Homework 1 for PHYS 7688 – Due Thursday September 19

Exercise (Canonical momentum)
Show that the Lorentz-force equation can be derived from the Hamiltonian $H = c\sqrt{[\vec{p}_c - q\vec{A}(\vec{r},t)]^2 + (mc)^2} + q\Phi(\vec{r},t)$, where the canonical momentum $\vec{p}_c$ is related to the classical momentum by $\vec{p} = \vec{p}_c - q\vec{A}$.

Exercise (Symplecticity)
(a) A matrix $\underline{M}$ is symplectic when it satisfies $\underline{M}\underline{J}\underline{M}^T = \underline{J}$. Using $\underline{J}^-$
and $\underline{J}^T = -\underline{J}$, show that the following properties are also satisfied:

$$\underline{M}^{-1} = -\underline{J}\underline{M}^T\underline{J}\ , \quad \underline{M}^T\underline{J}\underline{M} = \underline{J}\ .$$

(b) The linear transport map of a quadrupole is given by

$$\begin{pmatrix} x \\ p_x \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k}s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}s) \\ -\sqrt{k}\sin(\sqrt{k}s) & \cos(\sqrt{k}s) \end{pmatrix} \begin{pmatrix} x_0 \\ p_{x0} \end{pmatrix}$$

when k is the strength of the quadrupole field and p is the momentu
particle. Derive a Hamiltonian $H(x, p_x)$ that represents this map.

Exercise (Curvi-linear system)
Given a reference trajectory that is a helix around the z-axis with
Exercise (Beta functions)
Find the general form of the beta function in a drift:
(a) by solving the differential equation for $\beta(s)$ with the initial conditions $\beta(0) = \beta_0$ and $\alpha(0) = \alpha_0$.
(b) by solving the differential equation for $x(s)$ and $x'(s)$. The initial x_0 and x'_0 have to be expressed in terms of the initial Twiss parameters, and the Twiss parameters at s will then be expressed in terms of the solutions $x(s)$ and $x'(s)$.
(c) Find the general form of a beta function in a quadrupole of focusing strength k in terms of the distance s along the quadrupole and in terms of the initial Twiss parameters before the quadrupole.

Exercise (Phase space distribution):
(a) Given the Twiss parameters α, β, γ: specify the transformation from the amplitude and phase variables J and ϕ to the Cartesian phase space variables x and x'.
(b) Specify the inverse transformation.
(c) Given the Gaussian beam distribution in amplitude and phase variables, $\rho(J,\phi) = \frac{1}{2\pi\epsilon}e^{-\frac{J}{\epsilon}}$. What is the projection $\rho(x)$ of this distribution on the x axis. Check that the rms width of this distribution leads to $\sqrt{< x^2 >} = \sqrt{\beta\epsilon}$.

Exercise (Propagation of Twiss parameters)
Characterize Twiss parameters by $\{\beta(s), \alpha(s), \psi(s)\}$. Imagine two sections of a beam line where the first section transports Twiss parameters $\{\beta_0, \alpha_0, 0\}$ to $\{\beta_1, \alpha_1, \psi_1\}$ and the second transports $\{\beta_1, \alpha_1, 0\}$ to $\{\beta_2, \alpha_2, \psi_2\}$. Show that the total beam-line transports $\{\beta_0, \alpha_0, 0\}$ to $\{\beta_2, \alpha_2, \psi_1 + \psi_2\}$.