Invariant of Motion

$$x(s) = \sqrt{2J\beta(s)}\sin(\psi(s)+\phi_0)$$

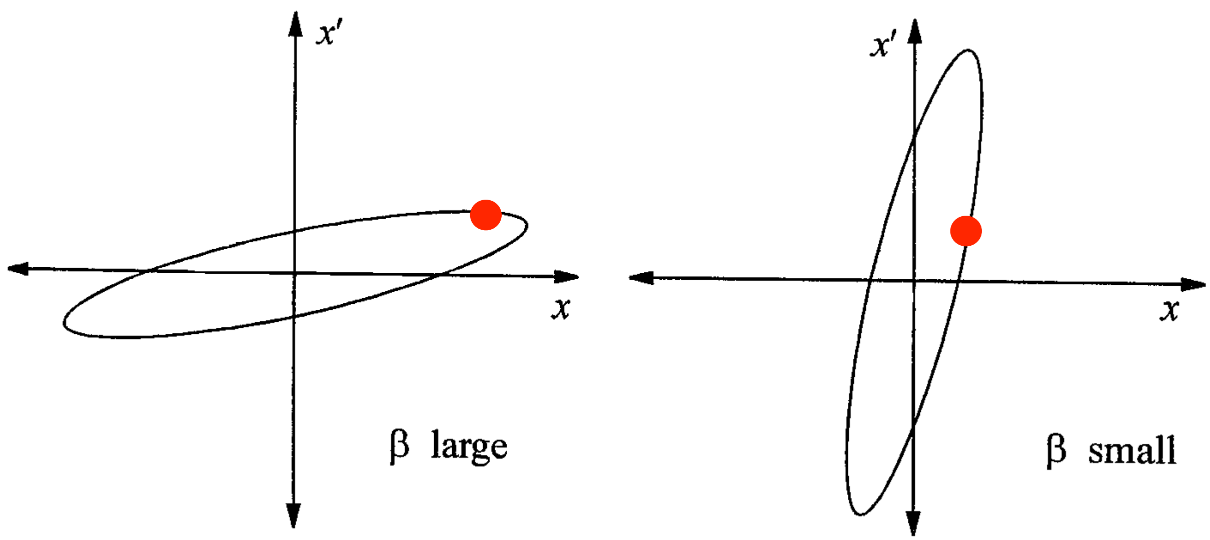
Where J and ϕ are given by the starting conditions x_0 and x'_0.

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = 2J$$

Leads to the invariant of motion:

$$f(x,x',s) = \gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 \quad \Rightarrow \quad \tfrac{d}{ds} f = 0$$

It is called the Courant-Snyder invariant.



Twiss Parameters, e.g. drift

CHESS & LEPP

$$\gamma = \frac{1+\alpha^2}{\beta}$$

$$\beta' = -2\alpha$$

$$\alpha' = k\beta - \gamma$$

$$\beta'' = 2\gamma = 2\frac{1+\frac{1}{4}\beta'^2}{\beta} = \frac{d\beta'}{d\beta}\frac{d\beta}{ds}$$

$$\frac{\beta'}{1+\frac{1}{4}\beta'^2}d\beta' = 2\frac{d\beta}{\beta}$$

$$\log(1+\tfrac{1}{4}\beta'^2) = \log(\beta/\beta_0)$$

$$\beta' = 2\sqrt{\beta/\beta_0 - 1}$$

$$\frac{d\beta}{2\sqrt{\beta/\beta_0 - 1}} = ds$$

$$\beta_0\sqrt{\beta/\beta_0 - 1} = s - s_0$$

$$\beta(s) = \beta_0\left(1+\left(\frac{s-s_0}{\beta_0}\right)^2\right)$$

Propagation of Twiss Parameters

$$(x_0, x_0') \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = 2J$$

$$(x, x') \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = 2J = (x_0, x_0') \underline{M}^T \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \underline{M} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} = \underline{M}^{-T} \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \underline{M}^{-1}$$

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \underline{M} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \underline{M}^T$$

Twiss Parameters in a Drift

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} \beta_0 - 2\alpha_0 s + \gamma_0 s^2 & \gamma_0 s - \alpha_0 \\ \gamma_0 s - \alpha_0 & \gamma_0 \end{pmatrix}$$

$$\beta = \beta_0^*[1 + (\tfrac{s}{\beta_0^*})^2] \quad \text{for} \quad \alpha_0^* = 0$$