Symplectic Flows => Hamilton Function

For every symplectic transport map there is a Hamilton function

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

Force vector: $\vec{h}(\vec{z}, s) = -\underline{J}\left[\frac{d}{ds}\vec{M}(s, \vec{z}_0)\right]_{\vec{z}_0 = \vec{M}^{-1}(\vec{z}, s)}$

Since then: $\frac{d}{ds}\vec{z} = \underline{J}\vec{h}(\vec{z}, s)$

There is a Hamilton function H with: $\vec{h} = \vec{\partial} H$

If and only if: $\partial_{z_j} h_i = \partial_{z_i} h_j \quad \Rightarrow \quad \underline{h} = \underline{h}^T$

$$\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J} \quad \Rightarrow \quad \begin{cases} \frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T = -\underline{M}\,\underline{J}\,\frac{d}{ds}\underline{M}^T \\ \underline{M}^{-1} = -\underline{J}\,\underline{M}^T\,\underline{J} \end{cases}$$

$$\vec{h} \circ \vec{M} = -\underline{J}\frac{d}{ds}\vec{M}$$

$$\underline{h}(\vec{M})\underline{M} = -\underline{J}\frac{d}{ds}\underline{M}$$

$$\underline{h}(\vec{M}) = -\underline{J}\frac{d}{ds}\underline{M}\,\underline{M}^{-1} = \underline{J}\frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T\underline{J} = -\underline{J}\,\underline{M}\,\underline{J}\frac{d}{ds}\underline{M}^T\underline{J} = \underline{M}^{-T}\frac{d}{ds}\underline{M}^T\underline{J} = \underline{h}^T$$

Generating Functions

The motion of particles can be represented by Generating Functions

Each flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

With a Jacobi Matrix : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = (\vec{\partial}_0 \vec{M}^T)^T$

That is Symplectic: $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

Can be represented by a Generating Function:

$$F_1(\vec{q}, \vec{q}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1 \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1$$

$$F_2(\vec{p}, \vec{q}_0, s) \quad \text{with} \quad \vec{q} = \vec{\partial}_p F_2 \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_2$$

$$F_3(\vec{q}, \vec{p}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_3 \quad , \quad \vec{q}_0 = -\vec{\partial}_{p_0} F_3$$

$$F_4(\vec{p}, \vec{p}_0, s) \quad \text{with} \quad \vec{q} = \vec{\partial}_q F_4 \quad , \quad \vec{q}_0 = -\vec{\partial}_{p_0} F_4$$

6-dimensional motion needs only one function ! But to obtain the transport map this has to be inverted.

Symplectic Representations

Hamiltonian

$$\vec{z}\,' = \underline{J}\,\vec{\partial}\,H(\vec{z},s)$$

ODE

$$\vec{z}\,' = \vec{F}\,,\quad \underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = 0$$

Generating Functions

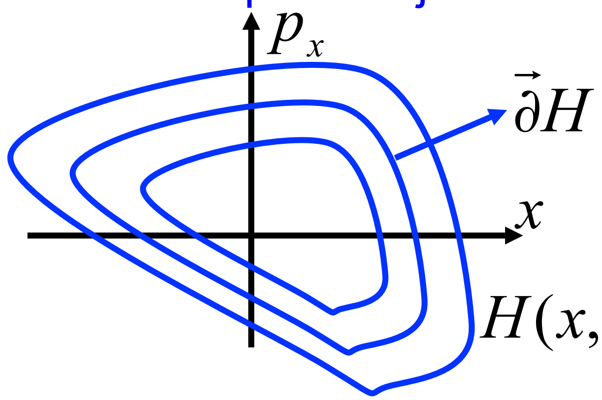
$$(\vec{p},\vec{p}_0) = -\underline{J}\,\vec{\partial}\,F_1(\vec{q},\vec{q}_0,s)$$

Symplectic transport map

$$\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$$

Phase space density in 2D

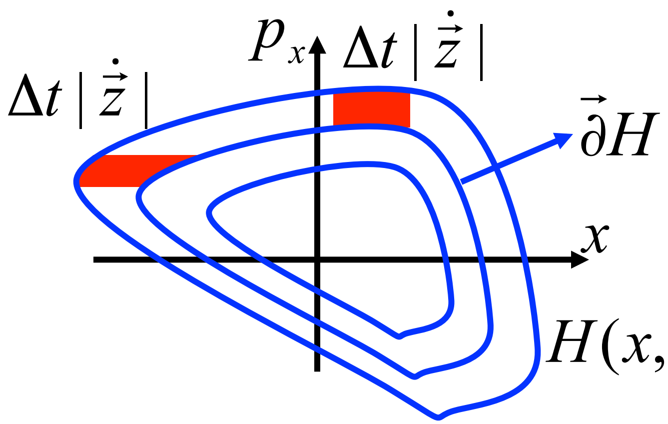
- Phase space trajectories move on surfaces of constant energy



$H(x, p_x) = \text{const}$

$$\frac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial}H \quad \Rightarrow \quad \frac{d}{ds}\vec{z} \perp \vec{\partial}H$$

- A phase space volume does not change when it is transported by Hamiltonian motion. $\Delta E = d\left|\vec{\partial}H\right|$



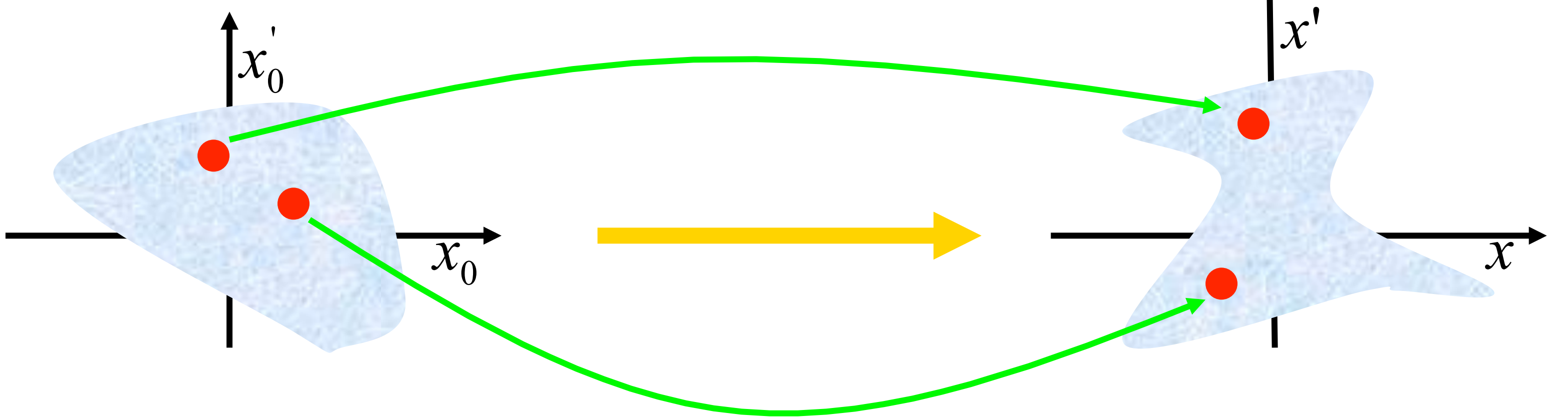
$H(x, p_x) = \text{const}$

The distance d of lines with equal energy is proportional to $d \propto 1/|\vec{\partial}H| \propto |\dot{\vec{z}}\,|^{-1}$

$$\boxed{d * \Delta t\,|\dot{\vec{z}}\,| = \text{const}}$$

Lioville's Theorem

- A phase space volume does not change when it is transported by Hamiltonian motion. $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0$ with $\det[\underline{M}(s)] = +1$



Volume = $V = \iint_V d^n\vec{z} = \iint_{V_0} \left|\frac{\partial \vec{z}}{\partial \vec{z}_0}\right| d^n\vec{z}_0 = \iint_{V_0} |\underline{M}| d^n\vec{z}_0 = \iint_{V_0} d^n\vec{z}_0 = V_0$

Hamiltonian Motion → $V = V_0$

But Hamiltonian requires symplecticity, which is much more than just $\det[\underline{M}(s)] = +1$

Eigenvalues of a Symplectic Matrix

For matrices with real coefficients:

If there is an eigenvector and eigenvalue: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$

then the complex conjugates are also eigenvector and eigenvalue: $\underline{M}\vec{v}_i^{\,*} = \lambda_i^* \vec{v}_i^{\,*}$

For symplectic matrices:

If there are eigenvectors and eigenvalues: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$ with $\underline{J} = \underline{M}^T \underline{J}\,\underline{M}$

then $\vec{v}_i^{\,T} \underline{J}\vec{v}_j = \vec{v}_i^{\,T} \underline{M}^T \underline{J}\,\underline{M}\vec{v}_j = \lambda_i \lambda_j \vec{v}_i^{\,T} \underline{J}\vec{v}_j \quad \Rightarrow \quad \vec{v}_i^{\,T} \underline{J}\vec{v}_j (\lambda_i \lambda_j - 1) = 0$

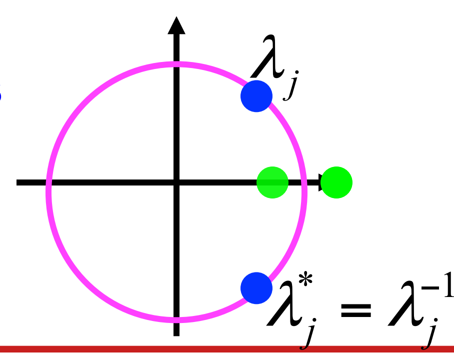
Therefore $\underline{J}\vec{v}_j$ is orthogonal to all eigenvectors with eigenvalues that are not $1/\lambda_j$. Since it cannot be orthogonal to all eigenvectors, there is at least one eigenvector with eigenvalue $1/\lambda_j$

Two dimensions: λ_j is eigenvalue

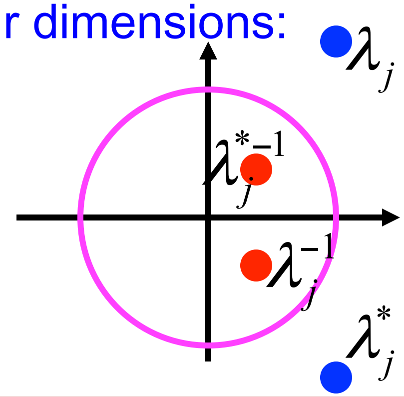
Then $1/\lambda_j$ and λ_j^* are eigenvalues

$\lambda_2 = 1/\lambda_1 = \lambda_1^* \quad \Rightarrow \quad |\lambda_j| = 1$

$\lambda_2 = 1/\lambda_1 = \lambda_2^*$



Four dimensions:



Advantages of Symplecticity

- Transfer matrix of linear motion with

$$\vec{z}(s) = \underline{M}(s)\cdot\vec{z}_0 \quad \text{with} \quad \det(\underline{M}(s)) = +1 \quad \text{and} \quad \vec{v}_i^{\,T}\underline{J}\vec{v}_j(\lambda_i\lambda_j - 1) = 0$$

- One function suffices to compute the total nonlinear transfer map:

$$F_1(\vec{q},\vec{q}_0,s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s) \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s)$$

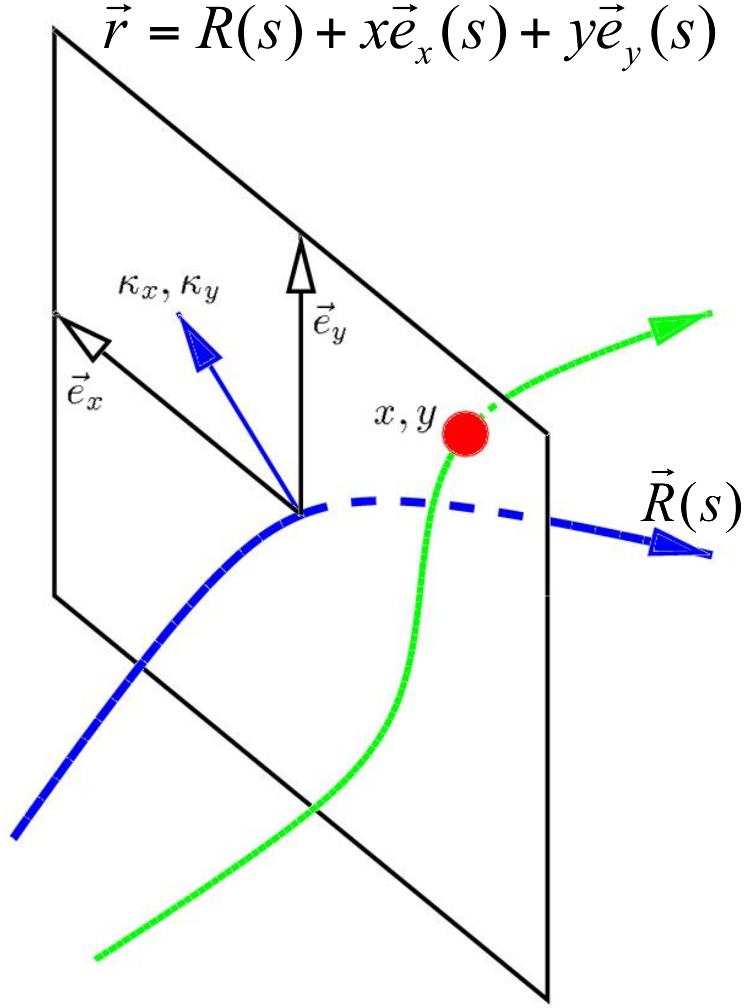
$$\left.\begin{aligned} \vec{z} &= \begin{pmatrix}\vec{q}\\ \vec{p}\end{pmatrix} = \begin{pmatrix}\vec{q}\\ -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s)\end{pmatrix} = \vec{f}(\vec{Q},s) \\ \vec{z}_0 &= \begin{pmatrix}\vec{q}_0\\ \vec{p}_0\end{pmatrix} = \begin{pmatrix}\vec{q}_0\\ \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s)\end{pmatrix} = \vec{g}(\vec{Q},s) \end{aligned}\right\} \quad \begin{aligned} \vec{z} &= \vec{f}(\vec{g}^{-1}(\vec{z}_0,s),s) \\ \vec{M} &= \vec{f}\circ\vec{g}^{-1} \end{aligned}$$

- Therefore Taylor Expansion coefficients of the transport map are related.
- Computer codes can numerically approximate $\vec{M}(s,\vec{z}_0)$ with exact symplectic symmetry.
- Liouville's Theorem for phase space densities holds.

The Frenet Coordinate System

CHESS & LEPP

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$



$$\left|d\vec{R}\right| = ds$$

$$\vec{e}_s \equiv \tfrac{d}{ds}\vec{R}(s)$$

$$\vec{e}_\kappa \equiv -\tfrac{d}{ds}\vec{e}_s \Big/ \left|\tfrac{d}{ds}\vec{e}_s\right|$$

$$\vec{e}_b \equiv \vec{e}_s \times \vec{e}_\kappa$$

$$\tfrac{d}{ds}\vec{e}_s = -\kappa\,\vec{e}_\kappa \quad \text{with} \quad \kappa = \frac{1}{\rho}$$

$$0 = \tfrac{d}{ds}\left(\vec{e}_\kappa \cdot \vec{e}_s\right) = \vec{e}_s \cdot \tfrac{d}{ds}\vec{e}_\kappa - \kappa$$

Accumulated torsion angle T

$$\tfrac{d}{ds}\vec{e}_\kappa = \kappa\,\vec{e}_s + T'\vec{e}_b$$

$$0 = \tfrac{d}{ds}\left(\vec{e}_b \cdot \vec{e}_\kappa\right) = \vec{e}_\kappa \cdot \tfrac{d}{ds}\vec{e}_b + T'$$

$$\tfrac{d}{ds}\vec{e}_b = -T'\vec{e}_\kappa$$

$$\vec{r}\,' = (x' - yT')\vec{e}_\kappa + (y' + xT')\vec{e}_b + (1 + x\kappa)\vec{e}_s$$