If the solenoids field was perpendicular to the particle’s motion, its bending radius would be 
\[ \rho_z = \frac{p}{qB_z} \]

\[ i^2 = -\left( \frac{qB_z}{2m\gamma} \right)^2 r = -\frac{qv_z}{m\gamma} B_z \frac{r}{4\rho_z} \]

Solenoid focusing is weak compared to the deflections created by a transverse magnetic field.

Transverse fields: 
\[ \vec{B} = B_x \hat{e}_x + B_y \hat{e}_y \]

\[ m\gamma \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \times \begin{pmatrix} B_x \\ B_y \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \frac{qv_z}{m\gamma} \begin{pmatrix} -B_y \\ B_x \end{pmatrix} \]

Weak focusing < Strong focusing by about \[ \frac{r}{\rho} \]
Solenoid Focusing

Solenoid magnets are used in detectors for particle identification via

\[ \rho = \frac{p}{qB} \]

The solenoid’s rotation \( \dot{\varphi} = -\frac{qB_z}{2m\gamma} \) of the beam is often compensated by a reversed solenoid called compensator.

**Solenoid or Weak Focusing:**

Solenoids are also used to focus low \( \gamma \) beams:

\[ \ddot{w} = -\left( \frac{qB_z}{2m\gamma} \right)^2 w \]

**Weak focusing from natural ring focusing:**

\[ \Delta r = r - R \]

\[ [(R + \Delta r) \cos \varphi - \Delta x_0]^2 + [(R + \Delta r) \sin \varphi - \Delta y_0]^2 = R^2 \]

Linearization in \( \Delta \):

\[ \Delta r = (\cos \varphi \Delta x_0 + \sin \varphi \Delta y_0) \]

\[ \partial^2_{\varphi} \Delta r = -\Delta r \quad \Rightarrow \quad \Delta \dddot{r} = -\dot{\varphi}^2 \Delta r = -\left( \frac{\nu}{\rho} \right)^2 \Delta r = -\left( \frac{qB}{m\gamma} \right)^2 \Delta r \]
\[ \psi = \Psi_1 \text{Im}\{x - iy\} = -\Psi_1 \cdot y \Rightarrow \vec{B} = -\nabla \psi = \Psi_1 \hat{e}_y \]

C\textsubscript{1} Symmetry

(+,-) in \(\Psi\)

(S,N) in \(B\)

Dipole magnets are used for steering the beams direction

\[ dp = p \, d\varphi \]

\[ \frac{d\vec{p}}{dt} = q \vec{v} \times \vec{B} \Rightarrow \frac{dp}{dt} = qvB_\perp \Rightarrow \rho = \frac{dl}{d\varphi} = \frac{vdt}{dp/p} = \frac{p}{qB_\perp} \]

Bending radius:

\[ \rho = \frac{p}{qB} \]
Different Dipoles

C-shape magnet:

H-shape magnet:

Window frame magnet:

\[ \vec{B}_\perp (\text{out}) = \vec{B}_\perp (\text{in}) \]
\[ \vec{H}_\perp (\text{out}) = \mu_r \vec{H}_\perp (\text{in}) \]

\[ 2nI = \oint \vec{H} \cdot d\vec{s} = H_{Fe} l_{Fe} + H_0 2a \]
\[ = \frac{1}{\mu_r} H_0 l_{Fe} + H_0 2a \approx H_0 2a \]

\[ B_0 = \mu_0 \frac{nI}{a} \]

Dipole strength:
\[ \frac{1}{\rho} = \frac{q\mu_0}{p} \frac{nI}{a} \]
Dipole Fields

B = 2 T: Typical limit, since the field becomes dominated by the coils, not the iron.

Limiting $j$ for Cu is about $100 \text{A/mm}^2$

B < 1.5 T: Typically used region

B < 1 T: Region in which $B_0 = \mu_0 \frac{nI}{a}$

Shims reduce the space that is open to the beam, but they also reduce the fringe field region.

\[
\Delta B \over B = 2 \times 10^{-4}
\]
Where is the vertical Dipole?
In a quadrupole particles are focused in one plane and defocused in the other plane. Other modes of strong focusing are not possible.
\[
\psi = -\Psi_2 \cdot 2xy \Rightarrow \text{Equipotential: } x = \frac{\text{const.}}{y}
\]

\[
\vec{B} = 2\Psi_2 \begin{pmatrix} y \\ x \end{pmatrix} \Rightarrow \vec{B}(0 \leftrightarrow 1) = 2\Psi_2 r \hat{e}_r
\]

Quadrupole strength:

\[
nI = \oint H \cdot d\vec{s} \approx \int_0^a H_x dr = \Psi_2 \frac{a^2}{\mu_0}
\]

\[
k_1 = \frac{q}{p} \frac{\partial_x B_y}{B_y} \bigg|_0 = \frac{q\mu_0}{p} \frac{2nI}{a^2}
\]
The coils show that this is an upright quadrupole not a rotated or skew quadrupole.
\[ \psi = \Psi_3 \text{Im}\{(x - iy)^3\} = \Psi_3 \cdot (y^3 - 3x^2y) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_3 \cdot 3\left(\begin{array}{c} 2xy \\ x^2 - y^2 \end{array}\right) \]

C\text{\textsubscript{3}} Symmetry

\begin{align*}
\text{i} & \quad \text{Sextupole fields hardly influence the particles close to the center, where one can linearize in } x \text{ and } y. \\
\text{ii} & \quad \text{In linear approximation a by } \Delta x \text{ shifted sextupole has a quadrupole field.} \\
\text{iii} & \quad \text{When } \Delta x \text{ depends on the energy, one can build an energy dependent quadrupole.}
\end{align*}

\[ \vec{B} = -\vec{\nabla}\psi = \Psi_3 \cdot 3\left(\begin{array}{c} 2xy \\ x^2 - y^2 \end{array}\right) \]

\[ x \mapsto \Delta x + x \]

\[ \vec{B} \approx \Psi_3 \cdot 3\left(\begin{array}{c} 2xy \\ x^2 - y^2 \end{array}\right) + 6\Psi_3\Delta x\left(\begin{array}{c} y \\ x \end{array}\right) + O(\Delta x^2) \]
Sextupole Fields

\[ \psi = \Psi_2 \cdot (y^3 - 3x^2y) \Rightarrow \text{Equipotential:} \quad x = \sqrt{\frac{\text{const.}}{y}} + \frac{1}{3} y^2 \]

 Equipotential:

\[ B_y \bigg|_{x=0} = -\Psi_3 3y^2 \]

 quadrupole strength:

\[ k_2 = \frac{q}{p} \frac{\partial^2 B_y}{\partial x^2} \bigg|_0 = \frac{q \mu_0}{p} \frac{6nI}{a^3} \]
Real Sextupoles
The CESR Tunnel
Higher order Multipoles

\[ \psi = \Psi_n \text{Im} \{(x - iy)^n\} = \Psi_n \cdot (\ldots - in x^{n-1}y) \quad \Rightarrow \quad \tilde{B}(y = 0) = \Psi_n n \begin{pmatrix} 0 \\ x^{n-1} \end{pmatrix} \]

Multipole strength: \[ k_n = \frac{q}{p} \frac{\partial^n B_y}{\partial x^n y} \bigg|_{x,y=0} = \frac{q}{p} \Psi_{n+1} (n+1)! \text{ units: } \frac{1}{m^{n+1}} \]

p/q is also called Bp and used to describe the energy of multiply charge ions

Names: dipole, quadrupole, sextupole, octupole, decapole, duodecapole, …

Higher order multipoles come from

- Field errors in magnets
- Magnetized materials
- From multipole magnets that compensate such erroneous fields
- To compensate nonlinear effects of other magnets
- To stabilize the motion of many particle systems
- To stabilize the nonlinear motion of individual particles