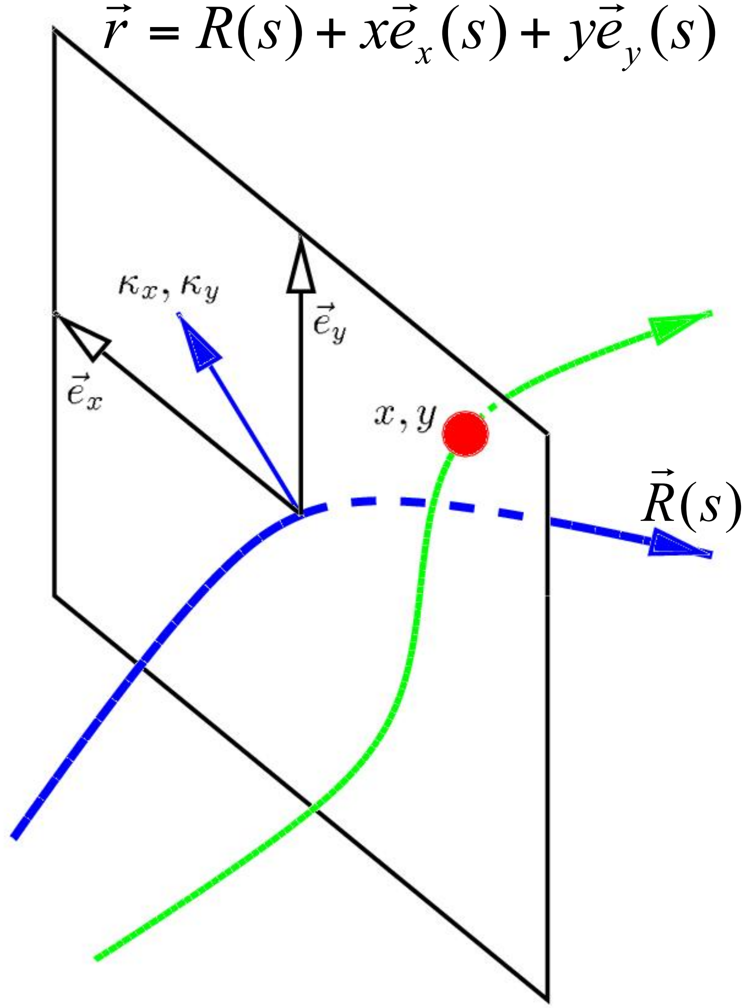
The Frenet Coordinate System

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$



$$|d\vec{R}| = ds$$

$$\vec{e}_s \equiv \tfrac{d}{ds}\vec{R}(s)$$

$$\vec{e}_\kappa \equiv -\tfrac{d}{ds}\vec{e}_s \Big/ \left|\tfrac{d}{ds}\vec{e}_s\right|$$

$$\vec{e}_b \equiv \vec{e}_s \times \vec{e}_\kappa$$

$$\tfrac{d}{ds}\vec{e}_s = -\kappa\,\vec{e}_\kappa \quad \text{with} \quad \kappa = \frac{1}{\rho}$$

$$0 = \tfrac{d}{ds}\left(\vec{e}_\kappa \cdot \vec{e}_s\right) = \vec{e}_s \cdot \tfrac{d}{ds}\vec{e}_\kappa - \kappa$$

Accumulated torsion angle T

$$\tfrac{d}{ds}\vec{e}_\kappa = \kappa\,\vec{e}_s + T'\vec{e}_b$$

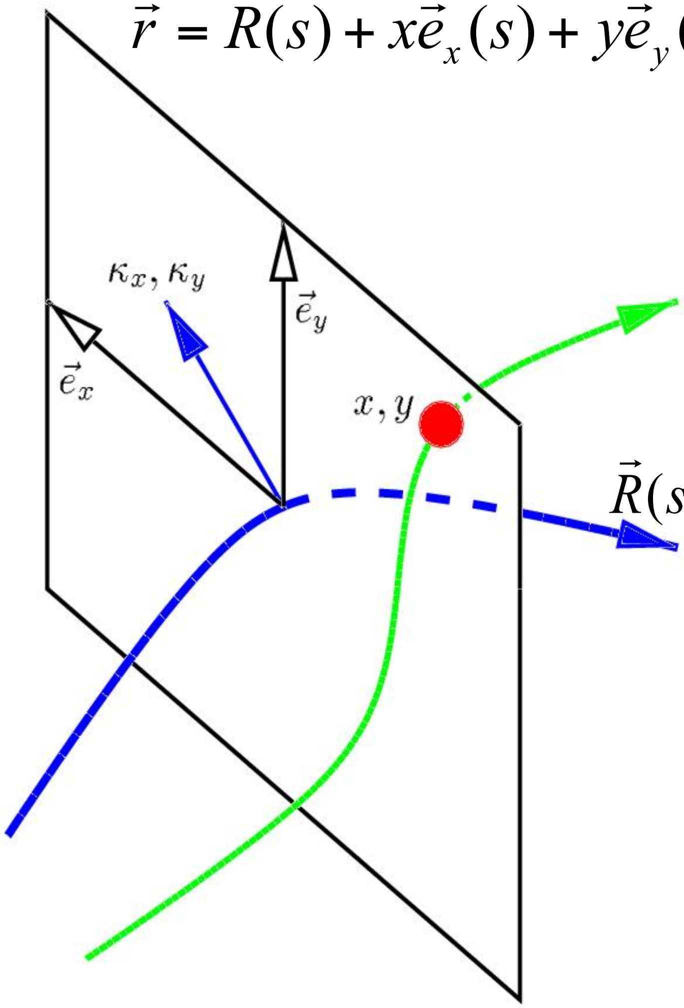
$$0 = \tfrac{d}{ds}\left(\vec{e}_b \cdot \vec{e}_\kappa\right) = \vec{e}_\kappa \cdot \tfrac{d}{ds}\vec{e}_b + T'$$

$$\tfrac{d}{ds}\vec{e}_b = -T'\vec{e}_\kappa$$

$$\vec{r}\,' = (x' - yT')\vec{e}_\kappa + (y' + xT')\vec{e}_b + (1 + x\kappa)\vec{e}_s$$

The Curvi-linear System

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$



$$\vec{e}_x \equiv \vec{e}_\kappa \cos(T) - \vec{e}_b \sin(T)$$
$$\vec{e}_y \equiv \vec{e}_\kappa \sin(T) + \vec{e}_b \cos(T)$$

$$\tfrac{d}{ds}\vec{e}_s = -\kappa_x \vec{e}_x - \kappa_y \vec{e}_y$$
$$\tfrac{d}{ds}\vec{e}_x = \kappa \cos(T)\vec{e}_s = \kappa_x \vec{e}_s$$
$$\tfrac{d}{ds}\vec{e}_y = \kappa \sin(T)\vec{e}_s = \kappa_y \vec{e}_s$$

$$\tfrac{d}{ds}\vec{r} = x'\vec{e}_x + y'\vec{e}_y + (1 + x\kappa_x + y\kappa_y)\vec{e}_s$$

Phase Space ODE

$$\frac{d}{ds}\vec{r} = x'\vec{e}_x + y'\vec{e}_y + \underbrace{(1 + x\kappa_x + y\kappa_y)}_{h}\vec{e}_s$$

$$\frac{d}{dt}\vec{p} = \vec{F}$$

$$\frac{d}{ds}\vec{r} = \dot{s}^{-1}\frac{d}{dt}\vec{r} = \dot{s}^{-1}\frac{1}{m\gamma}\vec{p} = \frac{h}{p_s}\vec{p}$$

$$\frac{d}{ds}\vec{p} = (p_x' - p_s\kappa_x)\vec{e}_x + (p_y' - p_s\kappa_y)\vec{e}_y + (p_s' + \kappa_x p_x + \kappa_y p_y)\vec{e}_s$$

$$= \dot{s}^{-1}\frac{d}{dt}\vec{p} = \dot{s}^{-1}\vec{F} = \frac{m\gamma h}{p_s}\vec{F}$$

$$\begin{pmatrix} x' \\ y' \\ p_x' \\ p_y' \end{pmatrix} = \begin{pmatrix} \frac{h}{p_s}p_x \\ \frac{h}{p_s}p_y \\ \frac{m\gamma h}{p_s}F_x + p_s\kappa_x \\ \frac{m\gamma h}{p_s}F_y + p_s\kappa_y \end{pmatrix}$$

$$t' = \dot{s}^{-1} = \frac{hm\gamma}{p_s}$$

$$E = \sqrt{(pc)^2 + (mc^2)^2}$$

$$E' = \frac{d}{dp}\sqrt{(pc)^2 + (mc^2)^2}\,\frac{d}{ds}p = c^2\frac{\vec{p}}{E}\frac{d}{ds}\vec{p} = \frac{h}{p_s}\vec{p}\cdot\vec{F}$$

6 Dimensional Phase Space

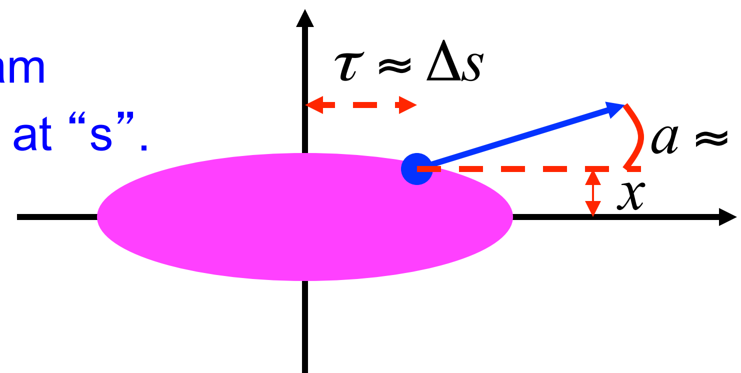
Using a reference momentum p_0 and a reference time t_0:

$$\vec{z} = (x, a, y, b, \tau, \delta)$$

$$a = \frac{p_x}{p_0}, \quad b = \frac{p_y}{p_0}, \quad \delta = \frac{E - E_0}{E_0}, \quad \tau = (t_0 - t)\frac{c^2}{v_0} = (t_0 - t)\frac{E_0}{p_0}$$

Usually p_0 is the design momentum of the beam

And t_0 is the time at which the bunch center is at "s".



$$\left.\begin{array}{l} x' = \partial_{p_x} K \\ p_x' = -\partial_x K \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} x' = \partial_a K/p_0 \,, & a' = -\partial_x K/p_0 \\ y' = \partial_b K/p_0 \,, & b' = -\partial_y K/p_0 \end{array}\right.$$

$$-t' = \partial_E K \quad \Rightarrow \quad \tau' = \tfrac{c^2}{v_0}\partial_\delta K/E_0 = \partial_\delta K/p_0$$

$$E' = -\partial_{-t} K \quad \Rightarrow \quad \delta' = -\tfrac{1}{E_0}\partial_\tau K \tfrac{c^2}{v_0} = -\partial_\tau K/p_0$$

New Hamiltonian:

$$\tilde{H} = K/p_0$$

Simplified Equation of Motion

Only magnetic fields: $\vec{E} = 0$

Mid-plane symmetry: $B_x(x,y,s) = -B_x(x,-y,s)\,,\quad B_y(x,y,s) = B_y(x,-y,s)$

Linearization in : $B_x(x,y,s) = \frac{p_0}{q} ky\,,\quad B_y(x,y,s) = \frac{p_0}{q}\frac{1}{\rho} + \frac{p_0}{q} kx$

Highly relativistic : $\frac{p-p_0}{p_0} \to \frac{E-E_0}{E_0}\quad,\quad \frac{v-v_0}{v_0} \to 0$

$$x' = \frac{p_x}{p_z} = \frac{p_x}{\sqrt{(p_0+dp)^2 - p_x^2 - p_y^2}} =_1 \frac{p_x}{p_0} = a \qquad \Rightarrow \qquad y' = b$$

$$a' = \frac{(\vec{p}' + p_s \kappa \vec{e}_x)_x}{p_0} = \frac{h}{p_0 v_s}\frac{d}{dt} p_x + \frac{p_s \kappa}{p_0} = -\frac{1+x\kappa}{p_0 v_s} q v_s B_y + \frac{p_s \kappa}{p_0}$$

$$=_1 -(1+x\kappa)(\kappa + kx) + (1+\delta)\kappa = -x(\kappa^2 + k) + \delta\kappa \qquad \Rightarrow \qquad b' = ky$$

$$\tau' = \frac{d(t-t_0)}{ds}\frac{E_0}{p_0} = \left(\frac{1}{v_0} - \frac{h}{v_s}\right)\frac{E_0}{p_0} =_1 -x\kappa \qquad , \qquad \delta' = 0$$

Hamiltonian:

$$H = \frac{1}{2}a^2 + \frac{1}{2}b^2 + \frac{1}{2}k(x^2 - y^2) + \frac{1}{2}\kappa^2 x^2 - \kappa x \delta$$

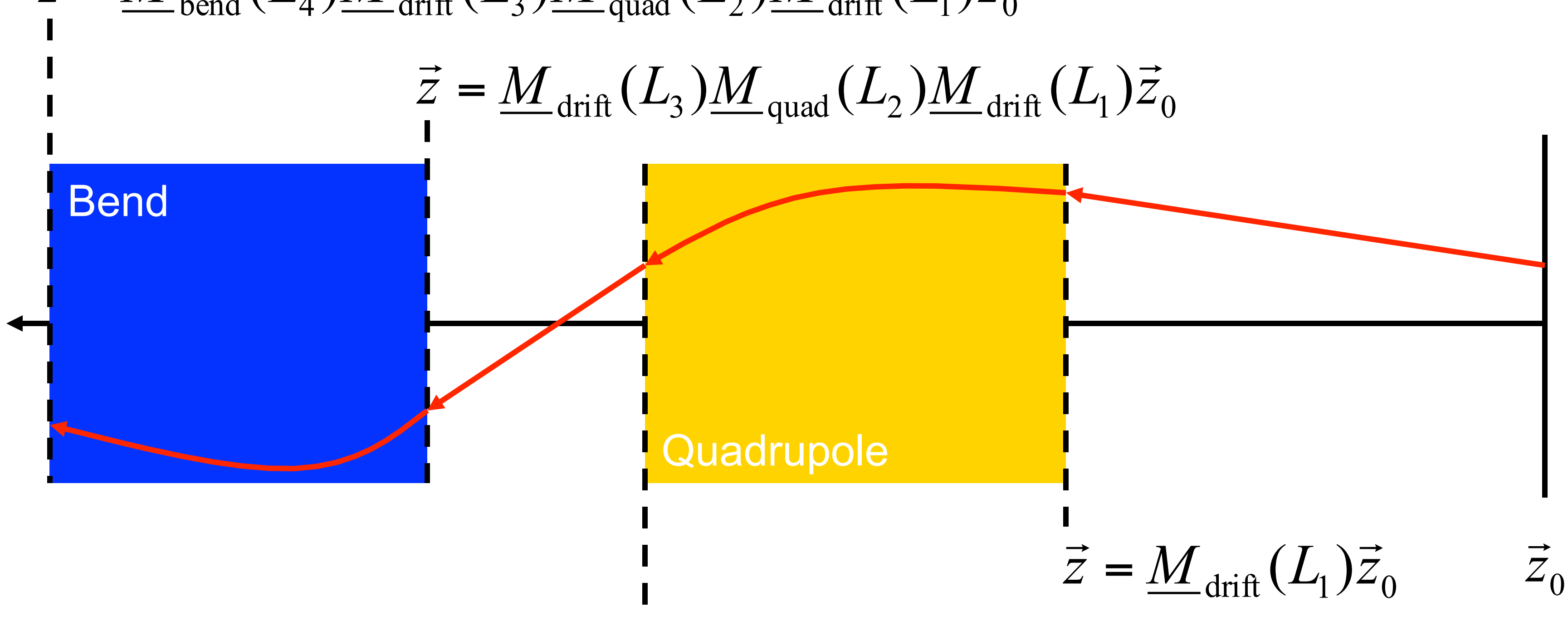
Matrix Solutions

Linear equation of motion: $\vec{z}\,' = \underline{F}(s)\vec{z}$

Matrix solution of the starting condition $\vec{z}(0) = \vec{z}_0$

$$\vec{z} = \underline{M}_{\text{bend}}(L_4)\underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$



$$\vec{z} = \underline{M}_{\text{drift}}(L_1)\vec{z}_0 \qquad \vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

The Drift

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Note that in nonlinear expansion $x' \neq a$ so that the drift does not have a linear transport map even though $x(s) = x_0 + x_0' s$ is completely linear.

$$\begin{pmatrix} x \\ a \\ y \\ b \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} x_0 + s a_0 \\ a \\ y_0 + s b_0 \\ b_0 \\ \tau_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \underline{0} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \vec{z}_0$$

The Dipole Equation of Motion

$$x'' + x\kappa^2 = \delta\kappa$$

$$y'' = 0$$

$$\tau' + x\kappa = 0$$

Homogeneous solution:

$$x_H'' = -x_H\kappa^2 \quad \Rightarrow \quad x_H = A\cos(\kappa s) + B\sin(\kappa s)$$ (natural ring focusing)

Variation of constants:

$$x = A(s)\cos(\kappa s) + B(s)\sin(\kappa s)$$

$$x' = -A\kappa\sin(\kappa s) + B\kappa\cos(\kappa s) + \underbrace{A'\cos(\kappa s) + B'\sin(\kappa s)}_{\equiv 0}$$

$$x'' = -\kappa^2 x \underbrace{- A'\kappa\sin(\kappa s) + B'\kappa\cos(\kappa s)}_{=\delta\kappa} = -\kappa^2 x + \delta\kappa$$

$$\begin{pmatrix} \cos(\kappa s) & \sin(\kappa s) \\ -\sin(\kappa s) & \cos(\kappa s) \end{pmatrix} \begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

The Dipole

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} \cos(\kappa s) & -\sin(\kappa s) \\ \sin(\kappa s) & \cos(\kappa s) \end{pmatrix} \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \delta\, \kappa^{-1} \begin{pmatrix} \cos(\kappa s) \\ \sin(\kappa s) \end{pmatrix} + \begin{pmatrix} A_H \\ B_H \end{pmatrix} \quad \text{with} \quad x = A\cos(\kappa s) + B\sin(\kappa s)$$

$$\tau' = -x\kappa$$

$$\underline{M} = \begin{pmatrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) & 0 & 0 & 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ -\kappa\sin(\kappa s) & \cos(\kappa s) & 0 & 0 & 0 & \sin(\kappa s) \\ 0 & 0 & 1 & s & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] & 0 & 0 & 1 & \kappa^{-1}[\sin(\kappa s)-s\kappa] \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$