Quasi-periodic Perturbation

$$J' = \cos(\psi + \phi_0)\sqrt{2J\beta}\Delta f \quad , \quad \phi_0' = -\sin(\psi + \phi_0)\sqrt{\tfrac{\beta}{2J}}\Delta f$$

$$\tilde{\psi} = \psi - \mu\tfrac{s}{L} \, , \; \varphi = \mu\tfrac{s}{L} + \phi_0$$

$$J' = \cos(\tilde{\psi} + \varphi)\sqrt{2J\beta}\Delta f \quad , \quad \varphi' = \mu\tfrac{1}{L} - \sin(\tilde{\psi} + \varphi)\sqrt{\tfrac{\beta}{2J}}\Delta f$$

New independent variable $\vartheta = 2\pi\frac{s}{L}$

$$\tfrac{d}{d\vartheta}J = \cos(\tilde{\psi} + \varphi)\sqrt{2J\beta}\Delta f\,\tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi = \upsilon - \sin(\tilde{\psi} + \varphi)\sqrt{\tfrac{\beta}{2J}}\Delta f\,\tfrac{L}{2\pi}$$

$$\Delta f(x) = \Delta f(\sqrt{2J\beta}\sin(\tilde{\psi} + \varphi))$$

The perturbations are 2π periodic in ϑ and in φ

φ is approximately $\varphi \approx \upsilon \cdot \vartheta$

For irrational ν, the perturbations are quasi-periodic.

Tune Shift with Amplitude

$$\frac{d}{d\vartheta}J = \cos(\Psi + \varphi)\sqrt{2J\beta}\,\Delta f\,\frac{L}{2\pi} \quad , \quad \frac{d}{d\vartheta}\varphi = \upsilon - \sin(\Psi + \varphi)\sqrt{\frac{\beta}{2J}}\,\Delta f\,\frac{L}{2\pi}$$

$$\frac{d}{d\vartheta}\varphi = \partial_J H \quad , \quad \frac{d}{d\vartheta}J = -\partial_\phi H \quad , \quad H(\varphi, J, \vartheta) = \upsilon \cdot J - \frac{L}{2\pi}\int_0^x \Delta f(\hat{x}, s)\, d\hat{x}$$

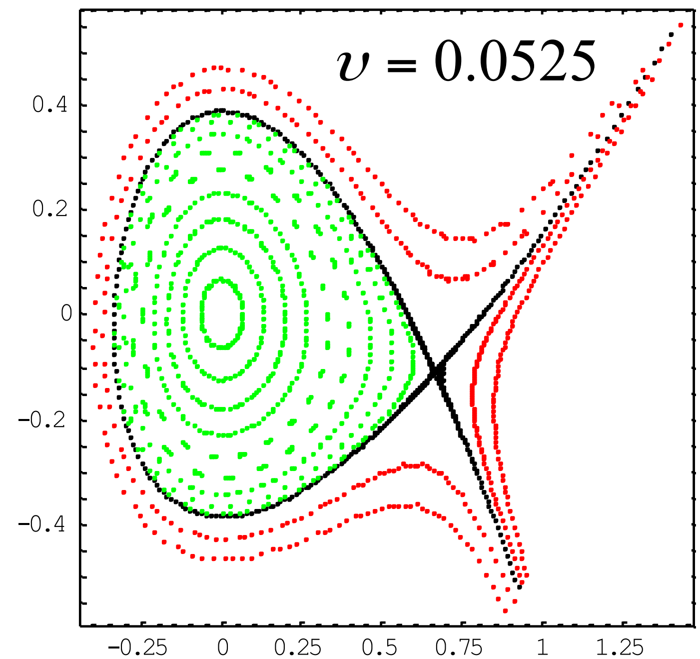
The motion remains Hamiltonian in the perturbed coordinates !

If there is a part in $\partial_J H$ that does not depend on φ, s $\Rightarrow$ Tune shift

The effect of other terms tends to average out.

$$\varphi(\vartheta) - \varphi_0 \approx \vartheta \cdot \partial_J \langle H \rangle_{\varphi,\vartheta}(J)$$

$$\upsilon(J) = \upsilon + \partial_J \langle \Delta H \rangle_{\varphi,\vartheta}(J)$$



Tune Shift Examples

$$H(\varphi,J)=\upsilon\cdot J-\tfrac{L}{2\pi}\int_0^x\Delta f(\hat{x},s)\,d\hat{x}\quad,\quad \Delta\upsilon(J)=\partial_J\langle\Delta H\rangle_{\varphi,\vartheta}$$

Quadrupole: $\Delta f=-\Delta k\,x$

$$\Delta H=\tfrac{L}{2\pi}\Delta k\tfrac{1}{2}x^2=\tfrac{L}{2\pi}\Delta k\,J\beta\sin^2(\tilde{\psi}+\varphi)$$

$$\langle\Delta H\rangle_{\varphi,\vartheta}=\tfrac{1}{2\pi}\int_0^{2\pi}\Delta k\beta\,d\vartheta\,L\frac{J}{4\pi}=\int_0^L\Delta k\beta\,ds\,\frac{J}{4\pi}\Rightarrow\Delta\upsilon=\tfrac{1}{4\pi}\oint\Delta k\beta\,ds$$

Sextupole: $\Delta f=-k_2\tfrac{1}{2}x^2$

$$\Delta H=\tfrac{L}{2\pi}k_2\tfrac{1}{3!}x^3=\tfrac{L}{2\pi}k_2\tfrac{1}{3!}\sqrt{2J\beta}^{\,3}\sin^3(\tilde{\psi}+\varphi)$$

$$\langle\Delta H\rangle_{\varphi,\vartheta}=0\quad\Rightarrow\quad\Delta\upsilon=0$$

Octupole: $\Delta f=-k_3\tfrac{1}{3!}x^3$

$$\Delta H=\tfrac{L}{2\pi}k_3\tfrac{1}{4!}x^4=\tfrac{L}{2\pi}k_3\tfrac{1}{3!}(J\beta)^2\sin^4(\tilde{\psi}+\varphi)$$

$$\langle\Delta H\rangle_{\varphi,\vartheta}=\tfrac{J^2}{3!2\pi}\oint k_3\beta^2ds\left\langle\tfrac{1}{2^4}(e^{i\varphi}-e^{-i\varphi})^4\right\rangle_\varphi\Rightarrow\Delta\upsilon=J\tfrac{1}{16\pi}\oint k_3\beta^2ds$$

Nonlinear Resonances

$$\frac{d}{d\vartheta}J=\cos(\tilde{\psi}+\varphi)\sqrt{2J\beta}\Delta f\,\frac{L}{2\pi}\quad,\quad\frac{d}{d\vartheta}\varphi=\upsilon-\sin(\tilde{\psi}+\varphi)\sqrt{\frac{\beta}{2J}}\Delta f\,\frac{L}{2\pi}$$

$$\frac{d}{d\vartheta}\varphi=\partial_J H\quad,\quad\frac{d}{d\vartheta}J=-\partial_\phi H\quad,\quad H(\varphi,J,\vartheta)=\upsilon\cdot J-\frac{L}{2\pi}\int_0^x\Delta f(\hat{x},s)\,d\hat{x}$$

The effect of the perturbation is especially strong when

$\cos(\tilde{\psi}+\varphi)\sqrt{\beta}\Delta f$ or $\sin(\tilde{\psi}+\varphi)\sqrt{\beta}\Delta f$

has contributions that hardly change, i.e. the change of

$\sqrt{\beta(\vartheta)}\Delta f(x(\vartheta),\vartheta)$ is in resonance with the rotation angle $\varphi(\vartheta)$.

Periodicity allows Fourier expansion:

$$H(\varphi,J,\vartheta)=\sum_{n,m=-\infty}^{\infty}\widehat{H}_{nm}(J)e^{i[n\vartheta+m\varphi]}=\sum_{n,m=-\infty}^{\infty}H_{nm}(J)\cos(n\vartheta+m\varphi+\Psi_{nm}(J))$$

$$H_{00}(J)=\langle H(\varphi,J,s)\rangle_{\varphi,s}\Rightarrow$$ Tune shift