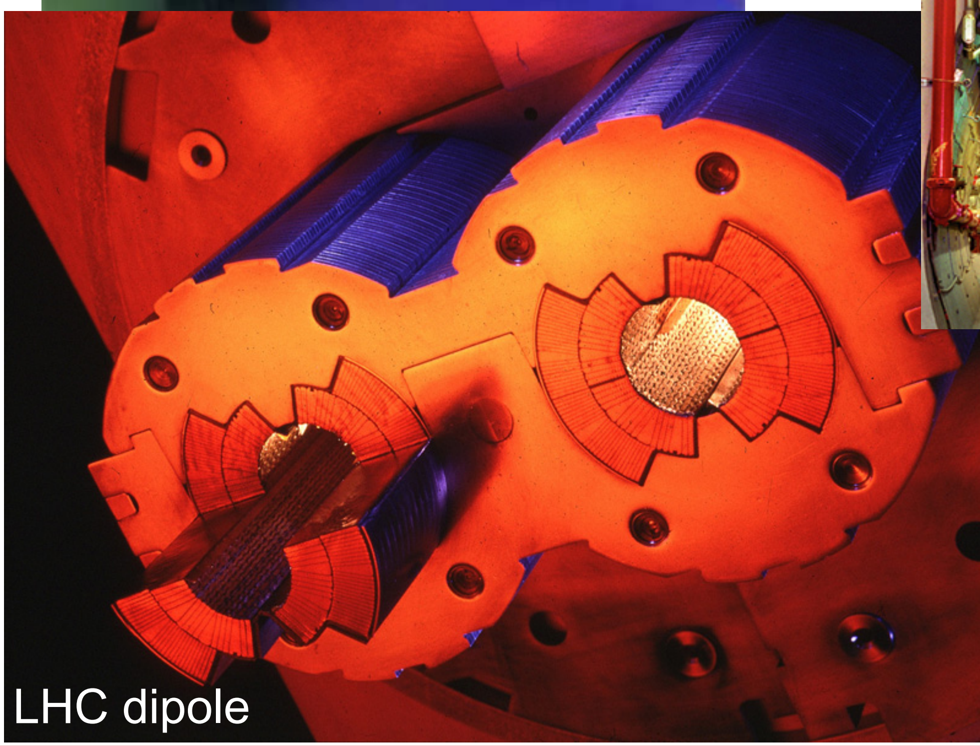


Approximate multipole

Real Air-coil Multipoles

Quadrupole corrector

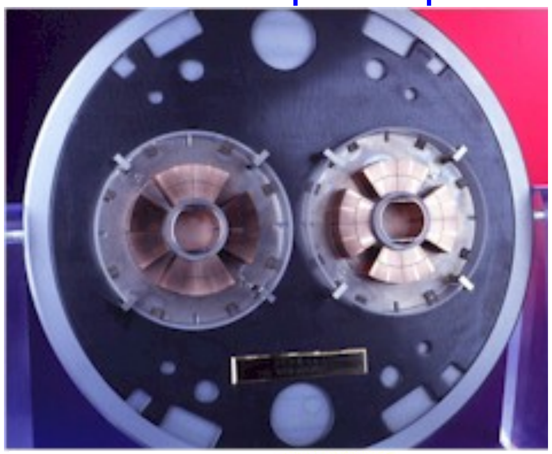
RHIC Tunne



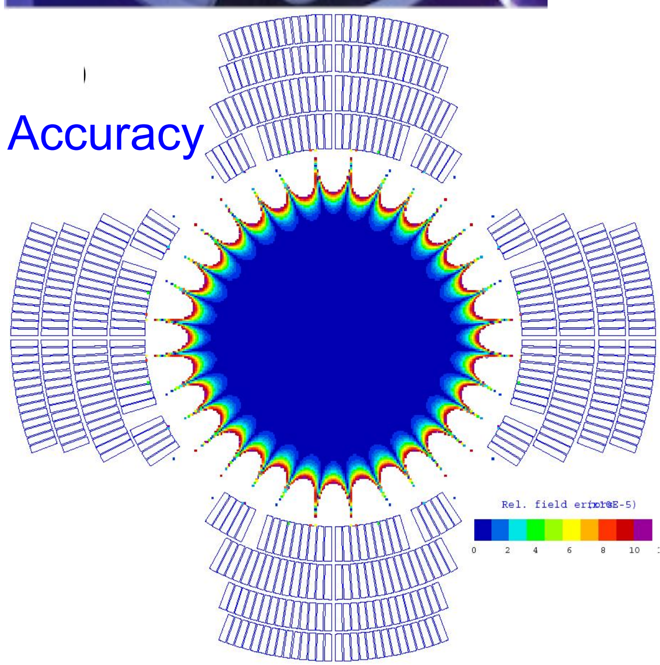
LHC dipole

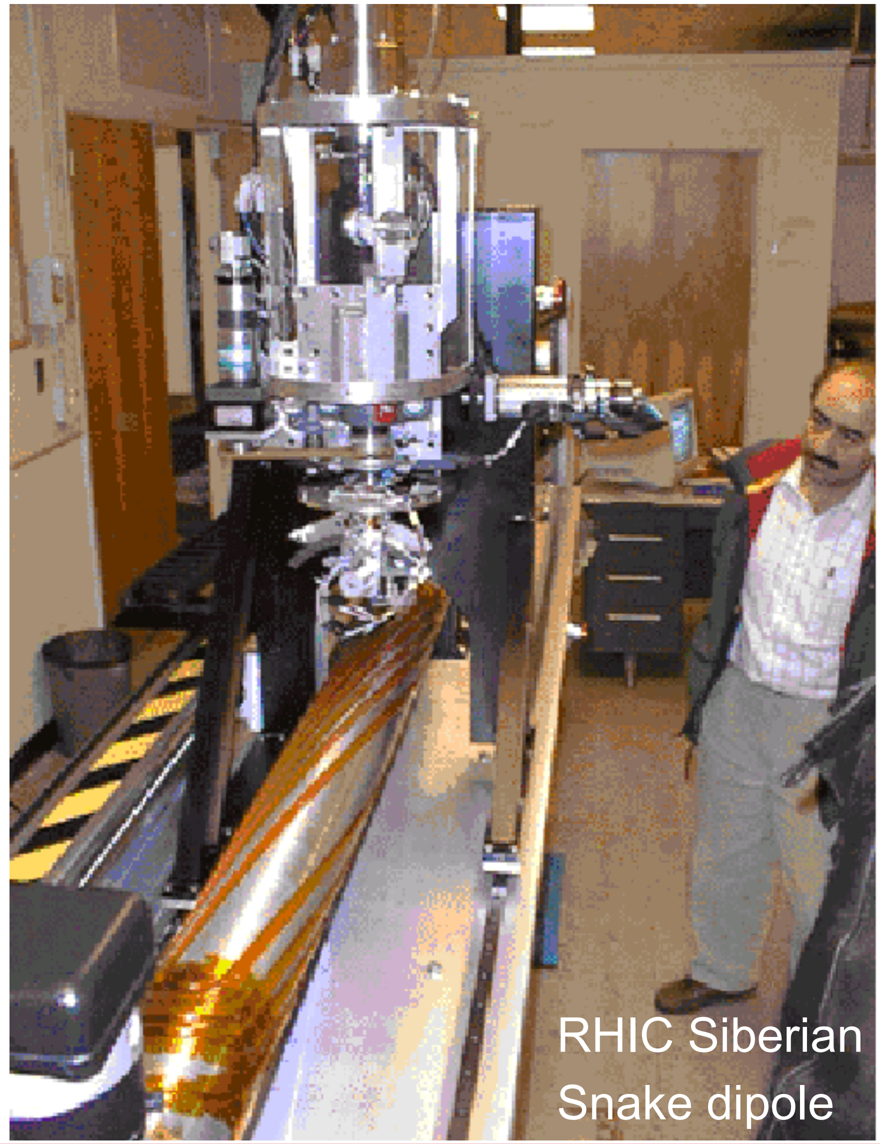
Special SC Air-coil Magnets

LHC double quadrupole



Accuracy





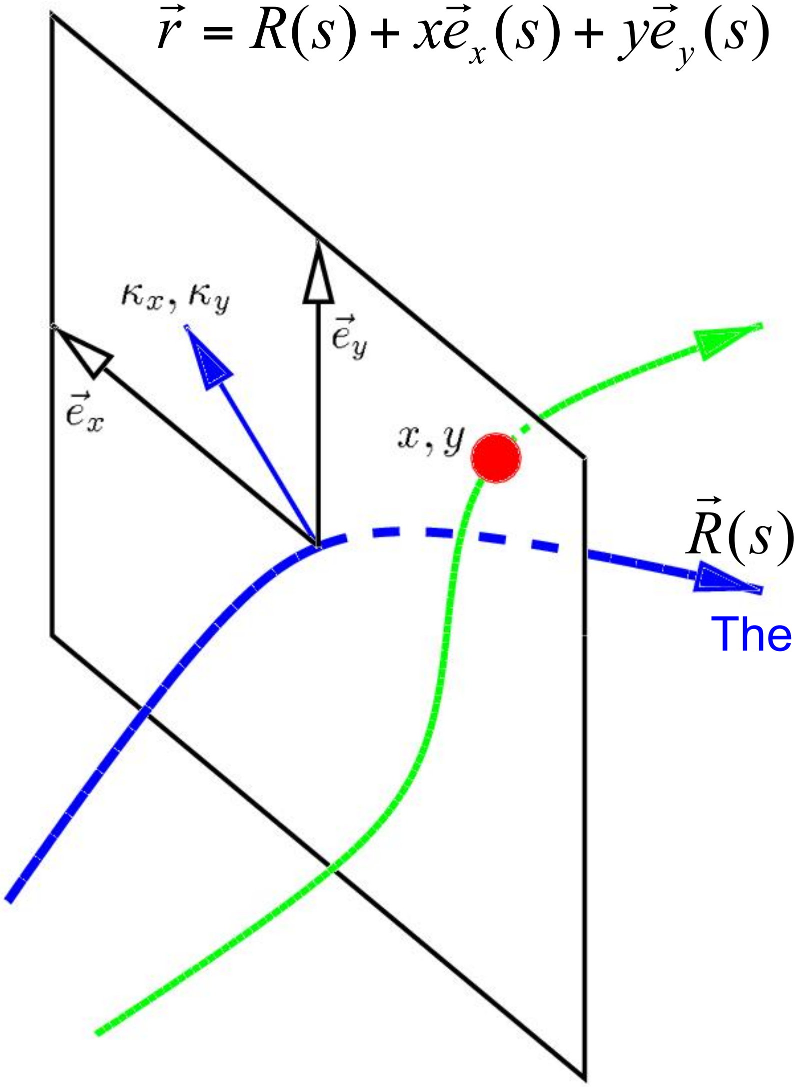
RHIC Siberian Snake dipole

The comoving Coordinate System

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$

$$|d\vec{R}| = ds$$

$$\vec{e}_s \equiv \frac{d}{ds}\vec{R}(s)$$



The time dependence of a particle's motion is often not as interesting as the trajectory along the accelerator length "s".

Matrix Solutions

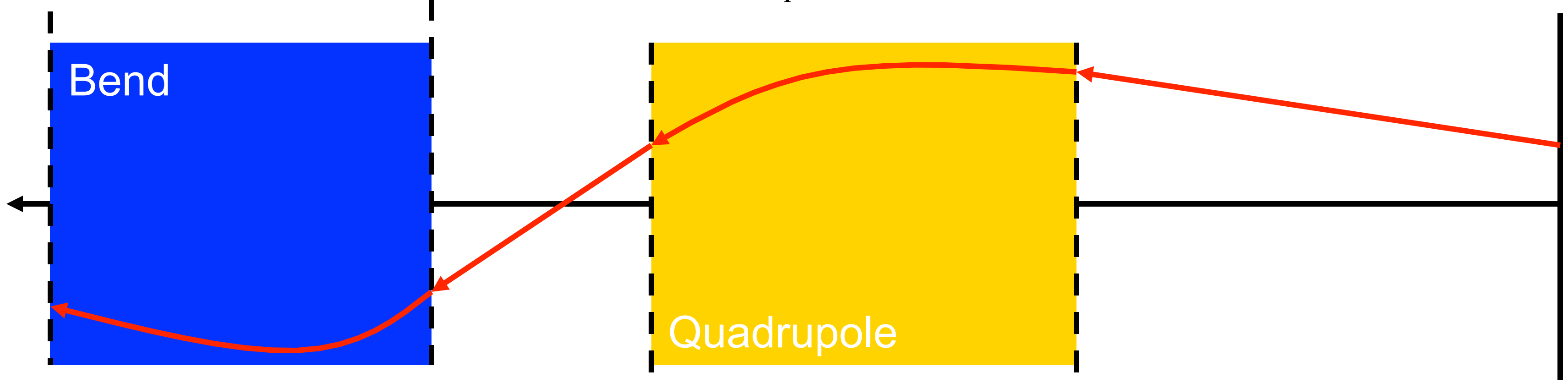
CHESS & LEPP

Linear equation of motion: $\vec{z}' = \underline{F}(s)\vec{z}$

Matrix solution of the starting condition $\vec{z}(0) = \vec{z}_0$

$$\vec{z} = \underline{M}_{\text{bend}}(L_4)\underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$



$$\vec{z} = \underline{M}_{\text{drift}}(L_1)\vec{z}_0 \qquad \vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

The Drift

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\ddot{x} = 0 \quad \Rightarrow \quad x'' = 0 \quad \Rightarrow \quad a = x', a' = 0$$

Linear solution:

$$x(s) = x_0 + x_0' s$$

$$\begin{pmatrix} x \\ a \\ y \\ b \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} x_0 + s a_0 \\ a \\ y_0 + s b_0 \\ b_0 \\ \tau_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \underline{0} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \vec{z}_0$$

The 4D Equation of Motion

$$\frac{d^2}{dt^2}\vec{r} = \vec{f}_r(\vec{r}, \tfrac{d}{dt}\vec{r}, t)$$

3 dimensional ODE of 2nd order can be changed to a

6 dimensional ODE of 1st order:

$$\left.\begin{array}{l} \frac{d}{dt}\vec{r} = \frac{1}{m\gamma}\vec{p} = \frac{c}{\sqrt{p^2-(mc)^2}}\vec{p} \\ \frac{d}{dt}\vec{p} = \vec{F}(\vec{r},\vec{p},t) \end{array}\right\} \quad \frac{d}{dt}\vec{Z} = \vec{f}_Z(\vec{Z},t)\,, \quad \vec{Z} = (\vec{r},\vec{p})$$

If the force does not depend on time, as in a typical beam line magnet, the energy is conserved so that one can reduce the dimension to 5. The equation of motion is then autonomous.

Furthermore, the time dependence is often not as interesting as the trajectory along the accelerator length "s". Using "s" as the independent variable reduces the dimensions to 4. The equation of motion is then no longer autonomous.

$$\frac{d}{ds}\vec{z} = \vec{f}_z(\vec{z},s)\,, \quad \vec{z} = (x, y, p_x, p_y)$$