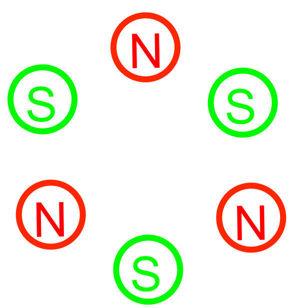
Sextupoles (revisited)

$$\psi = \Psi_3 \,\mathrm{Im}\{(x - iy)^3\} = \Psi_3 \cdot (y^3 - 3x^2 y) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

C_3 Symmetry



i) Sextupole fields hardly influence the particles close to the center, where one can linearize in x and y.

ii) In linear approximation a by Δx shifted sextupole has a quadrupole field.

iii) When Δx depends on the energy, one can build an energy dependent quadrupole.

$$\vec{B} = -\vec{\nabla}\psi = \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

$$x \mapsto \Delta x + x$$

$$\vec{B} \approx \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix} + 6\Psi_3 \Delta x \begin{pmatrix} y \\ x \end{pmatrix} + O(\Delta x^2)$$

$$k_2 = 3!\,\Psi_3 \Rightarrow k_1 = k_2 \Delta x$$

Chromaticity and its Correction

Chromaticity ξ = energy dependence of the tune

$$\upsilon(\delta) = \upsilon + \tfrac{\partial \upsilon}{\partial \delta}\delta + \ldots$$

$$\xi = \tfrac{\partial \upsilon}{\partial \delta} \quad \text{with} \quad \upsilon = \tfrac{\mu}{2\pi}$$

Natural chromaticity ξ_0 = energy dependence of the tune due to quadrupoles only

$$\xi_{x0} = -\frac{1}{4\pi}\oint \beta_x(\hat{s})k_1(\hat{s})d\hat{s} \qquad \xi_{y0} = \frac{1}{4\pi}\oint \beta_y(\hat{s})k_1(\hat{s})d\hat{s}$$

Particles with energy difference oscillate around the periodic dispersion leading to a quadrupole effect in sextupoles that also shifts the tune:

$$\boxed{\xi_x = \frac{1}{4\pi}\oint \beta_x(-k_1 + \eta_x k_2)d\hat{s}} \qquad \boxed{\xi_y = \frac{1}{4\pi}\oint \beta_y(k_1 - \eta_x k_2)d\hat{s}}$$

Typically the the chormaticity ξ is chosen to be slightly positive, between 0 and 3.

Perturbations

$$\begin{pmatrix} x' \\ a' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix}\begin{pmatrix} x \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta f \end{pmatrix}$$

$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi+\phi_0) \\ \cos(\psi+\phi_0) \end{pmatrix} = \sqrt{2J}\,\underline{\beta}\,\vec{S}$$

This would be a solution with constant J and ϕ when $\Delta f=0$.

Variation of constants:

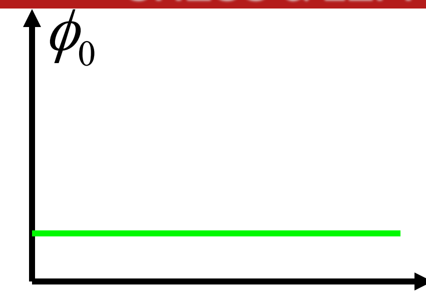
$$\frac{J'}{\sqrt{2J}}\underline{\beta}\,\vec{S} + \sqrt{2J}\,\phi_0'\begin{pmatrix} 0 & \sqrt{\beta} \\ -\frac{1}{\sqrt{\beta}} & -\frac{\alpha}{\sqrt{\beta}} \end{pmatrix}\vec{S} = \begin{pmatrix} 0 \\ \Delta f \end{pmatrix}$$

$$\frac{J'}{\sqrt{2J}}\vec{S} + \sqrt{2J}\,\phi_0'\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\vec{S} = \underline{\beta}^{-1}\begin{pmatrix} 0 \\ \Delta f \end{pmatrix} \quad \text{with} \quad \underline{\beta}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

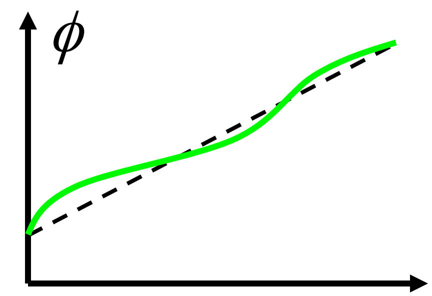
$$\frac{J'}{\sqrt{2J}} = \cos(\psi+\phi_0)\sqrt{\beta}\Delta f \quad , \quad \sqrt{2J}\,\phi_0' = -\sin(\psi+\phi_0)\sqrt{\beta}\Delta f$$

Simplification of linear motion

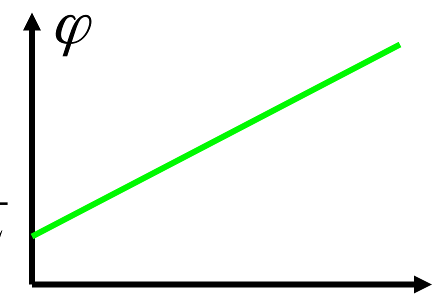
$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi + \phi_0) \\ \cos(\psi + \phi_0) \end{pmatrix} \Rightarrow \begin{matrix} J' = 0 \\ \phi_0' = 0 \end{matrix}$$



$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix} \Rightarrow \begin{matrix} J' = 0 \\ \phi' = \frac{1}{\beta} \end{matrix}$$



$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi - \mu\frac{s}{L} + \varphi) \\ \cos(\psi - \mu\frac{s}{L} + \varphi) \end{pmatrix} \Rightarrow \begin{matrix} J' = 0 \\ \varphi' = \mu\frac{1}{L} \end{matrix}$$



$$\tilde{\psi} = \psi - \mu\tfrac{s}{L} \Rightarrow \tilde{\psi}(s+L) = \tilde{\psi}(s)$$

Corresponds to Floquet' s Theorem