Quadrupole Errors for One-pass Accelerators

$$\vec{z}\,' = \underline{L}(s)\,\vec{z} + \Delta\vec{f}(\vec{z},s)$$

$$\vec{z}(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z},\hat{s})\,d\hat{s} \approx \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_H,\hat{s})\,d\hat{s}$$

$$x'' = -(\kappa^2 + k)x - \Delta k(s)\,x \quad \Rightarrow \quad \begin{pmatrix} x' \\ a' \end{pmatrix} = \begin{pmatrix} a \\ -(\kappa^2 + k)x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \Delta k(s) & 0 \end{pmatrix}\begin{pmatrix} x \\ a \end{pmatrix}$$

$$\vec{z}(s) = \left\{\underline{M}(s) - \int_0^s \underline{M}(s,\hat{s})\begin{pmatrix} 0 & 0 \\ \Delta k(\hat{s}) & 0 \end{pmatrix}\underline{M}(\hat{s},0)\,d\hat{s}\right\}\vec{z}_0$$

One quadrupole error:

$$\Delta\underline{M}(s,\hat{s}) = -\underline{M}(s,\hat{s})\begin{pmatrix} 0 & 0 \\ \Delta kl(\hat{s}) & 0 \end{pmatrix}$$

Quadrupole Error and Phase advance

$$\Delta \underline{M}(s,\hat{s}) = -\underline{M}(s,\hat{s})\begin{pmatrix} 0 & 0 \\ \Delta kl(\hat{s}) & 0 \end{pmatrix} \qquad \underline{M}(s) = \begin{pmatrix} \dots & \sqrt{\beta_0\beta}\sin\tilde{\psi} \\ \dots & \sqrt{\frac{\beta_0}{\beta}}[\cos\tilde{\psi} - \alpha\sin\tilde{\psi}] \end{pmatrix}$$

$$\Delta \underline{M}(s,\hat{s}) = -\Delta kl(\hat{s})\begin{pmatrix} \sqrt{\hat{\beta}\beta}\sin\psi & 0 \\ \sqrt{\frac{\hat{\beta}}{\beta}}[\cos\psi - \alpha\sin\psi] & 0 \end{pmatrix} , \qquad \psi = \psi(s) - \psi(\hat{s})$$

$$= \begin{pmatrix} \frac{\frac{1}{2}\Delta\beta[\cos\psi + \hat{\alpha}\sin\psi] + \Delta\psi\beta[\hat{\alpha}\cos\psi - \sin\psi]}{\sqrt{\hat{\beta}\beta}} & \sqrt{\hat{\beta}}\left(\frac{\frac{\Delta\beta}{2}\sin\psi + \Delta\psi\beta\cos\psi}{\sqrt{\beta}}\right) \\ \dots & \dots \end{pmatrix}$$

$$\frac{1}{2}\Delta\beta\cos\psi + \frac{1}{2}\Delta\beta\frac{\sin^2\psi}{\cos\psi} = \frac{1}{2}\Delta\beta\frac{1}{\cos\psi} = -\Delta kl(\hat{s})\beta\hat{\beta}\sin\psi \qquad \Delta\psi = -\frac{\Delta\beta}{2\beta}\tan\psi$$

$$\boxed{\Delta\beta = -\Delta kl(\hat{s})\beta\hat{\beta}\sin 2\psi} \qquad \boxed{\Delta\psi = \Delta kl(\hat{s})\hat{\beta}\tfrac{1}{2}(1 - \cos 2\psi)}$$

Twiss functions correction in one pass accelerators

$$\Delta\psi = \Delta k l_j \beta_j \sin^2(\psi - \psi_j)$$

→ More focusing always increases the tune

$$\frac{\Delta\beta}{\beta} = -\Delta k l_j \beta_j \sin(2[\psi - \psi_j])$$

→ Beta beat oscillates twice as fast as orbit.

Notice the self consistency: $\Delta\psi = \int_0^s \left(\frac{1}{\hat\beta + \Delta\hat\beta} - \frac{1}{\hat\beta}\right) d\hat s = -\int_0^s \frac{\Delta\hat\beta}{\hat\beta} d\hat s$

$$\Delta\alpha = -\frac{\Delta\beta'}{2} = \Delta k l_j \beta_j [\cos(2[\psi - \psi_j]) - \alpha \sin(2[\psi - \psi_j])]$$

$$\frac{\Delta\beta}{\beta} = -\sum_j \Delta k l_j \beta_j \sin(2[\psi - \psi_j]) \qquad \Delta\psi = \sum_j \Delta k l_j \beta_j \tfrac{1}{2}[1 - \cos(2[\psi - \psi_j])]$$

When beta functions and betatron phases have been measured at many places, quadrupoles can be changed with these formulas to correct the Twiss errors.