



The 6D Equation of Motion



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Usually one prefers to compute the trajectory as a function of “s” along the accelerator even when the energy is not conserved, as when accelerating cavities are in the accelerator.

Then the energy “E” and the time “t” at which a particle arrives at the cavities are important. And the equations become 6 dimensional again:

$$\frac{d}{ds} \vec{z} = \vec{f}_z(\vec{z}, s), \quad \vec{z} = (x, y, p_x, p_y, -t, E)$$

But: $\vec{z} = (\vec{r}, \vec{p})$ is an especially suitable variable, since it is a phase space vector so that its equation of motion comes from a Hamiltonian, or by variation principle from a Lagrangian.

$$\delta \int [p_x \dot{x} + p_y \dot{y} + p_s \dot{s} - H(\vec{r}, \vec{p}, t)] dt = 0 \quad \Rightarrow \quad \text{Hamiltonian motion}$$

$$\delta \int [p_x x' + p_y y' - H t' + p_s (x, y, p_x, p_y, t, H)] ds = 0 \quad \Rightarrow \quad \text{Hamiltonian motion}$$

The new canonical coordinates are: $\vec{z} = (x, y, p_x, p_y, -t, E)$ with $E = H$

The new Hamiltonian is: $K = -p_s(\vec{z}, s)$



The equations of motion can be determined by one function:

$$\frac{d}{ds} x = \partial_{p_x} H(\vec{z}, s), \quad \frac{d}{ds} p_x = -\partial_x H(\vec{z}, s), \quad \dots$$

$$\frac{d}{ds} \vec{z} = \underline{J} \vec{\partial} H(\vec{z}, s) = \vec{F}(\vec{z}, s) \quad \text{with} \quad \underline{J} = \text{diag}(\underline{J}_2), \quad \underline{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The force has a **Hamiltonian Jacobi Matrix**:

A linear force: $\vec{F}(\vec{z}, s) = \underline{F}(s) \cdot \vec{z}$

The **Jacobi Matrix** of a linear force: $\underline{F}(s)$

The general Jacobi Matrix : $F_{ij} = \partial_{z_j} F_i \quad \text{or} \quad \underline{F} = \left(\vec{\partial} \vec{F}^T \right)^T$

Hamiltonian Matrices: $\underline{F} \underline{J} + \underline{J} \underline{F}^T = 0$

Prove : $F_{ij} = \partial_{z_j} F_i = \partial_{z_j} J_{ik} \partial_{z_k} H = J_{ik} \partial_k \partial_j H \Rightarrow \underline{F} = \underline{J} \underline{D} \underline{H}$

$$\underline{F} \underline{J} + \underline{J} \underline{F}^T = \underline{J} \underline{D} \underline{J} \underline{H} + \underline{J} \underline{D}^T \underline{J}^T \underline{H} = 0$$



The flow of a Hamiltonian equation of motion has a **symplectic Jacobi Matrix**

The **flow or transport map**: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

A linear flow: $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0$

The Jacobi Matrix of a linear flow: $\underline{M}(s)$

The general **Jacobi Matrix** : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = \left(\vec{\partial}_0 \vec{M}^T \right)^T$

The **Symplectic Group SP(2N)** : $\underline{M} \underline{J} \underline{M}^T = \underline{J}$

$$\frac{d}{ds} \vec{z} = \frac{d}{ds} \vec{M}(s, \vec{z}_0) = \underline{J} \vec{\nabla} H = \vec{F} \quad \frac{d}{ds} M_{ij} = \partial_{z_{0j}} F_i(\vec{z}, s) = \partial_{z_{0j}} M_k \partial_{z_k} F_i(\vec{z}, s)$$

$$\frac{d}{ds} \underline{M}(s, \vec{z}_0) = \underline{F}(\vec{z}, s) \underline{M}(s, \vec{z}_0)$$

$$\underline{K} = \underline{M} \underline{J} \underline{M}^T$$

$$\frac{d}{ds} \underline{K} = \frac{d}{ds} \underline{M} \underline{J} \underline{M}^T + \underline{M} \underline{J} \frac{d}{ds} \underline{M}^T = \underline{F} \underline{M} \underline{J} \underline{M}^T + \underline{M} \underline{J} \underline{M}^T \underline{F}^T = \underline{F} \underline{K} + \underline{K} \underline{F}^T$$

$\underline{K} = \underline{J}$ is a solution. Since this is a linear ODE, $\underline{K} = \underline{J}$ is the unique solution.



For every symplectic transport map there is a **Hamilton function**

The **flow or transport map**: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

Force vector: $\vec{h}(\vec{z}, s) = -\underline{J} \left[\frac{d}{ds} \vec{M}(s, \vec{z}_0) \right]_{\vec{z}_0 = \vec{M}^{-1}(\vec{z}, s)}$

Since then: $\frac{d}{ds} \vec{z} = \underline{J} \vec{h}(\vec{z}, s)$

There is a Hamilton function H with: $\vec{h} = \vec{\partial} H$

If and only if: $\partial_{z_j} h_i = \partial_{z_i} h_j \implies \underline{h} = \underline{h}^T$

$$\underline{M} \underline{J} \underline{M}^T = \underline{J} \implies \begin{cases} \frac{d}{ds} \underline{M} \underline{J} \underline{M}^T = -\underline{M} \underline{J} \frac{d}{ds} \underline{M}^T \\ \underline{M}^{-1} = -\underline{J} \underline{M}^T \underline{J} \end{cases}$$

$$\vec{h} \circ \vec{M} = -\underline{J} \frac{d}{ds} \vec{M}$$

$$\underline{h}(\vec{M}) \underline{M} = -\underline{J} \frac{d}{ds} \underline{M}$$

$$\underline{h}(\vec{M}) = -\underline{J} \frac{d}{ds} \underline{M} \underline{M}^{-1} = \underline{J} \frac{d}{ds} \underline{M} \underline{J} \underline{M}^T \underline{J} = -\underline{J} \underline{M} \underline{J} \frac{d}{ds} \underline{M}^T \underline{J} = \underline{M}^{-T} \frac{d}{ds} \underline{M}^T \underline{J} = \underline{h}^T$$