Answer:
The Hamiltonian

$$H = c\sqrt{[\vec{p}_c - q\vec{A}(\vec{r},t)]^2 + (mc)^2} + q\Phi(\vec{r},t) \tag{9}$$

leads to the equations of motion

$$\dot{\vec{r}} = \frac{d}{dt}\vec{r} = c\frac{\vec{p}_c - q\vec{A}}{\sqrt{(\vec{p}_c - q\vec{A})^2 + (mc)^2}} \ , \tag{10}$$

$$\dot{\vec{p}_c} = \frac{d}{dt}\vec{p}_c = cq\frac{(p_{ci} - qA_i)\vec{\partial}A_i}{\sqrt{(\vec{p}_c - q\vec{A})^2 + (mc)^2}} - q\vec{\partial}\Phi \ , \tag{11}$$

where a sum over the index i is implied. The first of these equations can be used to compute the relativistic factor γ as

$$\gamma = \frac{1}{\sqrt{1 - (\frac{\dot{\vec{r}}}{c})^2}} = \frac{1}{mc}\sqrt{(\vec{p}_c - q\vec{A})^2 + (mc)^2} \ . \tag{12}$$

With this the equations of motion can be simplified to

$$\dot{\vec{r}} = \frac{\vec{p}_c - q\vec{A}}{m\gamma} \ , \tag{13}$$

$$\dot{\vec{p}_c} = q\frac{(p_{ci} - qA_i)\vec{\partial}A_i}{m\gamma} - q\vec{\partial}\Phi(\vec{r},t) \ . \tag{14}$$

With $A_i\vec{B}C_i = \vec{A}\times(\vec{B}\times\vec{C}) + (\vec{A}\cdot\vec{B})\vec{C}$ one obtains

$$\dot{\vec{r}} = \frac{\vec{p}}{m\gamma} \ , \quad \vec{p} = \vec{p}_c - q\vec{A} \ , \tag{15}$$

$$\dot{\vec{p}_c} = q\dot{\vec{r}}\times(\vec{\partial}\times\vec{A}) - q[\vec{\partial}\Phi(\vec{r},t) + \partial_t\vec{A}] + q\frac{d}{dt}\vec{A} \ , \tag{16}$$

where $\frac{d}{dt}\vec{A} = (\dot{\vec{r}}\cdot\vec{\partial})\vec{A} + \partial_t\vec{A}$ was used. Taking into account that $\vec{E} = -[\vec{\partial}\Phi + \partial_t\vec{A}]$ and $\vec{B} = \vec{\partial}\times\vec{A}$, we obtain the Lorentz-force equation

$$\dot{\vec{p}} = m\gamma\dot{\vec{r}} \ , \quad \dot{\vec{p}} = q\dot{\vec{r}}\times\vec{B} + q\vec{E} \ . \tag{17}$$

Exercise (Symplecticity)
(a) A matrix M is symplectic when it satisfies $MJM^T = J$. Using $J^{-1} = -J$ and $J^T = -J$, show that the following properties are also satisfied:

$$M^{-1} = -JM^TJ \ , \quad M^TJM = J \ . \tag{6}$$

(b) The linear transport map of a quadrupole is given by

$$\begin{pmatrix} x \\ p_x \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k}s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}s) \\ -\sqrt{k}\sin(\sqrt{k}s) & \cos(\sqrt{k}s) \end{pmatrix} \begin{pmatrix} x_0 \\ p_{x0} \end{pmatrix} \tag{7}$$

when k is the strength of the quadrupole field and p is the momentum of the particle. Derive a Hamiltonian $H(x, p_x)$ that represents this map.

Answer:

(a) Multiplying $MJM^T = J$ by $-J$ leads to $-MJM^TJ = I$ so that $-JM^TJ$ is the inverse of M. Since the right and the left inverse for matrices is the same, we can write $-JM^TJM = I$. Multiplying this by J leads to $M^TJM = J$.

(b):

First we need to create a differential equation with this general solution. For this we write

$$\begin{aligned}
\begin{pmatrix} x' \\ p'_x \end{pmatrix} &= \begin{pmatrix} -\sqrt{k}\sin(\sqrt{k}s) & \cos(\sqrt{k}s) \\ -k\cos(\sqrt{k}s) & -\sqrt{k}\sin(\sqrt{k}s) \end{pmatrix} \begin{pmatrix} x_0 \\ p_{x0} \end{pmatrix} && (8) \\
&= J \begin{pmatrix} k\cos(\sqrt{k}s) & \sqrt{k}\sin(\sqrt{k}s) \\ -\sqrt{k}\sin(\sqrt{k}s) & \cos(\sqrt{k}s) \end{pmatrix} \begin{pmatrix} x_0 \\ p_{x0} \end{pmatrix} && (9) \\
&= J \begin{pmatrix} k\cos(\sqrt{k}s) & \sqrt{k}\sin(\sqrt{k}s) \\ -\sqrt{k}\sin(\sqrt{k}s) & \cos(\sqrt{k}s) \end{pmatrix} \begin{pmatrix} \cos(\sqrt{k}s) & -\frac{1}{\sqrt{k}}\sin(\sqrt{k}s) \\ \sqrt{k}\sin(\sqrt{k}s) & \cos(\sqrt{k}s) \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix} \\
&= J \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix} = J \begin{pmatrix} \partial_x \\ \partial_{p_x} \end{pmatrix} \left(\frac{1}{2}(kx^2 + p_x^2) \right) && (10)
\end{aligned}$$

The Hamiltonian is therefore

$$H = \frac{1}{2}(kx^2 + p_x^2). \tag{11}$$

Exercise (Curvi-linear system)
Given a reference trajectory that is a helix around the z-axis with

$$\vec{R}(z) = r\cos(kz)\vec{e}_X + r\sin(kz)\vec{e}_Y + z\vec{e}_Z, \tag{13}$$

with the Cartesian coordinate vectors $\vec{e}_X$, $\vec{e}_Y$ and $\vec{e}_Z$.
(a) Show that z is not the pathlength s with which the reference trajector is parametrized. Then compute the path length $s(z)$ and specify $\vec{R}(s)$ so that $|d\vec{R}| = ds$ and compute $\vec{e}_s$, $\vec{e}_\kappa$, and $\vec{e}_b$.
(b) Compute $\vec{e}_x$ and $\vec{e}_y$ of the curvilinear system and check that $\frac{d}{ds}\vec{e}_x$ and $\frac{d}{ds}\vec{e}_y$ are what they are specified to be in the handouts.

Answer:
(a)

$$ds = |d\vec{R}| = |-rk\sin(kz)\vec{e}_X + rk\cos(kz)\vec{e}_Y + \vec{e}_Z|dz = \sqrt{1+(rk)^2}dz\ . \tag{14}$$

Therefore $s(z) = z/\epsilon$ and with $\epsilon = \frac{1}{\sqrt{1+(rk)^2}}$,

$$\vec{R}(s) = r\cos(k\epsilon s)\vec{e}_X + r\sin(k\epsilon s)\vec{e}_Y + \epsilon s\vec{e}_Z\ . \tag{15}$$

$$\begin{aligned}
\vec{e}_s &= \partial_s\vec{R}(s) = -rk\epsilon\sin(k\epsilon s)\vec{e}_X + rk\epsilon\cos(k\epsilon s)\vec{e}_Y + \epsilon\vec{e}_Z\ , && (16)\\
\vec{\kappa} &= -\partial_s\vec{e}_s = r(k\epsilon)^2[\cos(k\epsilon s)\vec{e}_X + \sin(k\epsilon s)\vec{e}_Y]\ , && (17)\\
\vec{e}_\kappa &= \vec{\kappa}/|\vec{\kappa}| = \cos(k\epsilon s)\vec{e}_X + \sin(k\epsilon s)\vec{e}_Y\ , && (18)\\
\vec{e}_b &= \vec{e}_s\times\vec{e}_\kappa = -\epsilon\sin(k\epsilon s)\vec{e}_X + \epsilon\cos(k\epsilon s)\vec{e}_Y - rk\epsilon\vec{e}_Z\ . && (19)
\end{aligned}$$

(b)
The torsion T' is computed by

$$T' = \vec{e}_b\cdot\partial_s\vec{e}_\kappa = k\epsilon^2\ . \tag{20}$$

The new coordinate vectors are therefore (Note that the $\vec{e}_x$ and $\vec{e}_y$ here are of the new curvilinear system, which are NOT the $\vec{e}_X$ and $\vec{e}_Y$ from part (a))

$$\begin{aligned}
\vec{e}_x &= \cos(k\epsilon^2 s)\vec{e}_\kappa - \sin(k\epsilon^2 s)\vec{e}_b \qquad (21)\\
&= [\cos(k\epsilon^2 s)\cos(k\epsilon s) + \epsilon\sin(k\epsilon^2 s)\sin(k\epsilon s)]\vec{e}_X \\
&+ [\cos(k\epsilon^2 s)\sin(k\epsilon s) - \epsilon\sin(k\epsilon^2 s)\cos(k\epsilon s)]\vec{e}_Y + rk\epsilon\sin(k\epsilon^2 s)\vec{e}_Z \ , \\
\vec{e}_y &= \sin(k\epsilon^2 s)\vec{e}_\kappa + \cos(k\epsilon^2 s)\vec{e}_b \qquad (22)\\
&= [\sin(k\epsilon^2 s)\cos(k\epsilon s) - \epsilon\cos(k\epsilon^2 s)\sin(k\epsilon s)]\vec{e}_X \\
&+ [\sin(k\epsilon^2 s)\sin(k\epsilon s) + \epsilon\cos(k\epsilon^2 s)\cos(k\epsilon s)]\vec{e}_Y - rk\epsilon\cos(k\epsilon^2 s)\vec{e}_Z \ . \qquad (23)
\end{aligned}$$

A differentiation leads to

$$\begin{aligned}
\partial_s\vec{e}_x &= k\epsilon(-1+\epsilon^2)\cos(k\epsilon^2 s)\sin(k\epsilon s)\vec{e}_X \\
&+ k\epsilon(1-\epsilon^2)\cos(k\epsilon^2 s)\cos(k\epsilon s)\vec{e}_Y + r(k\epsilon)^2\epsilon\cos(k\epsilon^2 s)\vec{e}_Z \ , \qquad (24)\\
\partial_s\vec{e}_y &= k\epsilon(-1+\epsilon^2)\sin(k\epsilon^2 s)\sin(k\epsilon s)\vec{e}_X \\
&+ k\epsilon(1-\epsilon^2)\sin(k\epsilon^2 s)\cos(k\epsilon s)\vec{e}_Y + r(k\epsilon)^2\sin(k\epsilon^2 s)\vec{e}_Z \ . \qquad (25)\\
& \qquad (26)
\end{aligned}$$

With $1-\epsilon^2 = (rk\epsilon)^2$ this proves the desired relation

$$\partial_s\vec{e}_x = r(k\epsilon)^2\cos(k\epsilon^2 s)\vec{e}_s = \kappa_x\vec{e}_s \ , \qquad (27)$$

$$\partial_s\vec{e}_y = r(k\epsilon)^2\sin(k\epsilon^2 s)\vec{e}_s = \kappa_y\vec{e}_s \ . \qquad (28)$$

The right hand relations hold since $\vec{\kappa}\cdot\vec{e}_x = r(k\epsilon)^2\cos(k\epsilon^2 s)$ and $\vec{\kappa}\cdot\vec{e}_y = r(k\epsilon)^2\sin(k\epsilon^2 s)$.

Answer:

(a) The differential equation for $k = 0$ is

$$\beta' = -2\alpha \ , \quad \alpha' = -\gamma \tag{18}$$

leading to

$$\frac{d\beta'^2}{d\beta} = 2\frac{d\beta'}{d\beta}\beta' = 2\beta'' = 4\gamma = \frac{4+\beta'^2}{\beta} \tag{19}$$

This can be integrated leading to

$$\int \frac{1}{4+\beta'^2} d\beta'^2 = \int \frac{1}{\beta} d\beta \ \Rightarrow \ln(4+\beta'^2) = \ln\beta + \ln\frac{4}{\beta^*} \ , \tag{20}$$

with an integration constant $\ln\frac{4}{\beta^*}$ which is chosen so that $\beta = \beta^*$ when $\beta' = 0$. This leads to another differential equation

$$\beta' = \pm 2\sqrt{\frac{\beta}{\beta^*} - 1} \ \Rightarrow \int \frac{1}{\pm 2\sqrt{\frac{\beta}{\beta^*} - 1}} d\beta = \int ds \ . \tag{21}$$

With two integration constants one obtains $\pm\beta^*\sqrt{\frac{\beta}{\beta^*} - 1} = s - s_0$, which is $\beta(s) = \beta^*[1 + (\frac{s-s_0}{\beta^*})^2]$. This leads to $\beta_0 = \beta^*[1 + (\frac{s_0}{\beta^*})^2]$ and $\alpha_0 = -\frac{1}{2}\beta_0' = \frac{s_0}{\beta^*}$, which results in $\beta^* = \beta_0/[1 + \alpha_0^2] = \gamma_0$. Therefore one obtains

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2 \ . \tag{22}$$

(b) The solution of the equation of motion in a drift is

$$x(s) = x_0 + s x_0' \ , \ x'(s) = x_0' \ . \tag{23}$$

The initial phase space coordinates can be expressed in terms of the initial Twiss parameter as

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta_0} & 0 \\ -\frac{\alpha_0}{\sqrt{\beta_0}} & \frac{1}{\sqrt{\beta_0}} \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix} \ . \tag{24}$$

Similarly, the Twiss parameters at s are related to the phase space coordinates at s by

$$\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\Phi & \sin\Psi \\ -\sin\Psi & \cos\Psi \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix} = \frac{1}{\sqrt{2J}} \begin{pmatrix} x \\ x' \end{pmatrix}$$

$$= \quad \frac{1}{\sqrt{2J}} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} \frac{\beta_0 - \alpha_0 s}{\sqrt{\beta_0}} & \frac{s}{\sqrt{\beta_0}} \\ -\frac{\alpha_0}{\sqrt{\beta_0}} & \frac{1}{\sqrt{\beta_0}} \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix} \ . \tag{25}$$

Here the transport matrix of a drift has been used and could be replaced by that of another element within which the beta functions are sought. This leads to

$$\begin{pmatrix} \sqrt{\beta}\cos\Psi & \sqrt{\beta}\sin\Psi \\ -\frac{\alpha\cos\Psi+\sin\Psi}{\sqrt{\beta}} & -\frac{\alpha\sin\Psi-\cos\Psi}{\sqrt{\beta}} \end{pmatrix} = \frac{1}{\sqrt{\beta_0}} \begin{pmatrix} \beta_0 - \alpha_0 s & s \\ -\alpha_0 & 1 \end{pmatrix} . \tag{26}$$

Eliminating the betatron phase by adding the quadrature of the top two matrix elements leads to

$$\beta = \frac{1}{\beta_0}([\beta_0 - \alpha_0 s]^2 + s^2) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2 , \tag{27}$$

$$\alpha = \alpha_0 - \gamma_0 s , \tag{28}$$

$$\gamma = \frac{1+\alpha^2}{\beta} = \gamma_0 , \tag{29}$$

$$\tan\Psi = \frac{s}{\beta_0 - \alpha_0 s} . \tag{30}$$

(c) With $\tilde{c} = \cos(\sqrt{k}s)$ and $\tilde{s} = \sin(\sqrt{k}s)$ one obtains

$$\begin{pmatrix} \sqrt{\beta}\cos\Psi & \sqrt{\beta}\sin\Psi \\ -\frac{\alpha\cos\Psi+\sin\Psi}{\sqrt{\beta}} & -\frac{\alpha\sin\Psi-\cos\Psi}{\sqrt{\beta}} \end{pmatrix} = \frac{1}{\sqrt{\beta_0}} \begin{pmatrix} \tilde{c}\beta_0 - \frac{\tilde{s}}{\sqrt{k}}\alpha_0 & \frac{\tilde{s}}{\sqrt{k}} \\ -(\tilde{s}\sqrt{k}\beta_0 + \tilde{c}\alpha_0) & \tilde{c} \end{pmatrix} . \tag{31}$$

Eliminating the betatron phase by adding the quadrature of the top two matrix elements leads to

$$\begin{aligned} \beta &= \frac{1}{\beta_0}(\tilde{c}\beta_0 - \frac{\tilde{s}}{\sqrt{k}}\alpha_0)^2 + \frac{\tilde{s}^2}{k} \\ &= (\beta_0 + \frac{\gamma_0}{k})\frac{1}{2} + (\beta_0 - \frac{\gamma_0}{k})\frac{1}{2}\cos(2\sqrt{k}s) - \alpha_0\frac{1}{\sqrt{k}}\sin(2\sqrt{k}s) . \end{aligned} \tag{32}$$

Exercise (Phase space distribution):
(a) Given the Twiss parameters α, β, γ: specify the transformation from the amplitude and phase variables J and ϕ to the Cartesian phase space variables x and x'.
(b) Specify the inverse transformation.
(c) Given the Gaussian beam distribution in amplitude and phase variables, $\rho(J,\phi) = \frac{1}{2\pi\epsilon}e^{-\frac{J}{\epsilon}}$. What is the projection $\rho(x)$ of this distribution on the x axis. Check that the rms width of this distribution leads to $\sqrt{< x^2 >} = \sqrt{\beta\epsilon}$.

Answer:

(a)

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi(s) + \phi_0) \\ \cos(\psi(s) + \phi_0) \end{pmatrix} \tag{33}$$

(b) The inverse of this equation is obtained from

$$\begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sin(\psi(s) + \phi_0) \\ \cos(\psi(s) + \phi_0) \end{pmatrix} , \tag{34}$$

which leads to

$$J = \frac{1}{2}(\gamma x^2 + 2\alpha x x' + \beta x'^2) \; , \quad \phi_0 = \arctan(\frac{x}{\alpha x + \beta x'}) - \psi(s) \; . \tag{35}$$

(c) Since the Jacobi-Matrix of the transformation between (x, x') and (J, ϕ) is one,

$$\rho(x, x') = \rho(J(x, x'), \phi(x, x')) = \frac{1}{2\pi\epsilon} e^{-\frac{J(x,x')}{\epsilon}} = \frac{1}{2\pi\epsilon} e^{-\frac{\gamma x^2 + 2\alpha x x' + 2\beta x'^2}{2\epsilon}} \; . \tag{36}$$

The position distribution is then given by the projection along the x'-axis,

$$\begin{aligned} \rho(x) &= \int_{-\infty}^{\infty} \rho(x, x') dx' = \frac{1}{2\pi\epsilon} e^{-\frac{\gamma x^2 - \frac{\alpha^2}{\beta} x^2}{2\epsilon}} \int_{-\infty}^{\infty} e^{-\frac{\beta (x' + \frac{\alpha}{\beta} x)^2}{2\epsilon}} dx' \qquad (37) \\ &= \frac{1}{2\pi\epsilon} e^{-\frac{x^2}{2\beta\epsilon}} \int_{-\infty}^{\infty} e^{-\frac{\beta x'^2}{2\epsilon}} dx' = \frac{1}{2\pi\epsilon} \sqrt{\frac{2\epsilon}{\beta}} e^{-\frac{x^2}{2\beta\epsilon}} \int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \frac{1}{\sqrt{2\pi\beta\epsilon}} e^{-\frac{x^2}{2\beta\epsilon}} \; . \end{aligned}$$

This is a Gauss-function with the standard deviation $\sigma = \sqrt{\beta\epsilon}$.

Exercise (Propagation of Twiss parameters)

Characterize Twiss parameters by $\{\beta(s), \alpha(s), \psi(s)\}$. Imagine two sections of a beam line where the first section transports Twiss parameters $\{\beta_0, \alpha_0, 0\}$ to $\{\beta_1, \alpha_1, \psi_1\}$ and the second transports $\{\beta_1, \alpha_1, 0\}$ to $\{\beta_2, \alpha_2, \psi_2\}$. Show that the total beam-line transports $\{\beta_0, \alpha_0, 0\}$ to $\{\beta_2, \alpha_2, \psi_1 + \psi_2\}$.

Answer:

There are various ways to answer the question. The simplest is to observe that $\psi_1 = \int_{s_0}^{s_1} \frac{1}{\beta(s)} ds$ and $\psi_2 = \int_{s_1}^{s_2} \frac{1}{\beta(s)} ds$ so that the total phase advance is

$\psi = \int_{s_0}^{s_2} \frac{1}{\beta(s)} ds = \psi_1 + \psi_2$. One can also use the matrices

$$\underline{\beta} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} , \quad \underline{R}(\psi) = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} . \tag{8}$$

to represent the transport matrix of the two sections with

$$\underline{M}_1 = \underline{\beta}_1 \underline{R}(\psi_1) \underline{\beta}_0^{-1} , \quad \underline{M}_2 = \underline{\beta}_2 \underline{R}(\psi_2) \underline{\beta}_1^{-1} . \tag{9}$$

The transport matrix of the total beam line is then clearly

$$\underline{M} = \underline{M}_2 \, \underline{M}_1 = \underline{\beta}_2 \underline{R}(\psi_1 + \psi_2) \underline{\beta}_0^{-1} . \tag{10}$$