Third Integer Resonances

CHESS & LEPP

Sextupole: $\Delta f = -k_2 \frac{1}{2} x^2$

$$\Delta H = \frac{L}{2\pi} k_2 \frac{1}{3!} x^3 = \frac{L}{2\pi} k_2 \frac{1}{3!} \sqrt{2J\beta}^3 \sin^3(\tilde{\psi} + \varphi)$$

$$= \frac{L}{2\pi} k_2 \frac{1}{3!4} \sqrt{2J\beta}^3 [\sin(3[\tilde{\psi} + \varphi]) + 3\sin(\tilde{\psi} + \varphi)]$$

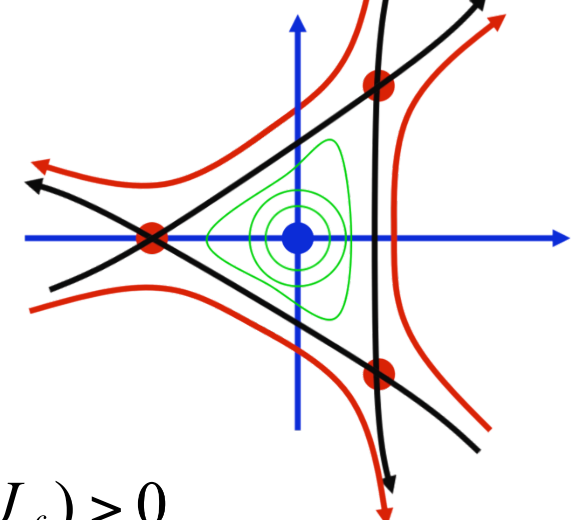
Simplification: one sextupole $k_2(\vartheta) = k_2\delta(\vartheta) = k_2 \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \cos(n\vartheta)$

$$\Delta H = \frac{L}{2\pi} k_2 \frac{1}{3!4} \sqrt{2J\beta}^3 \frac{1}{2\pi} \cos(-n\vartheta + 3\varphi + 3\tilde{\psi} - \tfrac{\pi}{2}) + \dots$$

for $\upsilon \approx \frac{n}{3}$

$$\Delta H \approx A_2 \sqrt{J}^3 \cos(3\Phi)$$

$$\left.\begin{array}{l} \Phi_f = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \dots \\ \delta \pm A_2 \frac{3}{2}\sqrt{J} = 0 \end{array}\right\} \begin{array}{l} \Phi_f = \frac{1}{3}\pi, \pi, \frac{5}{3}\pi \\ \text{for } \delta > 0 \end{array}$$



All these fixed points are instable since $H_{nm}(J_f) H''_{nm}(J_f) > 0$

Fourth Integer Resonances

Octupole: $\Delta f = -k_3 \frac{1}{3!} x^3$, $\Delta H = \frac{L}{2\pi} k_3 \frac{1}{4!} x^4 = \frac{L}{2\pi} k_3 \frac{1}{3!} J^2 \beta^2 \sin^4(\tilde{\psi} + \varphi)$

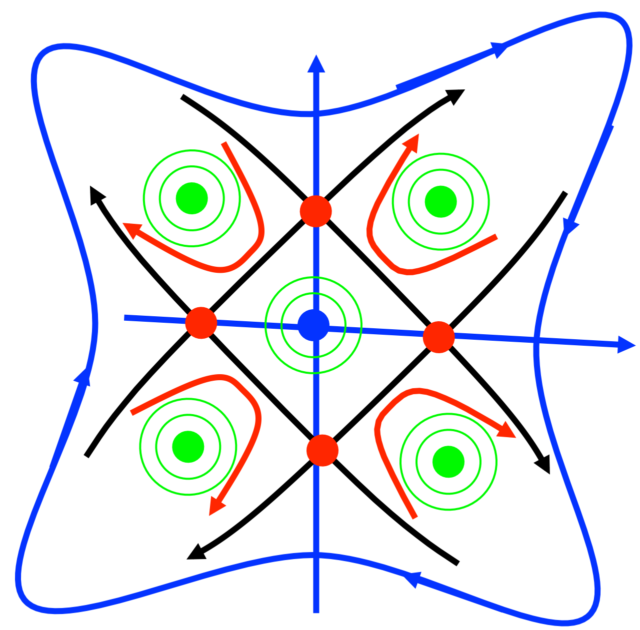
$$= \frac{L}{2\pi} k_3 \frac{1}{3!8} J^2 \beta^2 [\cos(4[\tilde{\psi} + \varphi]) - 4\cos(2[\tilde{\psi} + \varphi]) + 3]$$

Simplification: one octupole $k_3(\vartheta) = k_3 \delta(\vartheta) = k_3 \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \cos(n\vartheta)$

$\Delta H \approx A_3 J^2 [3 + \cos(4\Phi)]$ for $\upsilon \approx \frac{n}{4}$

$\Phi_f = 0, \frac{1}{4}\pi, \frac{2}{4}\pi, \ldots$ Either 8 fixed points: $\delta < 0$

$\delta + A_3 2J(3 \pm 1) = 0$ or none for: $\delta > 0$



$$H_{nm}(J_f)[H''_{nm}(J_f) \pm \Delta\upsilon'(J_f)] < 0$$

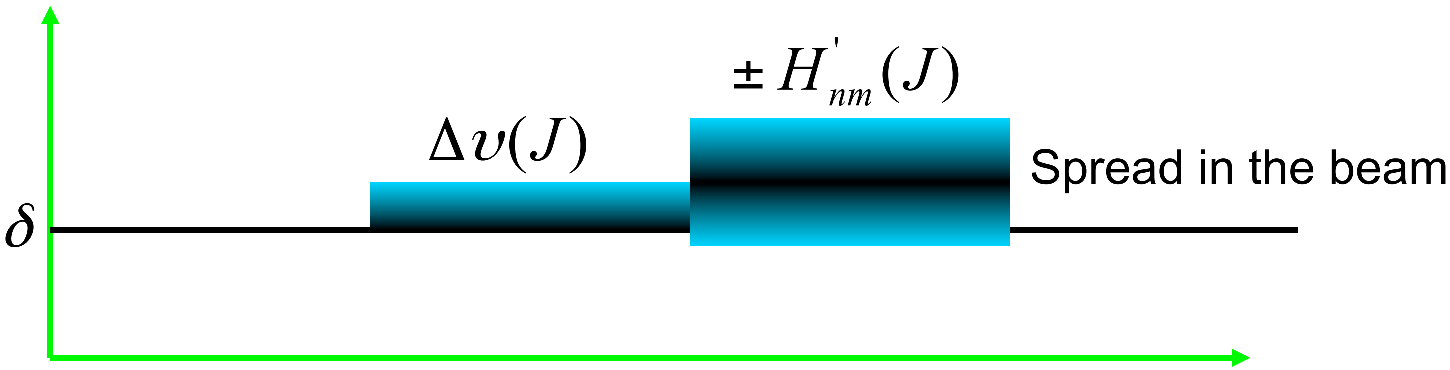
Stability for $(2A_3 J)^2 [1 \pm 3] < 0$,
i.e. for the 4 outer fixed points.

Resonance Width (Strength)

Fixed points: $\frac{d}{d\vartheta}J = mH_{nm}(J_f)\sin(m\Phi_f) = 0 \quad \Rightarrow \quad \Phi_f = \frac{k}{m}\pi$

If $\delta + \Delta\upsilon(J_f) \pm H'_{nm}(J_f) = 0$ has a solution.

δ has to avoid the region $\delta + \Delta\upsilon(J) \pm H'_{nm}(J) = 0$ for all particles.



Assuming that the tune shift and perturbation are monotonous in J:

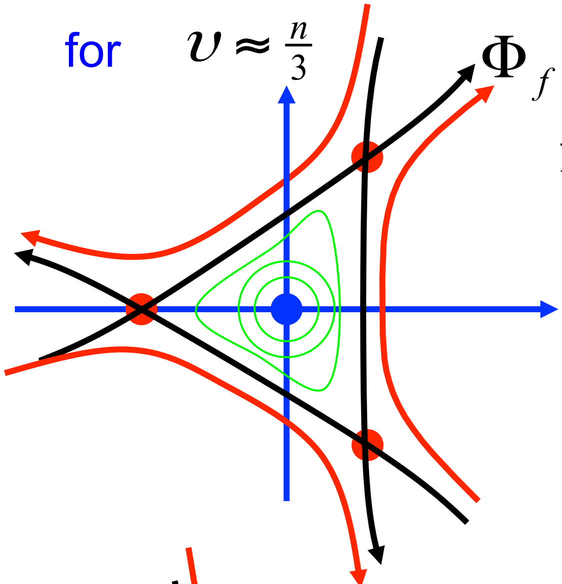
This tune region has the width $\Delta_{nm} = 2\,|\,H'_{nm}(J_{\max})|$ for strong resonances.

Δ_{nm} Is called Resonance Width, Resonance Strength, or Stop-Band Width

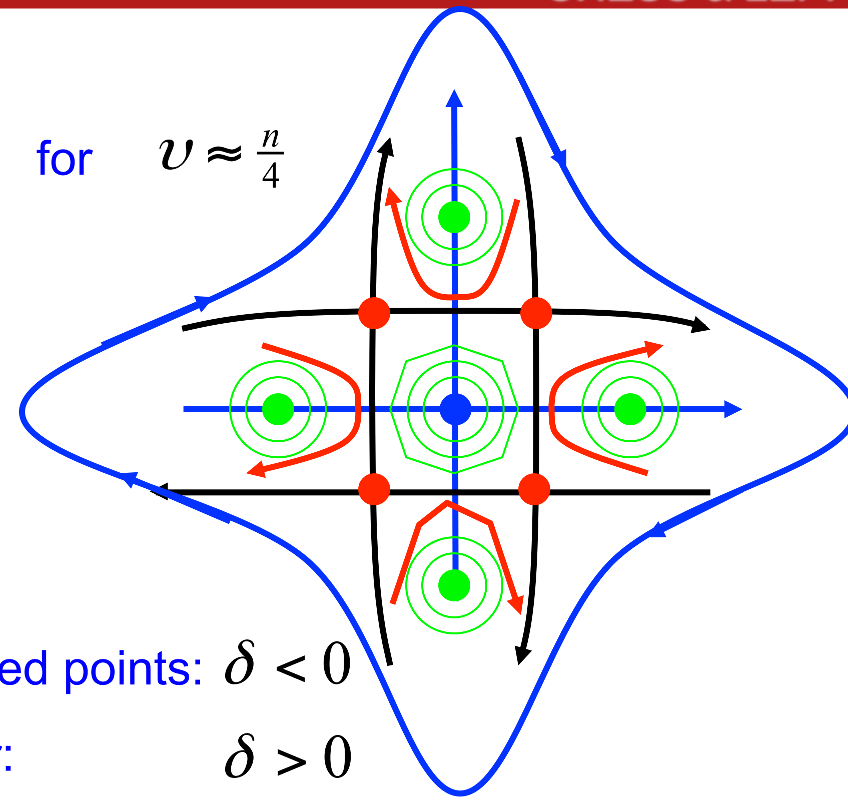
Particle motion in the single resonance model

for $\upsilon \approx \frac{n}{3}$

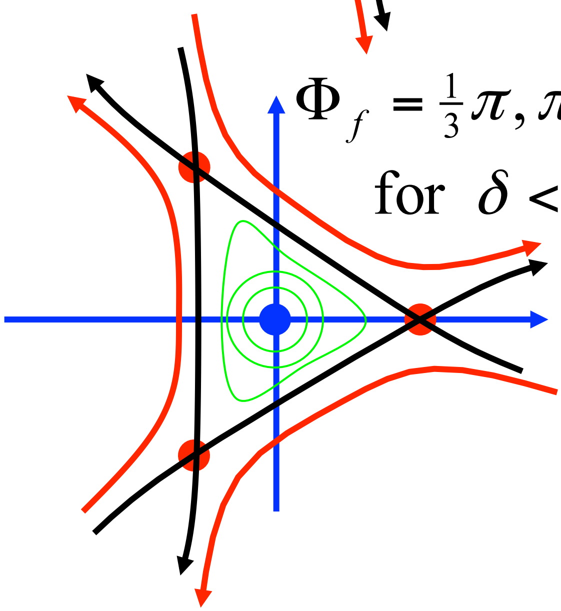
$\Phi_f = \frac{1}{3}\pi, \pi, \frac{5}{3}\pi$ for $\delta > 0$



for $\upsilon \approx \frac{n}{4}$



$\Phi_f = \frac{1}{3}\pi, \pi, \frac{5}{3}\pi$ for $\delta < 0$



Either 8 fixed points: $\delta < 0$

or none for: $\delta > 0$

How can the motion inside the fixed points be simplified for a real accelerator ?

→ Normal From Theory

Coupling Resonances

CHESS & LEPP

$$\tfrac{d}{d\vartheta}J_x = \cos(\tilde\psi_x+\varphi_x)\sqrt{2J_x\beta_x}\Delta f_x \tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi_x = \upsilon_x - \sin(\tilde\psi_x+\varphi_x)\sqrt{\tfrac{\beta_x}{2J_x}}\Delta f_x \tfrac{L}{2\pi}$$

$$\tfrac{d}{d\vartheta}J_y = \cos(\tilde\psi_y+\varphi_y)\sqrt{2J_y\beta_y}\Delta f_y \tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi_y = \upsilon_y - \sin(\tilde\psi_y+\varphi_y)\sqrt{\tfrac{\beta_y}{2J_y}}\Delta f_y \tfrac{L}{2\pi}$$

$$\tfrac{d}{d\vartheta}\vec\varphi = \vec\partial_J H \quad , \quad \tfrac{d}{d\vartheta}\vec J = -\vec\partial_\varphi H \quad , \quad H(\vec\varphi,\vec J,\vartheta) = \vec\upsilon\cdot\vec J - \tfrac{L}{2\pi}\int_0^{\vec x}\Delta\vec f(\hat{\vec x},s)d\hat{\vec x}$$

The integral form can be chosen since it is path independent. This is due to the Hamiltonian nature of the force: $\Delta f_{x,y}(x,y,s) = -\partial_{x,y}\Delta H(x,y,s)$

Single Resonance model for two dimensions means retaining only the amplitude dependent tune shift and one term in the two dimensional Fourier expansion:

$$H(\vec\varphi,\vec J,\vartheta) = \vec\upsilon\cdot\vec J + H_{00}(\vec J) + H_{n\vec m}(\vec J)\cos(n\vartheta + m_x\varphi_x + m_y\varphi_y + \Psi_{n\vec m}(\vec J))$$

For $n + m_x\upsilon_x + m_y\upsilon_y \approx 0$

$$m_x\varphi_x + m_y\varphi_y = \vec m\cdot\vec\varphi$$

Sum and Difference Resonances

$n + m_x \upsilon_x + m_y \upsilon_y \approx 0$ means that oscillations in y can drive oscillations in x in

$$x'' = -K\,x + \Delta f_x(x, y, s)$$

The resonance term in the Hamiltonian then changes only slowly:

$$H(\vec{\varphi}, \vec{J}, \vartheta) = \vec{\upsilon}\cdot\vec{J} + H_{00}(\vec{J}) + H_{n\vec{m}}(\vec{J})\cos(n\vartheta + \vec{m}\cdot\vec{\varphi} + \Psi_{n\vec{m}}(\vec{J}))$$

$$\tfrac{d}{d\vartheta}\vec{\varphi} = \vec{\partial}_J H \quad , \quad \tfrac{d}{d\vartheta}\vec{J} = -\vec{\partial}_\varphi H$$

$$J = \vec{m}\cdot\vec{J}$$

$$J_\perp = m_x J_x - m_y J_y = \vec{m}\times\vec{J} \quad \Rightarrow \quad \tfrac{d}{d\vartheta} J_\perp = 0$$

Difference resonances lead to stable motion since:

$$n + |m_x|\upsilon_x - |m_y|\upsilon_y \approx 0 \Rightarrow |m_x| J_x + |m_y| J_y = const.$$

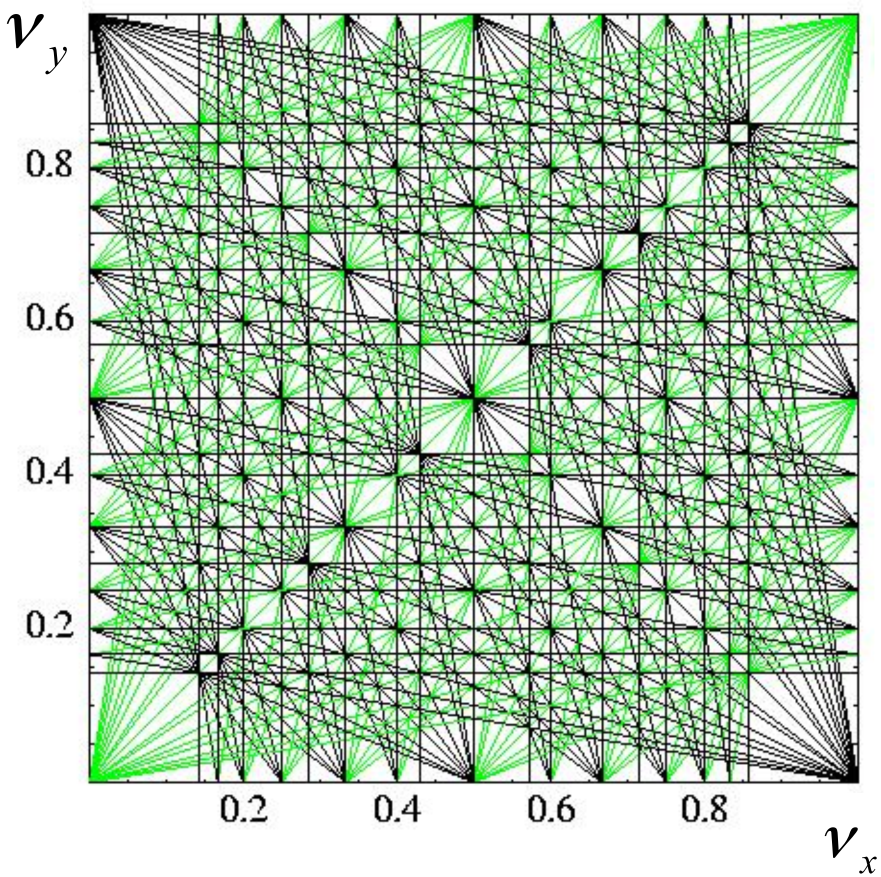
Sum resonances lead to unstable motion since:

$$n + |m_x|\upsilon_x + |m_y|\upsilon_y \approx 0 \Rightarrow |m_x| J_x - |m_y| J_y = const.$$

Resonances Diagram

$n + m_x \upsilon_x + m_y \upsilon_y \approx 0$ means that oscillations in y can drive oscillations in x in

$$x'' = -K\,x + \Delta f_x(x, y, s)$$



All these resonances have to be avoided by their respective resonance width.

The position of an accelerator in the tune plane s called its Working Point.