The Periodic Beta Function

If the particle distribution in a ring or any other periodic structure is stable, it is periodic from turn to turn.

$$\rho(x,x',s+L)=\rho(x,x',s)$$

To be matched to such a beam, the Twiss parameters α, β, γ must be the same after every turn.

$$\underline{M}(s,0)=\begin{pmatrix}\sqrt{\frac{\beta}{\beta_0}}[\cos\psi+\alpha_0\sin\psi] & \sqrt{\beta_0\beta}\sin\psi \\ \sqrt{\frac{1}{\beta_0\beta}}[(\alpha_0-\alpha)\cos\psi-(1+\alpha_0\alpha)\sin\psi] & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi-\alpha\sin\psi]\end{pmatrix}$$

$$\underline{M}_p(s)=\begin{pmatrix}\cos\mu+\alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu-\alpha\sin\mu\end{pmatrix}=\underline{1}\cos\mu+\begin{pmatrix}\alpha & \beta \\ -\gamma & -\alpha\end{pmatrix}\sin\mu$$

$$\mu=\psi(s+L)-\psi(s)$$

One Turn Matrix to Periodic Twiss

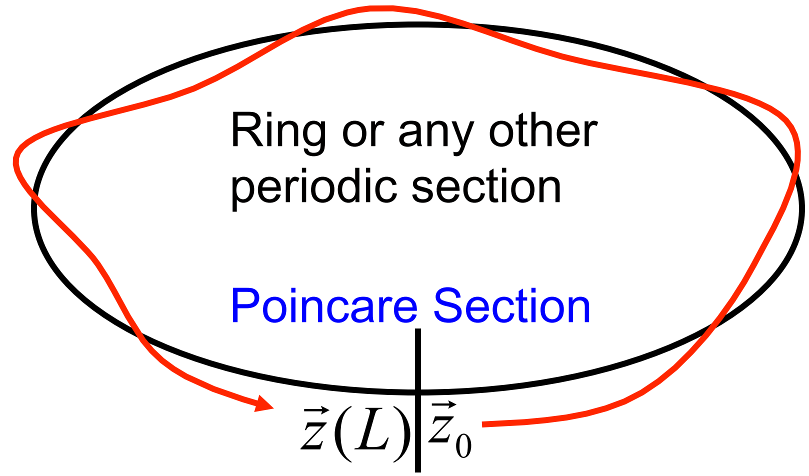
The periodic Twiss parameters are the solution of a nonlinear differential equation with periodic boundary conditions:

$$\beta' = -2\alpha \quad \text{with} \quad \beta(L) = \beta(0)$$

$$\alpha' = k\beta - \tfrac{1+\alpha^2}{\beta} \quad \text{with} \quad \alpha(L) = \alpha(0)$$

$$\mu = \int_0^L \tfrac{1}{\beta(\hat{s})} d\hat{s}$$

Note: $\beta(s) > 0$



$$\underline{M}_0(s) = \underline{1}\cos\mu + \underline{\beta}\sin\mu \; ; \; \underline{\beta} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\cos\mu = \tfrac{1}{2}\mathrm{Tr}[\underline{M}_0(s)]$$

$$\beta = \underline{M}_{0,12} \tfrac{1}{\sin\mu}$$

$$\alpha = (\underline{M}_{0,11} - \underline{M}_{0,22}) \tfrac{1}{2\sin\mu}$$

$$\gamma = \tfrac{1+\alpha^2}{\beta}$$

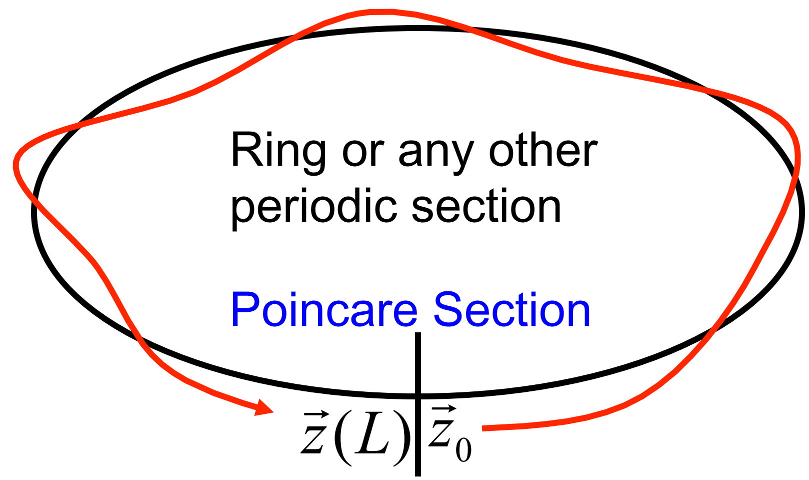
Stable beam motion and thus a periodic beta function can only exist when |Tr[M]| < 2.

The Tune of a Periodic Accelerator

The betatron phase advance per turn devided by 2π is called the TUNE.

$$\mu = 2\pi\nu = \psi(s+L) - \psi(s)$$

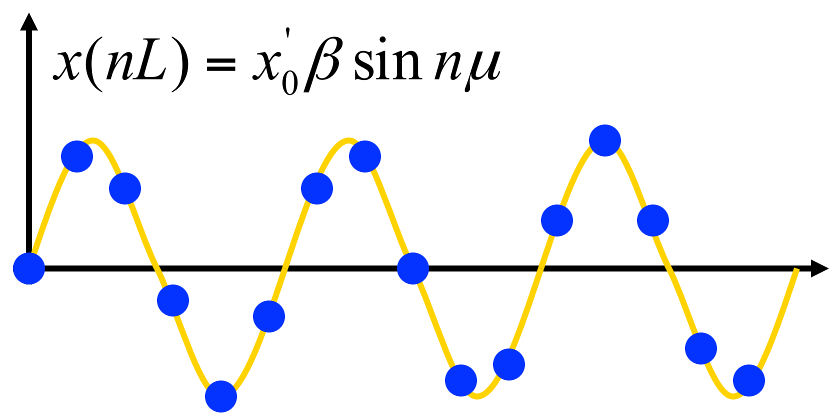
It is a property of the ring and does not depend on the azimuth s.



$$\underline{M}_0(s) = \underline{1}\cos\mu + \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} \sin\mu$$

$$2\cos\underline{\mu(s)} = \mathrm{Tr}[\underline{M}_0(s)] = \mathrm{Tr}[\underline{M}(s,0)\underline{M}_0(0)\underline{M}^{-1}(s,0)]$$

$$= \mathrm{Tr}[\underline{M}_0(0)] = 2\cos\underline{\mu(0)}$$



$$\underline{M}_0^n = \underline{1}\cos n\mu + \underline{\beta}\sin n\mu$$