Nonlinear Motion

Sextupoles cause nonlinear dynamics, which can be chaotic and unstable.

$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \underline{M}_0 \left[\begin{pmatrix} x_n \\ x'_n \end{pmatrix} - \frac{k_2 l_s}{2} \begin{pmatrix} 0 \\ x_n^{\ 2} \end{pmatrix} \right] \qquad \begin{pmatrix} x_n \\ x'_n \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \hat{x}_n \\ \hat{x}'_n \end{pmatrix}$$

$$\begin{pmatrix} \hat{x}_{n+1} \\ \hat{x}'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \left[\begin{pmatrix} \hat{x}_n \\ \hat{x}'_n \end{pmatrix} - \frac{k_2 l_s}{2} \sqrt{\beta} \begin{pmatrix} 0 \\ \beta \hat{x}_n^2 \end{pmatrix} \right]$$

$$\begin{pmatrix} \hat{x}_f \\ \hat{x}'_f \end{pmatrix} = \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \begin{pmatrix} 1-\cos\mu & \sin\mu \\ -\sin\mu & 1-\cos\mu \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \hat{x}_f^2 \end{pmatrix} = \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \frac{1}{2\sin\frac{\mu}{2}} \begin{pmatrix} -\cos\frac{\mu}{2} \\ \sin\frac{\mu}{2} \end{pmatrix} \hat{x}_f^2$$

$$\left.\begin{aligned} \hat{x}_f &= -\frac{4}{k_2 l_s} \beta^{-\frac{3}{2}} \tan\frac{\mu}{2} \\ \hat{x}'_f &= \frac{4}{k_2 l_s} \beta^{-\frac{3}{2}} \tan^2\frac{\mu}{2} \end{aligned}\right\} \hat{x} = \hat{x}_f + \Delta\hat{x} \qquad J_f = \frac{1}{2}(\hat{x}_f^2 + \hat{x}'^2_f) = \frac{1}{2\beta^3} \left(\frac{4}{k_2 l_s} \frac{\tan\frac{\mu}{2}}{\cos\frac{\mu}{2}}\right)^2$$

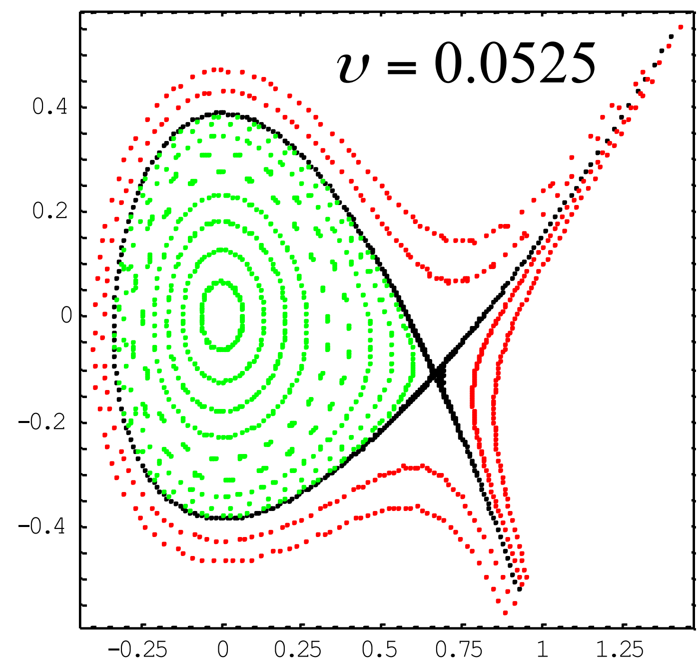
$$\begin{pmatrix} \Delta\hat{x}_{n+1} \\ \Delta\hat{x}'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \left[\begin{pmatrix} \Delta\hat{x}_n \\ \Delta\hat{x}'_n \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \Delta\hat{x}_n^2 - 4\tan\frac{\mu}{2} \Delta\hat{x}_n \end{pmatrix} \right]$$

The Dynamic Aperture

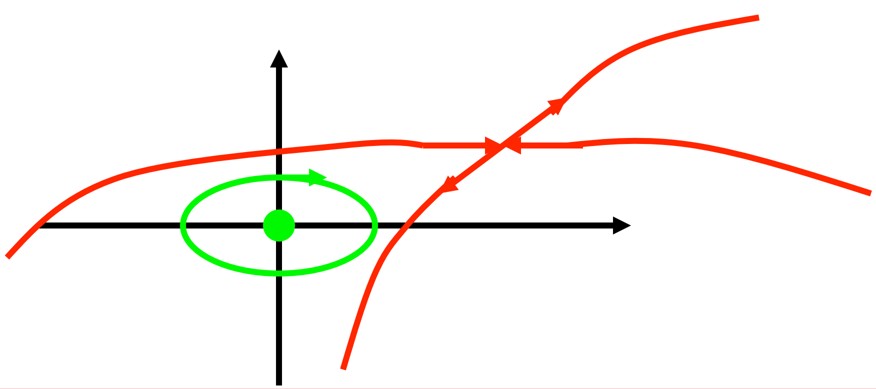
$$\begin{pmatrix}\Delta\hat{x}_{n+1}\\ \Delta\hat{x}'_{n+1}\end{pmatrix}=\begin{pmatrix}\cos\mu & \sin\mu\\ -\sin\mu & \cos\mu\end{pmatrix}\left[\begin{pmatrix}\Delta\hat{x}_n\\ \Delta\hat{x}'_n\end{pmatrix}-\begin{pmatrix}0\\ \frac{k_2 l_s}{2}\beta^{\frac{3}{2}}\Delta\hat{x}_n^2-4\tan\frac{\mu}{2}\Delta\hat{x}_n\end{pmatrix}\right]$$

$$\begin{pmatrix}\Delta\hat{x}_{n+1}\\ \Delta\hat{x}'_{n+1}\end{pmatrix}=\begin{pmatrix}\cos\mu+4\sin\mu\tan\frac{\mu}{2} & \sin\mu\\ -\sin\mu+4\cos\mu\tan\frac{\mu}{2} & \cos\mu\end{pmatrix}\left[\begin{pmatrix}\Delta\hat{x}_n\\ \Delta\hat{x}'_n\end{pmatrix}-\begin{pmatrix}0\\ \frac{k_2 l_s}{2}\beta^{\frac{3}{2}}\Delta\hat{x}_n^2\end{pmatrix}\right]$$

$$\tfrac{1}{2}Tr[\underline{M}]=1-2\sin^2\tfrac{\mu}{2}+4\sin^2\tfrac{\mu}{2}=1+2\sin^2\tfrac{\mu}{2}\geq 1$$



The additional fixed point is unstable !



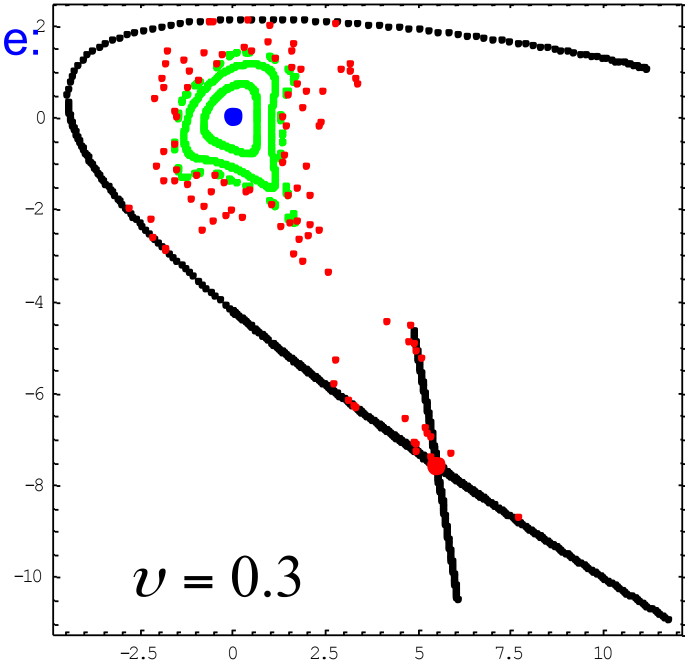
Sextupole Aperture

CHESS & LEPP

If the chormaticity is corrected by a single sextupole:

$$\xi_x = \xi_{0x} + \frac{1}{4\pi}\beta_x\eta_x k_2 l \approx 0$$

$$J_f = \frac{1}{2\beta^3}\left(\frac{4}{k_2 l_s}\frac{\tan\frac{\mu}{2}}{\cos\frac{\mu}{2}}\right)^2 \approx \frac{1}{2\beta}\left(\frac{\eta}{\xi_0\pi}\frac{\sin\frac{\mu}{2}}{\cos^2\frac{\mu}{2}}\right)^2$$



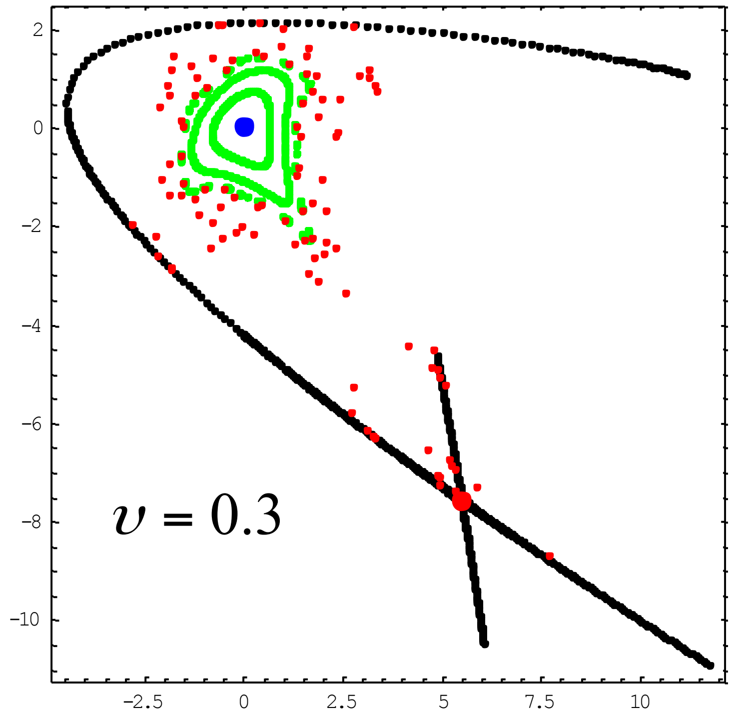
Often the dynamic aperture is much smaller than the fixed point indicates !

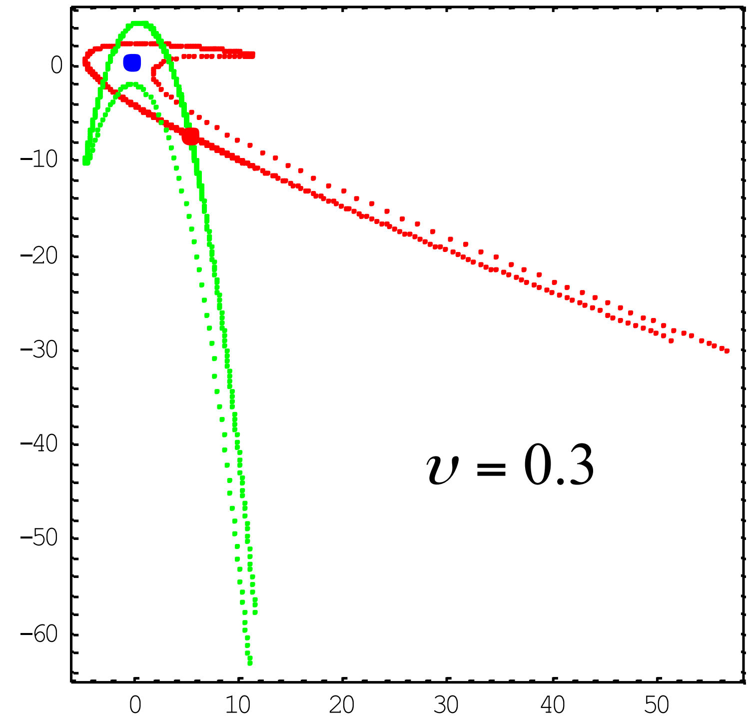
When many sextupoles are used: $\xi_{0x} + \frac{N}{4\pi}\beta_x\eta_x k_2 l \approx 0$

The sum of all k_2^2 is then reduced to about $\sum (k_2 l\beta)^2 \approx N(k_2 l\beta)^2 \approx \frac{1}{N}\left(\frac{4\pi}{\eta}\xi_0\right)^2$

The dynamic aperture is therefore greatly increased when distributed sextupoles are used.

Sextupole Extraction



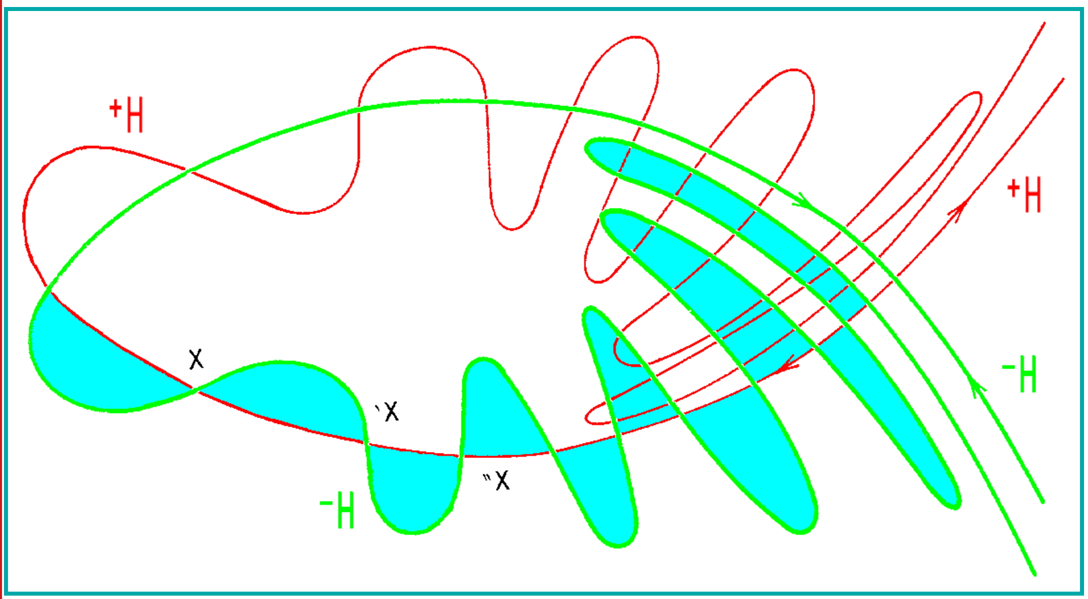


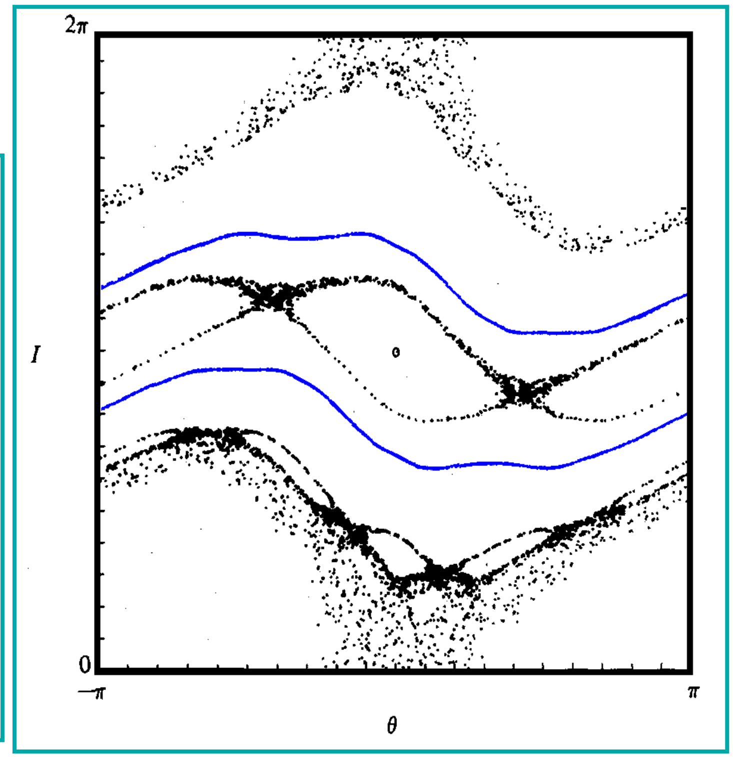
Due to the narrow region of unstable trajectories, sextupoles are used for slow particle extraction at a tune of 1/3.

The intersection of stable and unstable manifolds is a certain indication of chaos.

Homoclinic Points

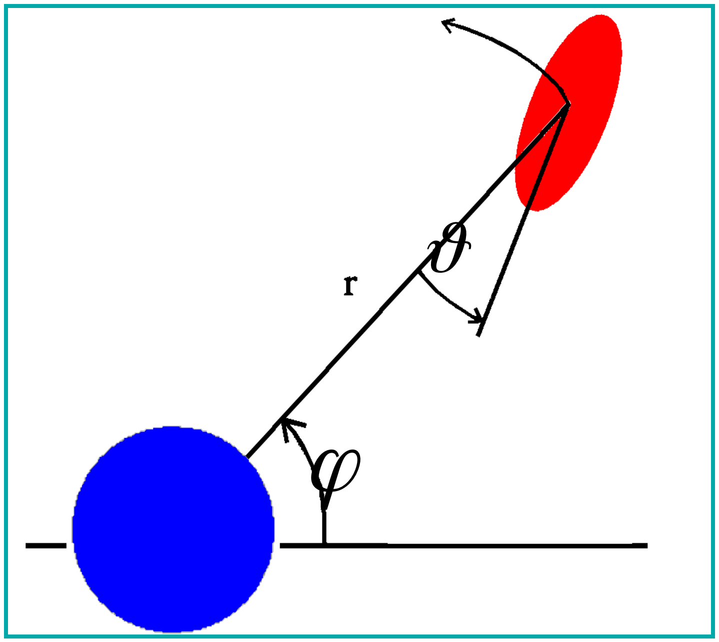
- At instable fixed points, there is a stable and an instabile invariant curve.
- Intersections of these curves (homoclinic points) lead to chaos.

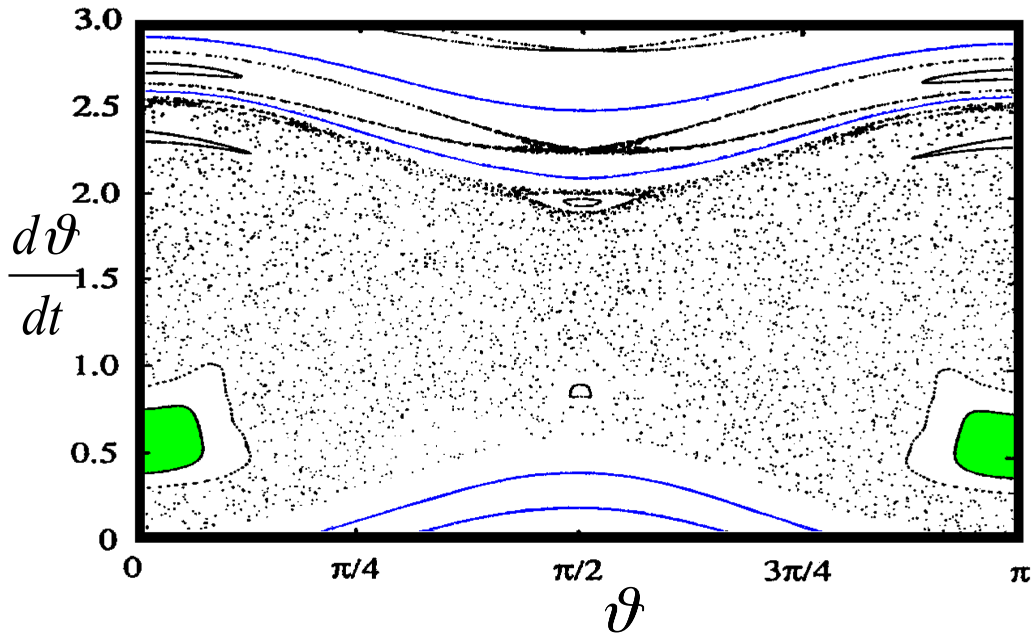




Hyperion: rotation around the vertical

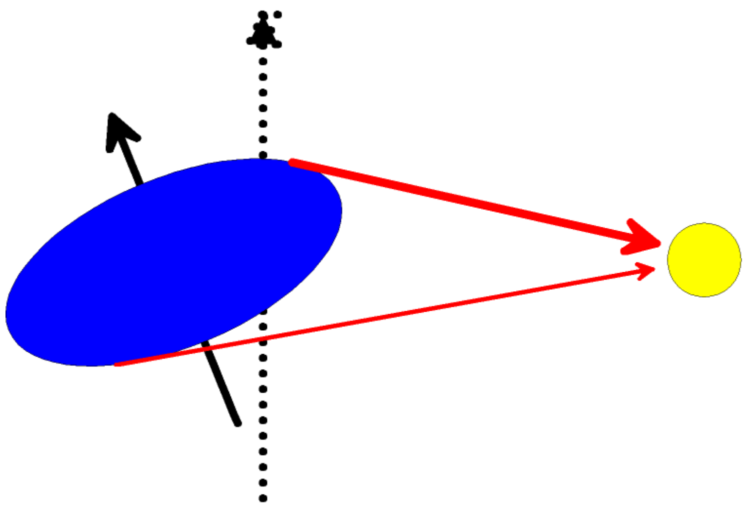
$$\frac{d^2(\vartheta+\varphi(t))}{dt^2} = -\alpha\left(\frac{a}{r(t)}\right)^3 \sin 2\vartheta$$

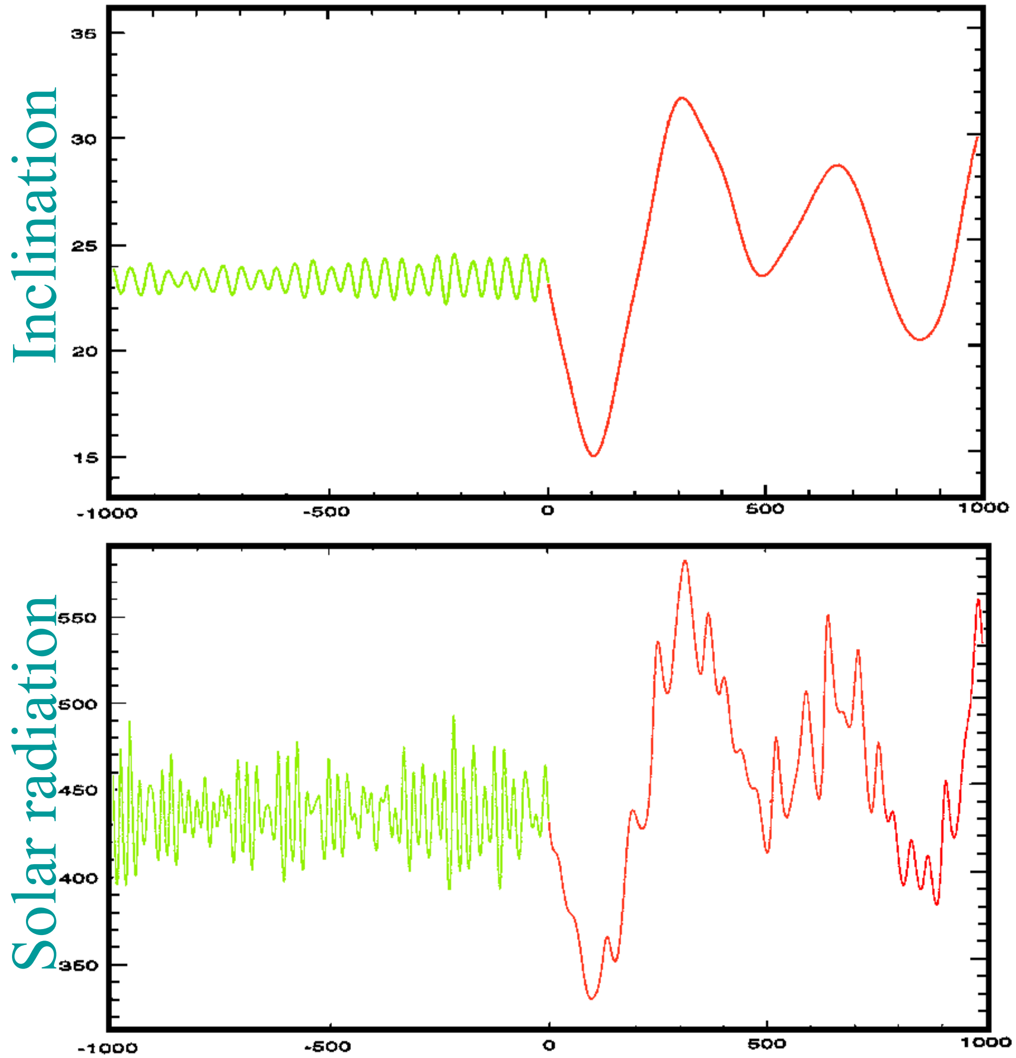




- On the path from Rotation to Libration around the Spin-Orbit-Coupling is a strong chaotic region.

Tilt of the earth





-1Ma with and without moon 1Ma

- Tidal forces from moon and sun cause a stabilization of the rotation axis.

The Single Resonance Model

$$\frac{d}{d\vartheta} J = \sum_{n,m=-\infty}^{\infty} m H_{nm}(J) \sin(n\vartheta + m\varphi + \Psi_{nm}(J))$$

$$\frac{d}{d\vartheta} \varphi = \upsilon + \partial_J \sum_{n,m=-\infty}^{\infty} H_{nm}(J) \cos(n\vartheta + m\varphi + \Psi_{nm}(J))$$

Strong deviation from: $J = J_0 \ , \quad \varphi = \upsilon\vartheta + \varphi_0$

Occur when there is coherence between the perturbation and the phase space rotation: $n + m \frac{d}{ds} \varphi \approx 0$

Resonance condition: tune is rational $\boxed{n + m\ \upsilon = 0}$

On resonance the integral would increases indefinitely !

Neglecting all but the most important term

$$H(\varphi, J, \vartheta) \approx \upsilon J + H_{00}(J) + H_{nm}(J) \cos(n\vartheta + m\varphi + \Psi_{nm}(J))$$

Fixed points

$$\tfrac{d}{d\vartheta} J = mH_{nm}(J)\sin(n\vartheta + m\varphi + \Psi_{nm}(J))$$

$$\tfrac{d}{d\vartheta} \varphi = \upsilon + \Delta\upsilon(J) + \partial_J [H_{nm}(J)\cos(n\vartheta + m\varphi + \Psi_{nm}(J))]$$

$$\Phi = \tfrac{1}{m}[n\vartheta + m\varphi + \Psi_{nm}(J)] \ , \quad \delta = \upsilon + \tfrac{n}{m}$$

$$\tfrac{d}{d\vartheta} J = mH_{nm}(J)\sin(m\Phi) \ , \quad \tfrac{d}{d\vartheta} \Phi = \delta + \Delta\upsilon(J) + H'_{nm}(J)\cos(m\Phi)$$

$$H(\Phi, J, \vartheta) \approx \delta J + H_{00}(J) + H_{nm}(J)\cos(m\Phi)$$

Fixed points: $\tfrac{d}{d\vartheta} J = mH_{nm}(J_f)\sin(m\Phi_f) = 0 \quad \Rightarrow \quad \Phi_f = \tfrac{k}{m}\pi$

If $\delta + \Delta\upsilon(J_f) \pm H'_{nm}(J_f) = 0$ has a solution.

$$\tfrac{d}{d\vartheta} \Delta J = \pm m^2 H_{nm}(J_f)\Delta\Phi \ , \quad \tfrac{d}{d\vartheta} \Delta\Phi = [\Delta\upsilon'(J_f) \pm H''_{nm}(J_f)]\Delta J$$

Stable fixed point for: $H_{nm}(J_f)[H''_{nm}(J_f) \pm \Delta\upsilon'(J_f)] < 0$