Simplified Equation of Motion

Only magnetic fields: $\vec{E} = 0$

Mid-plane symmetry: $B_x(x,y,s) = -B_x(x,-y,s)\,,\quad B_y(x,y,s) = B_y(x,-y,s)$

Linearization in : $B_x(x,y,s) = \frac{p_0}{q} ky\,,\quad B_y(x,y,s) = \frac{p_0}{q}\frac{1}{\rho} + \frac{p_0}{q} kx$

Highly relativistic : $\frac{p-p_0}{p_0} \to \frac{E-E_0}{E_0}\quad,\quad \frac{v-v_0}{v_0} \to 0$

$$x' = \frac{p_x}{p_z} = \frac{p_x}{\sqrt{(p_0+dp)^2 - p_x^2 - p_y^2}} =_1 \frac{p_x}{p_0} = a \qquad \Rightarrow \qquad y' = b$$

$$a' = \frac{(\vec{p}\,' + p_s\kappa\vec{e}_x)_x}{p_0} = \frac{h}{p_0 v_s}\frac{d}{dt}p_x + \frac{p_s\kappa}{p_0} = -\frac{1+x\kappa}{p_0 v_s} q v_s B_y + \frac{p_s\kappa}{p_0}$$

$$=_1 -(1+x\kappa)(\kappa+kx) + (1+\delta)\kappa = -x(\kappa^2+k) + \delta\kappa \qquad \Rightarrow \qquad b' = ky$$

$$\tau' = \frac{d(t-t_0)}{ds}\frac{E_0}{p_0} = \left(\frac{1}{v_0} - \frac{h}{v_s}\right)\frac{E_0}{p_0} =_1 -x\kappa \qquad , \qquad \delta' = 0$$

Hamiltonian:

$$H = \frac{1}{2}a^2 + \frac{1}{2}b^2 + \frac{1}{2}k(x^2 - y^2) + \frac{1}{2}\kappa^2 x^2 - \kappa x\delta$$

The Drift

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Note that in nonlinear expansion $x' \neq a$ so that the drift does not have a linear transport map even though $x(s) = x_0 + x_0' s$ is completely linear.

$$\begin{pmatrix} x \\ a \\ y \\ b \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} x_0 + s a_0 \\ a \\ y_0 + s b_0 \\ b_0 \\ \tau_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \underline{0} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \vec{z}_0$$

The Dipole Equation of Motion

$$x'' + x\,\kappa^2 = \delta\kappa$$

$$y'' = 0$$

$$\tau' + x\kappa = 0$$

Homogeneous solution:

$$x_H'' = -x_H\,\kappa^2 \quad \Rightarrow \quad x_H = A\cos(\kappa\,s) + B\sin(\kappa\,s)$$ (natural ring focusing)

Variation of constants:

$$x = A(s)\cos(\kappa\,s) + B(s)\sin(\kappa\,s)$$

$$x' = -A\kappa\sin(\kappa\,s) + B\kappa\cos(\kappa\,s) + \underbrace{A'\cos(\kappa\,s) + B'\sin(\kappa\,s)}_{\equiv 0}$$

$$x'' = -\kappa^2 x \underbrace{- A'\kappa\sin(\kappa\,s) + B'\kappa\cos(\kappa\,s)}_{=\delta\kappa} = -\kappa^2 x + \delta\kappa$$

$$\begin{pmatrix} \cos(\kappa\,s) & \sin(\kappa\,s) \\ -\sin(\kappa\,s) & \cos(\kappa\,s) \end{pmatrix}\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

The Dipole

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} \cos(\kappa s) & -\sin(\kappa s) \\ \sin(\kappa s) & \cos(\kappa s) \end{pmatrix} \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \delta\,\kappa^{-1} \begin{pmatrix} \cos(\kappa s) \\ \sin(\kappa s) \end{pmatrix} + \begin{pmatrix} A_H \\ B_H \end{pmatrix} \quad \text{with} \quad x = A\cos(\kappa s) + B\sin(\kappa s)$$

$$\tau' = -x\kappa$$

$$\underline{M} = \begin{pmatrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) & 0 & 0 & 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ -\kappa\sin(\kappa s) & \cos(\kappa s) & 0 & 0 & 0 & \sin(\kappa s) \\ 0 & 0 & 1 & s & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] & 0 & 0 & 1 & \kappa^{-1}[\sin(\kappa s)-s\kappa] \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The Quadrupole

$$x'' = -x\,k$$

$$y'' = \;\; y\,k$$

$$\underline{M}_4 = \begin{pmatrix} \cos(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}\,s) & \multicolumn{2}{c}{\underline{0}} \\ -\sqrt{k}\sin(\sqrt{k}\,s) & \cos(\sqrt{k}\,s) & & \\ \multicolumn{2}{c}{\underline{0}} & \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ & & \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix}$$

As for a drift:

$$\vec{D} = \vec{0} \quad \Rightarrow \quad \vec{T} = \vec{0}$$

$$M_{56} = 0$$

For k<0 one has to take into account that

$$\cos(\sqrt{k}\,s) = \cosh(\sqrt{|k|}\,s)\,, \quad \sin(\sqrt{k}\,s) = i\sinh(\sqrt{|k|}\,s)$$

$$\cosh(\sqrt{k}\,s) = \cos(\sqrt{|k|}\,s)\,, \quad \sinh(\sqrt{k}\,s) = i\sin(\sqrt{|k|}\,s)$$

The Combined Function Bend

$$x'' = -x\,(\underbrace{\kappa^2 + k}_{K}) + \delta\kappa$$

$$y'' = y\,k \quad , \quad \tau' = -\kappa\,x$$

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0}\,\vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{T} & \underline{0} & \underline{M}_\tau \end{pmatrix}$$

$$\underline{M}_\tau = \begin{pmatrix} 1 & M_{56} \\ 0 & 1 \end{pmatrix}$$

$$M_{56} = \tfrac{\kappa^2}{K\sqrt{K}}[\sin(\sqrt{K}s) - \sqrt{K}s]$$

$\underline{T}$ from symplecticity

$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K}\,s) & \tfrac{1}{\sqrt{K}}\sin(\sqrt{K}\,s) \\ -\sqrt{K}\sin(\sqrt{K}\,s) & \cos(\sqrt{K}\,s) \end{pmatrix}$$

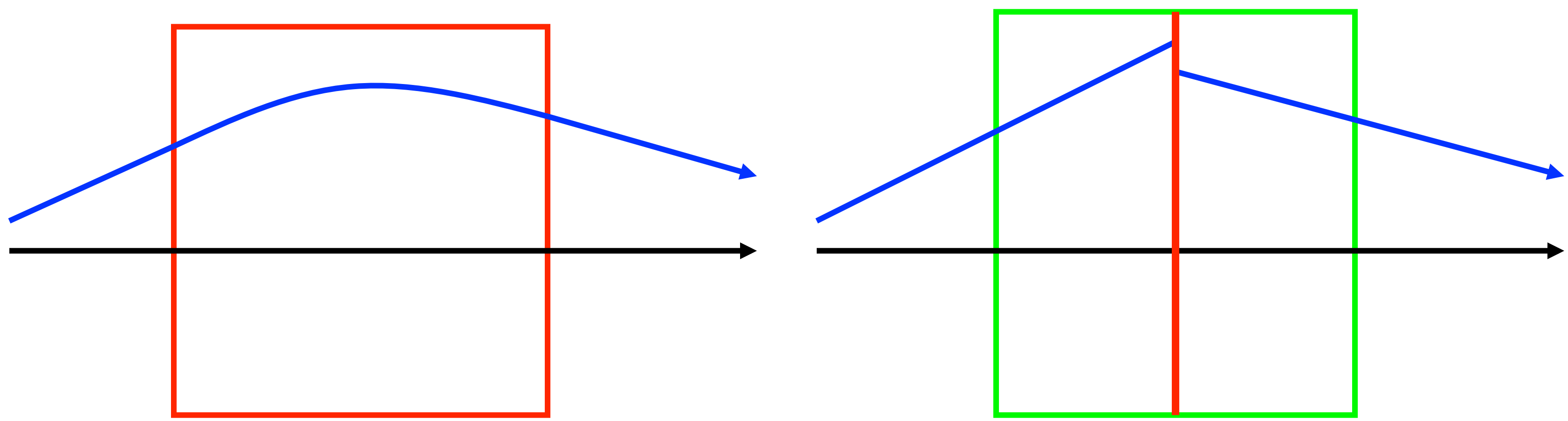
$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k}\,s) & \tfrac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \tfrac{\kappa}{K}[1 - \cos(\sqrt{K}s)] \\ \tfrac{\kappa}{\sqrt{K}}\sin(\sqrt{K}s) \end{pmatrix}$$

Options:

- For k>0:
  focusing in x, defocusing in y.
- For k<0, K<0:
  defocusing in x, focusing in y.
- For k<0, K>0:
  weak focusing in both planes.

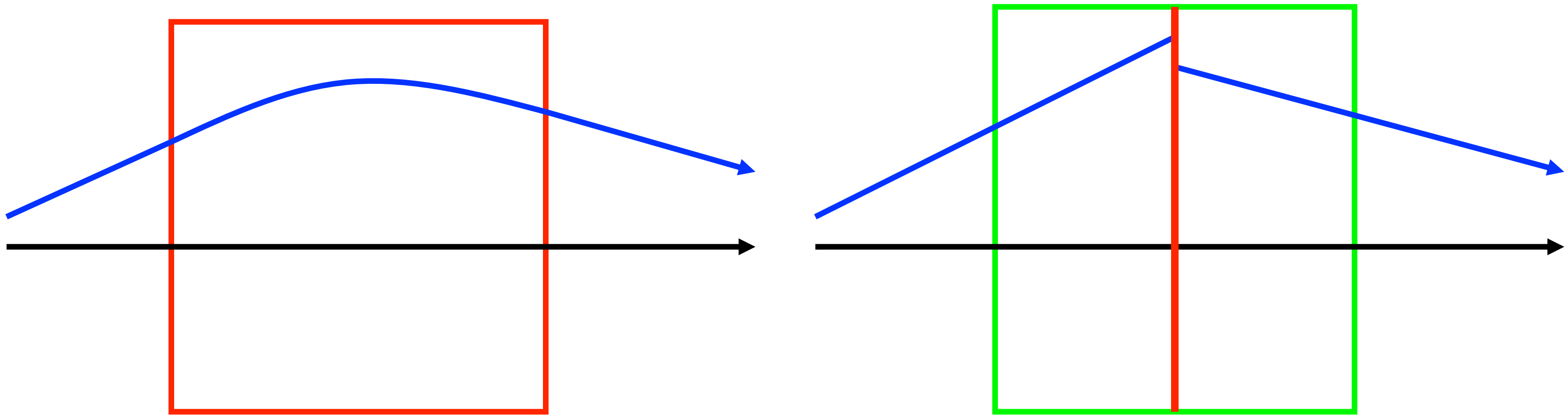
The Thin Lens Approximation



$$\vec{z}(s) = \underline{M}(s)\vec{z}_0 = \underline{D}(\tfrac{s}{2})\underline{D}^{-1}(\tfrac{s}{2})\underline{M}(s)\underline{D}^{-1}(\tfrac{s}{2})\underline{D}(\tfrac{s}{2})\vec{z}_0$$

Drift: $$\underline{M}^{\text{thin}}_{\text{drift}}(s) = \underline{D}^{-1}(\tfrac{s}{2})\underline{M}(s)\underline{D}^{-1}(\tfrac{s}{2}) = \underline{1}$$

The Thin Lens Quadrupole



$$\underline{M}^{\text{thin}}_{\text{quad,x}}(s) = \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\sqrt{k}s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}s) \\ -\sqrt{k}\sin(\sqrt{k}s) & \cos(\sqrt{k}s) \end{pmatrix} \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & s \\ -ks & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{s}{2} \\ -ks & 1+\frac{ks^2}{2} \end{pmatrix}$$

Weak magnet limit: $\sqrt{k}s << 1$

$$\underline{M}^{\text{thin}}_{\text{quad,x}}(s) \approx \begin{pmatrix} 1 & 0 \\ -ks & 1 \end{pmatrix}$$

The Thin Lens Dipole

$$\underline{M} = \begin{pmatrix} \begin{matrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) \\ -\kappa\sin(\kappa s) & \cos(\kappa s) \end{matrix} & \underline{0} & \begin{matrix} 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ 0 & \sin(\kappa s) \end{matrix} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \begin{matrix} -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] \\ 0 & 0 \end{matrix} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}$$

Weak magnet limit: $\kappa s << 1$

$$\underline{M}^{\text{thin}}_{\text{bend},x\tau}(s) = \underline{D}(-\tfrac{s}{2})\underline{M}_{\text{bend},x\tau}\underline{D}(-\tfrac{s}{2}) \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\kappa^2 s & 1 & 0 & \kappa s \\ -\kappa s & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Thin Combined Function Bend

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0}\,\vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{T} & \underline{0} & \underline{1} \end{pmatrix}$$

Weak magnet limit: $\kappa s << 1$

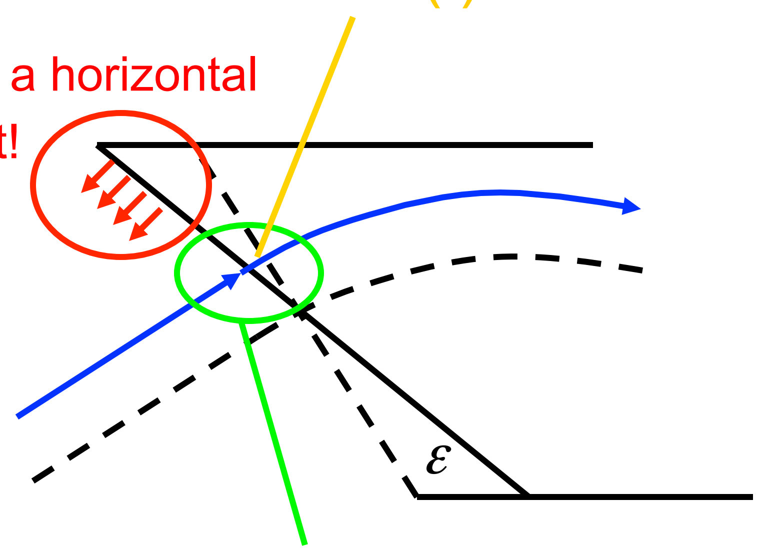
$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K}\, s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\, s) \\ -\sqrt{K}\sin(\sqrt{K}\, s) & \cos(\sqrt{K}\, s) \end{pmatrix} \qquad \underline{M}_x^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ -K\, s & 1 \end{pmatrix}$$

$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k}\, s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\, s) \\ \sqrt{k}\sinh(\sqrt{k}\, s) & \cosh(\sqrt{k}\, s) \end{pmatrix} \longrightarrow \underline{M}_y^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ k\, s & 1 \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \frac{\kappa}{K}[1-\cos(\sqrt{K} s)] \\ \frac{\kappa}{\sqrt{K}}\sin(\sqrt{K} s) \end{pmatrix} \qquad \vec{D} = \begin{pmatrix} 0 \\ \kappa\, s \end{pmatrix}$$

Edge Focusing

Top view :



Horizontal focusing with $\Delta x' = -x\dfrac{\tan(\varepsilon)}{\rho}$

$$B_x = \partial_y B_s\big|_{y=0}\, y\tan(\varepsilon) = \partial_s B_y\big|_{y=0}\, y\tan(\varepsilon)$$

$$y'' = \tfrac{q}{p}\,\partial_s B_y\big|_{y=0}\, y\tan(\varepsilon)$$

$$\Delta y' = \int y''\,ds = \tfrac{q}{p} B_y\, y\tan(\varepsilon) = y\frac{\tan(\varepsilon)}{\rho}$$

Quadrupole effect with

$$kl = \frac{\tan(\varepsilon)}{\rho}$$

$$\vec{z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\tan(\varepsilon)}{\rho} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\tan(\varepsilon)}{\rho} & 1 \end{pmatrix} \vec{z}_0$$

Orbit Distortions for a One-pass Accelerator

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \Delta f$$

The extra force can for example come from an erroneous dipole field or from a correction coil: $\Delta f = \frac{q}{p}\Delta B_y = \Delta\kappa$

Variation of constants: $\vec{z} = \underline{M}\vec{z}_0 + \Delta\vec{z}$ with $\Delta\vec{z} = \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \Delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$

$$\Delta\vec{z} = \int_0^L \begin{pmatrix} -\sqrt{\beta\hat{\beta}}\sin\hat{\psi} \\ \sqrt{\frac{\hat{\beta}}{\beta}}[\cos\hat{\psi} + \alpha\sin\hat{\psi}] \end{pmatrix} \Delta\kappa(\hat{s}) d\hat{s}$$

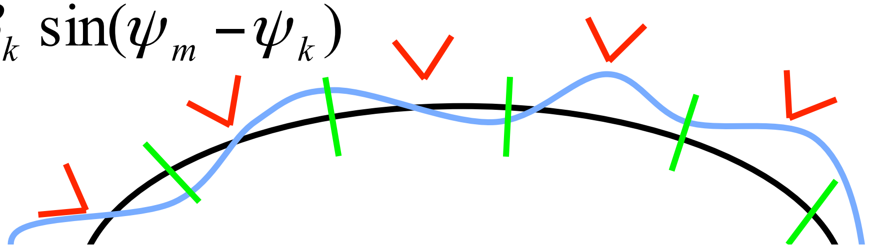
$$\boxed{\Delta x(s) = \sum_k \Delta\vartheta_k \sqrt{\beta(s)\beta_k}\sin(\psi(s) - \psi_k)}$$

Orbit Correction for a One-pass Accelerator

When the closed orbit $x_{co}^{old}(s_m)$ is measured at beam position monitors (BPMs, index m) and is influenced by corrector magnets (index k), then the monitor readings before and after changing the kick angles created in the correctors by $\Delta\vartheta_k$ are related by

$$x_{co}^{new}(s_m) = x_{co}^{old}(s_m) + \sum_k \Delta\vartheta_k \sqrt{\beta_m \beta_k}\sin(\psi_m - \psi_k)$$

$$= x_{co}^{old}(s_m) + \sum_k O_{mk}\Delta\vartheta_k$$



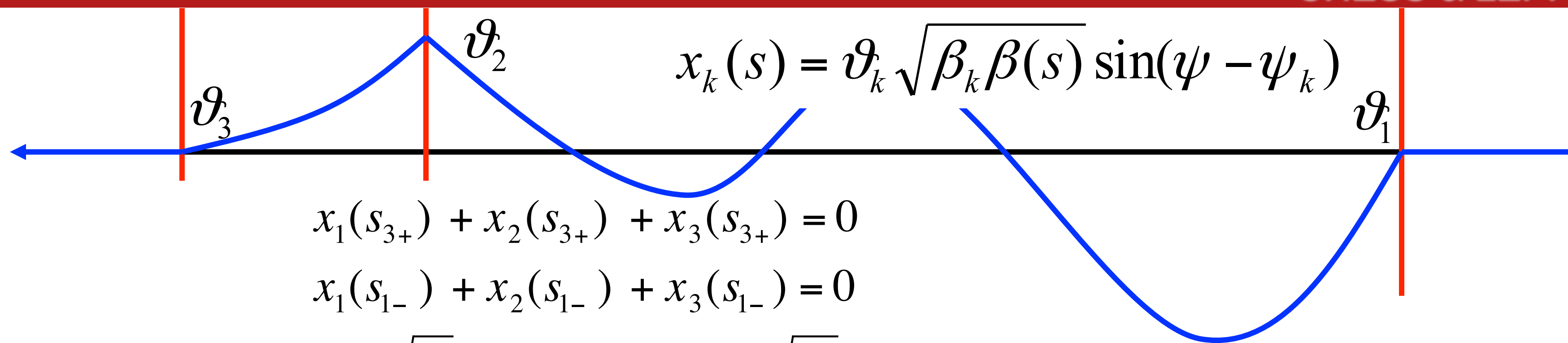
$$\vec{x}_{co}^{new} = \vec{x}_{co}^{old} + \underline{O}\Delta\vec{\vartheta}$$

$$\Delta\vec{\vartheta} = -\underline{O}^{-1}\vec{x}_{co}^{old} \quad \Rightarrow \quad \vec{x}_{co}^{new} = 0$$

It is often better not to try to correct the closed orbit at the the BPMs to zero in this way since

1. computation of the inverse can be numerically unstable, so that small errors in the old closed orbit measurement lead to a large error in the corrector coil settings.
2. A zero orbit at all BPMs can be a bad orbit inbetween BPMs

Closed Orbit Bumps



$$x_k(s) = \vartheta_k \sqrt{\beta_k \beta(s)} \sin(\psi - \psi_k)$$

$$x_1(s_{3+}) + x_2(s_{3+}) + x_3(s_{3+}) = 0$$

$$x_1(s_{1-}) + x_2(s_{1-}) + x_3(s_{1-}) = 0$$

$$\vartheta_1 \sqrt{\beta_1} \sin(\psi_3 - \psi_1) + \vartheta_2 \sqrt{\beta_2} \sin(\psi_3 - \psi_2) = 0$$

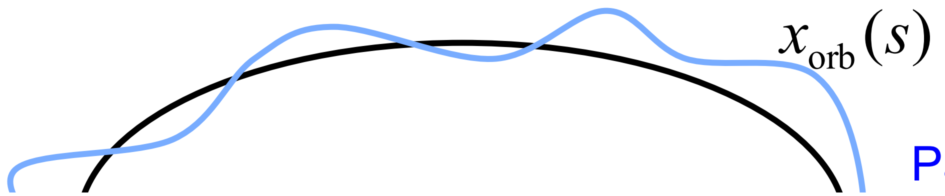
$$\vartheta_3 \sqrt{\beta_3} \sin(\psi_3 - \psi_1) + \vartheta_2 \sqrt{\beta_2} \sin(\psi_2 - \psi_1) = 0$$

$$\frac{\vartheta_1}{\vartheta_2} = -\frac{\sin(\psi_3 - \psi_2)/\sqrt{\beta_1}}{\sin(\psi_3 - \psi_1)/\sqrt{\beta_2}}$$

$$\frac{\vartheta_2}{\vartheta_3} = -\frac{\sin(\psi_3 - \psi_1)/\sqrt{\beta_2}}{\sin(\psi_2 - \psi_1)/\sqrt{\beta_3}}$$

$$\vartheta_1 : \vartheta_2 : \vartheta_3 = \beta_1^{-\frac{1}{2}} \sin\psi_{32} : -\beta_2^{-\frac{1}{2}} \sin\psi_{31} : \beta_3^{-\frac{1}{2}} \sin\psi_{21}$$

Oscillations around a distorted Orbit



Particles oscillate around this periodic orbit, not around the design orbit.

$$\vec{z} = \vec{z}_\beta + \vec{z}_\text{orb}$$

$$\vec{z}_\text{orb}(s) = \underline{M}\,\vec{z}_\text{orb}(0) + \Delta\vec{z}(s)$$

$$\vec{z}_\beta(s) + \vec{z}_\text{orb}(s) = \vec{z}(s) = \underline{M}\,\vec{z}(0) + \Delta\vec{z}(s) = \underline{M}[\vec{z}_\beta(0) + \vec{z}_\text{orb}(0)] + \Delta\vec{z}(s)$$

$$= \underline{M}\,\vec{z}_\beta(0) + \vec{z}_\text{orb}(s)$$

$$\boxed{\vec{z}_\beta(L) = \underline{M}_0\vec{z}_\beta(0)}$$

The distorted orbit does not change the linear transport matrix.