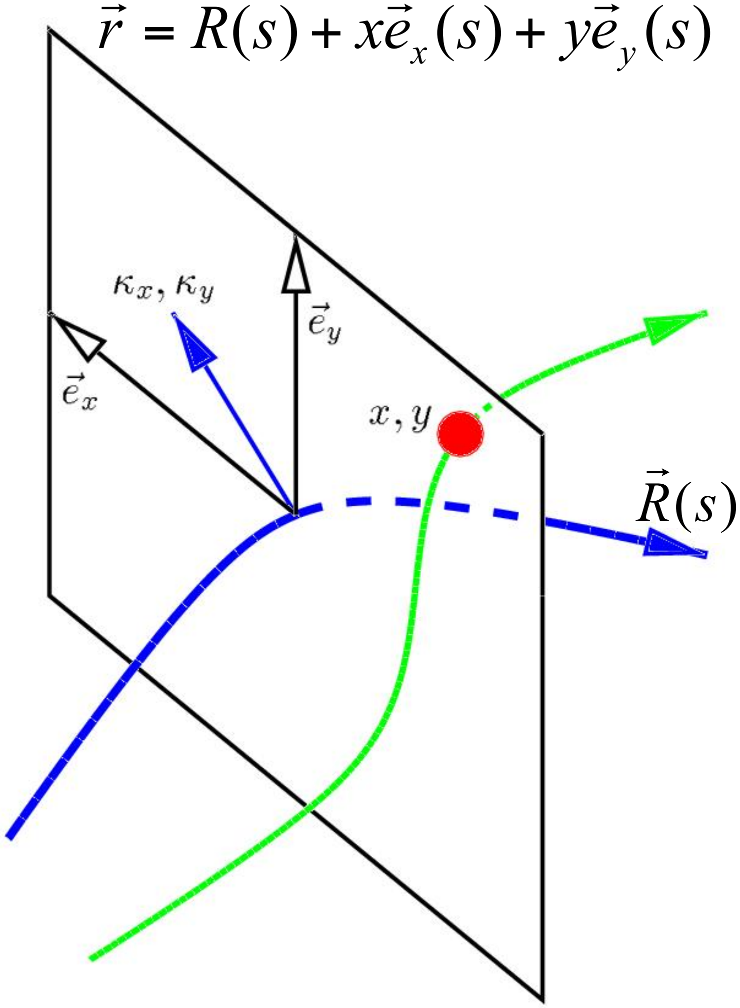
The Frenet Coordinate System

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$



$$\left|d\vec{R}\right| = ds$$

$$\vec{e}_s \equiv \tfrac{d}{ds}\vec{R}(s)$$

$$\vec{e}_\kappa \equiv -\tfrac{d}{ds}\vec{e}_s \Big/ \left|\tfrac{d}{ds}\vec{e}_s\right|$$

$$\vec{e}_b \equiv \vec{e}_s \times \vec{e}_\kappa$$

$$\tfrac{d}{ds}\vec{e}_s = -\kappa\,\vec{e}_\kappa \quad \text{with} \quad \kappa = \frac{1}{\rho}$$

$$0 = \tfrac{d}{ds}\left(\vec{e}_\kappa \cdot \vec{e}_s\right) = \vec{e}_s \cdot \tfrac{d}{ds}\vec{e}_\kappa - \kappa$$

Accumulated torsion angle T

$$\tfrac{d}{ds}\vec{e}_\kappa = \kappa\,\vec{e}_s + T'\vec{e}_b$$

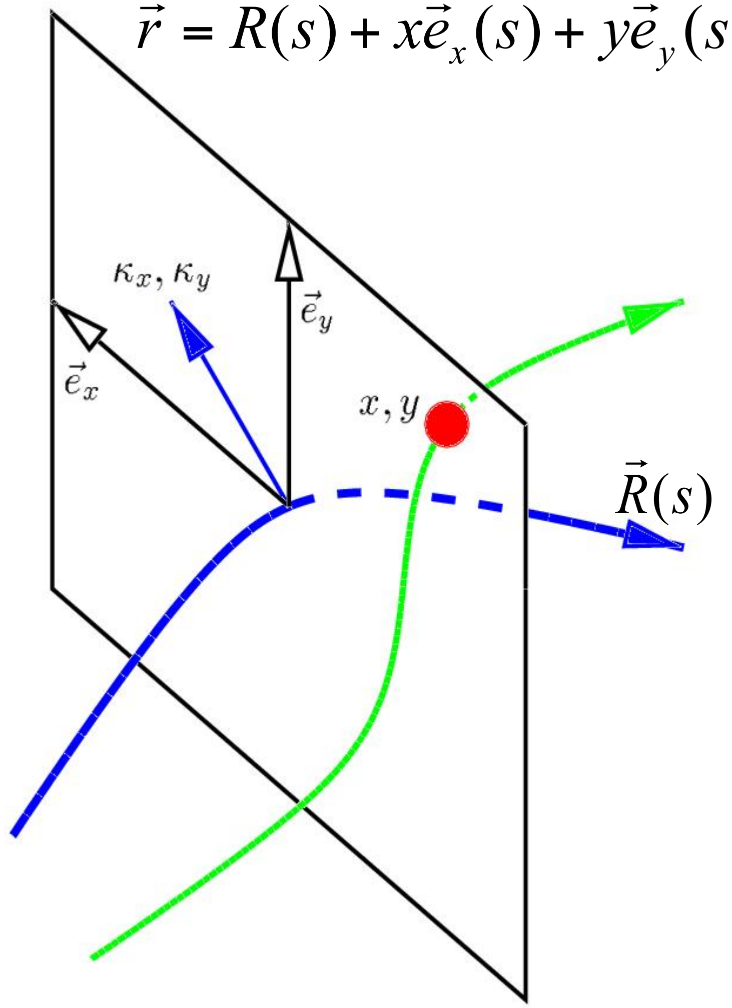
$$0 = \tfrac{d}{ds}\left(\vec{e}_b \cdot \vec{e}_\kappa\right) = \vec{e}_\kappa \cdot \tfrac{d}{ds}\vec{e}_b + T'$$

$$\tfrac{d}{ds}\vec{e}_b = -T'\vec{e}_\kappa$$

$$\vec{r}\,' = (x' - yT')\vec{e}_\kappa + (y' + xT')\vec{e}_b + (1 + x\kappa)\vec{e}_s$$

The Curvi-linear System

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$



$$\vec{e}_x \equiv \vec{e}_\kappa \cos(T) - \vec{e}_b \sin(T)$$

$$\vec{e}_y \equiv \vec{e}_\kappa \sin(T) + \vec{e}_b \cos(T)$$

$$\tfrac{d}{ds}\vec{e}_s = -\kappa_x \vec{e}_x - \kappa_y \vec{e}_y$$

$$\tfrac{d}{ds}\vec{e}_x = \kappa \cos(T)\vec{e}_s = \kappa_x \vec{e}_s$$

$$\tfrac{d}{ds}\vec{e}_y = \kappa \sin(T)\vec{e}_s = \kappa_y \vec{e}_s$$

$$\tfrac{d}{ds}\vec{r} = x'\vec{e}_x + y'\vec{e}_y + (1 + x\kappa_x + y\kappa_y)\vec{e}_s$$

Phase Space ODE

$$\tfrac{d}{ds}\vec{r} = x'\vec{e}_x + y'\vec{e}_y + \underbrace{(1 + x\kappa_x + y\kappa_y)}_{h}\vec{e}_s$$

$$\tfrac{d}{dt}\vec{p} = \vec{F}$$

$$\tfrac{d}{ds}\vec{r} = \dot{s}^{-1}\tfrac{d}{dt}\vec{r} = \dot{s}^{-1}\tfrac{1}{m\gamma}\vec{p} = \tfrac{h}{p_s}\vec{p}$$

$$\tfrac{d}{ds}\vec{p} = (p_x' - p_s\kappa_x)\vec{e}_x + (p_y' - p_s\kappa_y)\vec{e}_y + (p_s' + \kappa_x p_x + \kappa_y p_y)\vec{e}_s$$

$$= \dot{s}^{-1}\tfrac{d}{dt}\vec{p} = \dot{s}^{-1}\vec{F} = \tfrac{m\gamma h}{p_s}\vec{F}$$

$$\begin{pmatrix} x' \\ y' \\ p_x' \\ p_y' \end{pmatrix} = \begin{pmatrix} \frac{h}{p_s}p_x \\ \frac{h}{p_s}p_y \\ \frac{m\gamma h}{p_s}F_x + p_s\kappa_x \\ \frac{m\gamma h}{p_s}F_y + p_s\kappa_y \end{pmatrix}$$

$$t' = \dot{s}^{-1} = \frac{hm\gamma}{p_s}$$

$$E = \sqrt{(pc)^2 + (mc^2)^2}$$

$$E' = \tfrac{d}{dp}\sqrt{(pc)^2 + (mc^2)^2}\,\tfrac{d}{ds}p = c^2\frac{\vec{p}}{E}\tfrac{d}{ds}\vec{p} = \tfrac{h}{p_s}\vec{p}\cdot\vec{F}$$

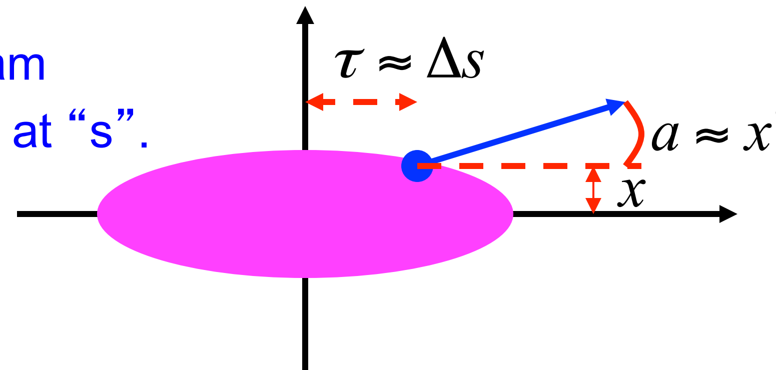
6 Dimensional Phase Space

Using a reference momentum p_0 and a reference time t_0:

$$\vec{z} = (x, a, y, b, \tau, \delta)$$

$$a = \frac{p_x}{p_0}, \quad b = \frac{p_y}{p_0}, \quad \delta = \frac{E - E_0}{E_0}, \quad \tau = (t_0 - t)\frac{c^2}{v_0} = (t_0 - t)\frac{E_0}{p_0}$$

Usually p_0 is the design momentum of the beam
And t_0 is the time at which the bunch center is at “s”.



$$\left.\begin{array}{l} x' = \partial_{p_x} K \\ p_x' = -\partial_x K \end{array}\right\} \Rightarrow \begin{cases} x' = \partial_a K/p_0 , & a' = -\partial_x K/p_0 \\ y' = \partial_b K/p_0 , & b' = -\partial_y K/p_0 \end{cases}$$

$$-t' = \partial_E K \Rightarrow \tau' = \tfrac{c^2}{v_0}\partial_\delta K/E_0 = \partial_\delta K/p_0$$

$$E' = -\partial_{-t} K \Rightarrow \delta' = -\tfrac{1}{E_0}\partial_\tau K \tfrac{c^2}{v_0} = -\partial_\tau K/p_0$$

New Hamiltonian:

$$\widetilde{H} = K/p_0$$