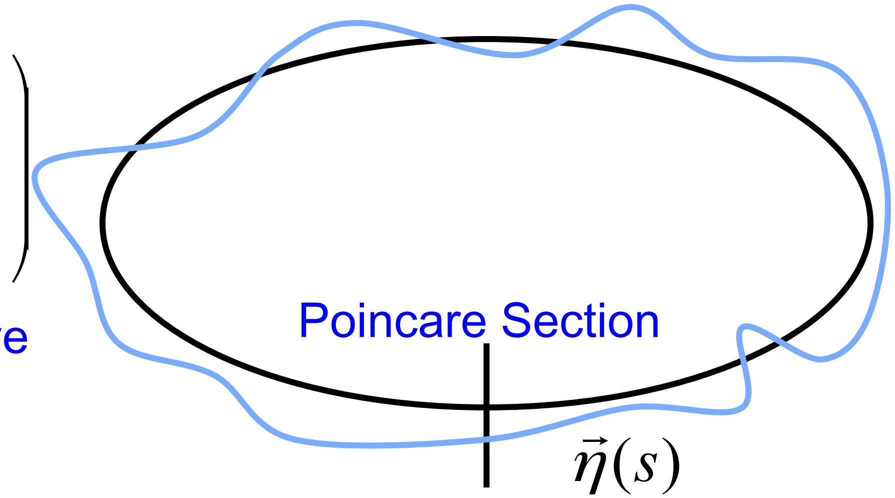
The Periodic Dispersion

$$\begin{pmatrix} \underline{M}_{0x}\vec{z}_0 + \vec{D}(L)\delta \\ M_{56}\delta \\ \delta \end{pmatrix} = \begin{pmatrix} \underline{M}_{0x} & \vec{0} & \vec{D}(L) \\ \vec{T}^T & 1 & M_{56} \\ \vec{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{z}_0 \\ 0 \\ \delta \end{pmatrix}$$



The periodic orbit for particles with relative energy deviation δ is

$$\vec{\eta}(0) = \underline{M}_0\vec{\eta}(0) + \vec{D}(L) \qquad \vec{\eta}(L) = \underline{M}_0\vec{\eta}(0) + \vec{D}(L) \quad \text{with} \quad \vec{\eta}(L) = \vec{\eta}(0)$$

$$\Downarrow$$

$$\boxed{\vec{\eta}(0) = [\underline{1} - \underline{M}_0(0)]^{-1}\vec{D}(L)}$$

Particles with energy deviation δ oscillates around this periodic orbit.

$$\vec{z} = \vec{z}_\beta + \delta\vec{\eta}$$

$$\underline{\vec{z}_\beta(L) + }\,\delta\vec{\eta}(L) = \vec{z}(L) = \underline{M}_0\vec{z}(0) + \vec{D}(L)\delta = \underline{M}_0[\vec{z}_\beta(0) + \delta\vec{\eta}(0)] + \vec{D}(L)\delta$$

$$= \underline{M}_0\underline{\vec{z}_\beta(0) +}\,\delta\vec{\eta}(L)$$

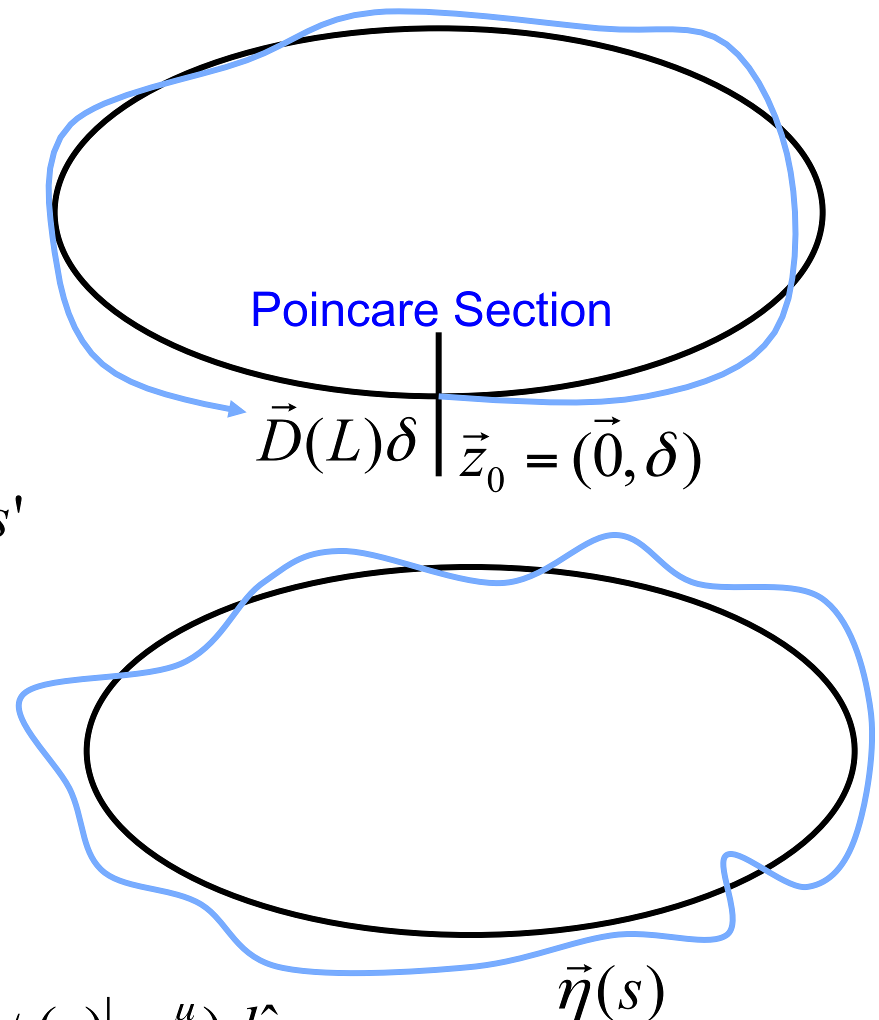
Periodic dispersion Integral

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \kappa\delta$$

$$\vec{z} = \underline{M}\vec{z}_0 + \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$$

$$\Rightarrow \quad \vec{D}(L) = \int_0^L \underline{M}(L-\hat{s})\begin{pmatrix} 0 \\ \kappa(\hat{s}) \end{pmatrix} ds'$$



$$\Delta\kappa = \delta\kappa$$

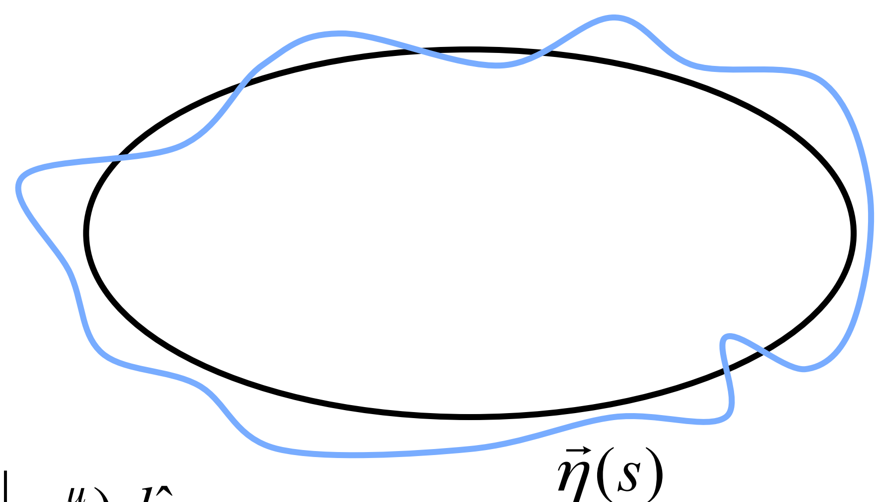
$$\eta(s) = \frac{\sqrt{\beta(s)}}{2\sin\frac{\mu}{2}} \oint \kappa(\hat{s})\sqrt{\beta(\hat{s})}\cos(|\psi(\hat{s}) - \psi(s)| - \tfrac{\mu}{2})d\hat{s}$$

Periodic dispersion correction

$$\eta''\delta = -(\kappa^2 + k)\eta\delta + \kappa\delta$$

$$(\eta + \Delta\eta)'' = -(\kappa^2 + k + \Delta k)(\eta + \Delta\eta) + \kappa$$

$$\Delta\eta = -(\kappa^2 + k)\Delta\eta + \kappa + \Delta k \cdot \eta$$



$$\Delta\kappa = \Delta k \cdot \eta$$

$$\eta(s) = \frac{\sqrt{\beta}}{2\sin\frac{\mu}{2}} \oint \Delta\hat{k} \cdot \hat{\eta}\sqrt{\hat{\beta}} \cos(|\hat{\psi} - \psi| - \tfrac{\mu}{2})\, d\hat{s}$$

Quadrupole Errors (repeat)

$$\vec{z}\,' = \underline{L}(s)\,\vec{z} + \Delta\vec{f}(\vec{z},s)$$

$$\vec{z}(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z},\hat{s})\,d\hat{s} \approx \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_H,\hat{s})\,d\hat{s}$$

$$x'' = -(\kappa^2 + k)x - \Delta k(s)\,x \quad \Rightarrow \quad \begin{pmatrix} x' \\ a' \end{pmatrix} = \begin{pmatrix} a \\ -(\kappa^2 + k)x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \Delta k(s) & 0 \end{pmatrix}\begin{pmatrix} x \\ a \end{pmatrix}$$

$$\vec{z}(s) = \left\{ \underline{M}(s) - \int_0^s \underline{M}(s,\hat{s}) \begin{pmatrix} 0 & 0 \\ \Delta k(\hat{s}) & 0 \end{pmatrix} \underline{M}(\hat{s},0)\,d\hat{s} \right\} \vec{z}_0$$

One quadrupole error:

$$\underline{M}(s,\hat{s}) + \Delta\underline{M}(s,\hat{s}) = \underline{M}(s,\hat{s}) - \underline{M}(s,\hat{s}) \begin{pmatrix} 0 & 0 \\ \Delta kl(\hat{s}) & 0 \end{pmatrix}$$

Quadrupole Error and Tune in periodic accelerators

$$\cos(\mu+\Delta\mu) = \tfrac{1}{2}\mathrm{Tr}[M_0(s_j)+\Delta M_0(s_j)] \approx \cos\mu - \Delta\mu\sin\mu$$

$$= \tfrac{1}{2}\mathrm{Tr}[\begin{pmatrix} 1 & 0 \\ -\Delta k l_j & 1 \end{pmatrix}\begin{pmatrix} \cos\mu+\alpha_j\sin\mu & \beta_j\sin\mu \\ -\gamma_j\sin\mu & \cos\mu-\alpha_j\sin\mu \end{pmatrix}]$$

$$= \cos\mu - \tfrac{1}{2}\Delta k l_j \beta_j \sin\mu$$

$$\boxed{\Delta\mu = \tfrac{1}{2}\Delta k l_j \beta_j}$$

Oscillation frequencies can be measured relatively easily and accurately.

Measurement of beta function: Change k and measure tune.

Quadrupole Error and periodic beta function

One pass accelerators:

$$\frac{\Delta\beta}{\beta} = -\Delta k l_j \beta_j \sin(2[\psi - \psi_j])$$

→ Beta beat oscillates twice as fast as orbit.

Periodic accelerators:

$$\frac{\Delta\beta}{\beta} = A\cos(2(\psi - \psi_j) + \varphi_0) \quad , \quad \psi(L) = \psi(0) + \mu \Rightarrow \varphi_0 = -\mu$$

$$\alpha' = k\beta - \gamma \quad , \quad \Delta\alpha' = k\Delta\beta - \Delta\gamma + \Delta k\beta$$

$$\Delta\alpha(s_j) - \Delta\alpha(s_j + L) = \Delta k l_j \beta_j$$

$$\left(\frac{\Delta\beta}{\beta}\right)'(s_j) - \left(\frac{\Delta\beta}{\beta}\right)'(s_j + L) = -2\Delta k l_j = \frac{4A\sin\mu}{\beta_j}$$

$$\frac{\Delta\beta}{\beta} = -\frac{\Delta k l_j \beta_j}{2\sin\mu}\cos(2|\psi - \psi_j| - \mu) , \quad \psi \in [0, \mu]$$

Quadrupole errors and betatron phase

$$M_{12} = \sqrt{\beta_j \beta} \sin(\psi - \psi_j)$$

$$(\frac{\Delta\beta_j}{2\beta_j} + \frac{\Delta\beta}{2\beta}) \sin(\psi - \psi_j) + (\Delta\psi - \Delta\psi_j)\cos(\psi - \psi_j) = 0$$

$$\frac{\Delta\beta}{\beta} = -\frac{\Delta k l_j \beta_j}{2\sin\mu} \cos(2(\psi - \psi_j) - \mu) \,, \quad \psi \in [\psi_j, \psi_j + \mu]$$

$$\Delta\psi = \Delta\psi_j + \frac{\Delta k l_j \beta_j}{4\sin\mu} [\cos(\mu) + \cos(2(\psi - \psi_j) - \mu)] \tan(\psi - \psi_j)$$

$$= \Delta\psi_j + \frac{\Delta k l_j \beta_j}{2\sin\mu} \cos(\psi - \psi_j - \mu) \sin(\psi - \psi_j)$$

$$\Delta\psi(s_j + L) = \Delta\psi_j + \frac{\Delta k l_j \beta_j}{2} = \Delta\psi_j + \mu$$

Quadrupole errors and betatron phase

$$\frac{\Delta\beta}{\beta} = -\frac{\Delta k l_j \beta_j}{2\sin\mu}\cos(2|\psi-\psi_j|-\mu)\,,\quad \psi\in[0,\mu]$$

$$\Delta\psi = -\int_0^s \frac{\Delta\hat\beta}{\hat\beta}d\hat s = \frac{\Delta k l_j \beta_j}{4\sin\mu}[\sin(2\psi_j-\mu)-\sin(2(\psi_j-\psi)-\mu)] \quad , \quad s<s_j$$

$$= \frac{\Delta k l_j \beta_j}{4\sin\mu}[\sin(2\psi_j-\mu)+\sin\mu+\int_{s_j}^s \frac{\Delta\hat\beta}{\hat\beta}d\hat s] \quad , \quad s>s_j$$

$$= \frac{\Delta k l_j \beta_j}{4\sin\mu}[\sin(2\psi_j-\mu)+\sin(2(\psi-\psi_j)-\mu)+2\sin\mu] \quad , \quad s>s_j$$

$$\Delta\psi(L) = \frac{\Delta k l_j \beta_j}{2} = \Delta\mu$$

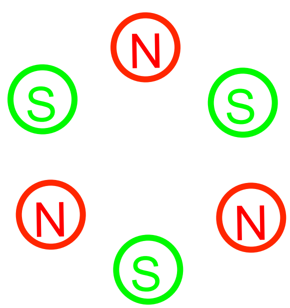
The phase change oscillates with twice the phase advance, while each quad error changes the center of the oscillation by

$$\delta\psi_j = \frac{\Delta k l_j \beta_j}{2}$$

Sextupoles (revisited)

$$\psi = \Psi_3 \,\mathrm{Im}\{(x - iy)^3\} = \Psi_3 \cdot (y^3 - 3x^2 y) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

C_3 Symmetry



i) Sextupole fields hardly influence the particles close to the center, where one can linearize in x and y.

ii) In linear approximation a by Δx shifted sextupole has a quadrupole field.

iii) When Δx depends on the energy, one can build an energy dependent quadrupole.

$$\vec{B} = -\vec{\nabla}\psi = \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

$$x \mapsto \Delta x + x$$

$$\vec{B} \approx \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix} + 6\Psi_3 \Delta x \begin{pmatrix} y \\ x \end{pmatrix} + O(\Delta x^2)$$

$$k_2 = \frac{q}{p} 3! \Psi_3 \Rightarrow k_1 = k_2 \Delta x$$

Chromaticity and its Correction

Chromaticity ξ = energy dependence of the tune

$$\upsilon(\delta) = \upsilon + \tfrac{\partial \upsilon}{\partial \delta}\delta + \dots$$

$$\xi = \tfrac{\partial \upsilon}{\partial \delta} \quad \text{with} \quad \upsilon = \tfrac{\mu}{2\pi}$$

Natural chromaticity ξ_0 = energy dependence of the tune due to quadrupoles only

$$\xi_{x0} = -\frac{1}{4\pi}\oint \beta_x(\hat{s}) k_1(\hat{s}) d\hat{s} \qquad \xi_{y0} = \frac{1}{4\pi}\oint \beta_y(\hat{s}) k_1(\hat{s}) d\hat{s}$$

Particles with energy difference oscillate around the periodic dispersion leading to a quadrupole effect in sextupoles that also shifts the tune:

$$\xi_x = \frac{1}{4\pi}\oint \beta_x(-k_1 + \eta_x k_2) d\hat{s} \qquad \xi_y = \frac{1}{4\pi}\oint \beta_y(k_1 - \eta_x k_2) d\hat{s}$$

Typically the the chormaticity ξ is chosen to be slightly positive, between 0 and 3.

Chromatic beta beat in periodic accelerators

One pass accelerators:

$$\frac{d\beta}{d\delta} = \beta(\Delta k_1 l - \Delta k_2 l \cdot \eta)\hat{\beta}\sin(2|\psi - \hat{\psi}| - \mu)$$

Periodic accelerators:

$$\frac{d\beta}{d\delta} = \frac{\beta}{2\sin\mu}(\Delta k_1 l - \Delta k_2 l \cdot \eta)\hat{\beta}\cos(2|\psi - \hat{\psi}| - \mu)$$

Sextupoles are used to compensate the chromaticity.

Several sextupoles are used to have their average compensate the chromaticity but have their regional variation compensate the chromatic beta beat on average and at critical sections, e.g. interaction points.

Chromatic dispersion

$$\eta'' = -(\kappa^2 + k)\eta + \kappa$$

$$(\eta + \eta_2\delta)'' = -(\kappa^2(1+\delta)^{-2} + k(1+\delta)^{-1} + k_s\eta\delta)(\eta + \eta_2\delta) + \kappa(1+\delta)^{-1}$$

$$(\eta + \eta_2\delta)'' = -(\kappa^2(1-2\delta) + k(1-\delta) + k_s\eta\delta)(\eta + \eta_2\delta) + \kappa(1-\delta)$$

$$\eta_2'' = -(\kappa^2 + k)\eta_2 + \kappa + [(\kappa^2 2 + k + k_s\eta)\eta - \kappa]$$

Second order dispersion is driven by sextupoles and energy dependent focusing and bending

$$\Delta\kappa = (\kappa^2 2 + k + k_s\eta)\eta - \kappa$$

One pass accelerators:

$$D_2(s) = \sqrt{\beta}\int_0^s [(\kappa^2 2 + k + k_s\eta)\eta - \kappa]\sqrt{\hat{\beta}}\sin(\psi - \hat{\psi})d\hat{s}$$

Periodic accelerators:

$$\eta_2(s) = \frac{\sqrt{\beta}}{2\sin\frac{\mu}{2}}\oint [(\kappa^2 2 + k + k_s\eta)\eta - \kappa]\sqrt{\hat{\beta}}\cos(|\hat{\psi} - \psi| - \tfrac{\mu}{2})\,d\hat{s}$$