Macroscopic Fields in Accelerators

$$\frac{d}{dt}\vec{p} = q(\vec{E} + \vec{v} \times \vec{B})$$

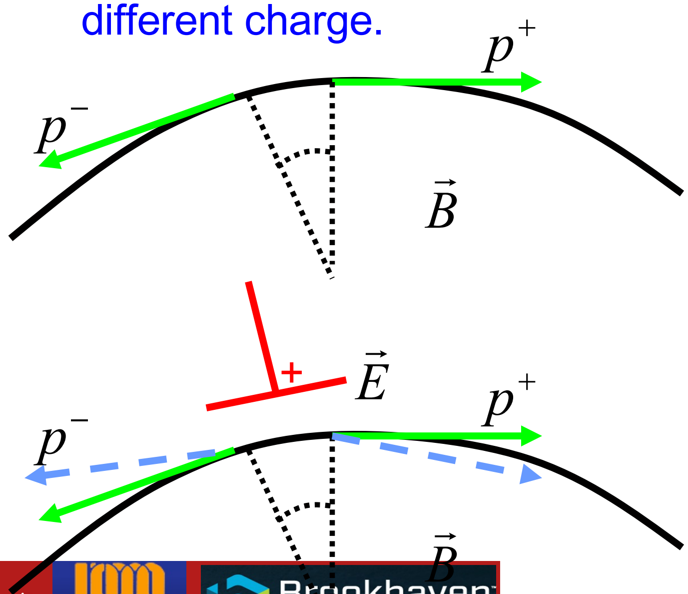
E has a similar effect as v B.

For relativistic particles B = 1T has a similar effect as

E = cB = 3 10^8 V/m , such an

Electric field is beyond technical limits.

- Electric fields are only used for very low energies or
- For separating two counter rotating beams with different charge.



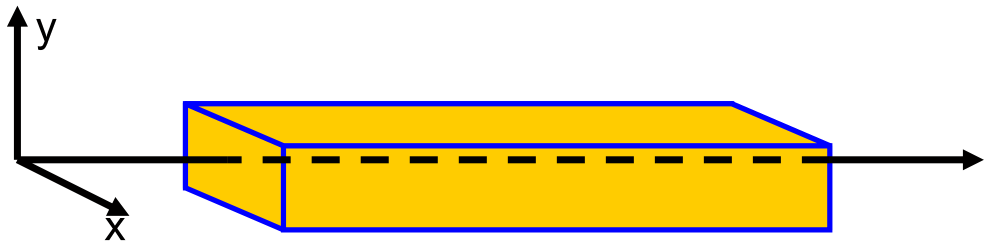
Electrostatic separators at CESR

Magnetic Fields in Accelerators

Static magnetic fileds: $\partial_t \vec{B} = 0\,;\quad \vec{E} = 0$ Charge free space: $\vec{j} = 0$

$$\vec{\nabla}\times\vec{B} = \mu_0(\vec{j} + \varepsilon_0 \partial_t \vec{E}) = 0 \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi(\vec{r})$$

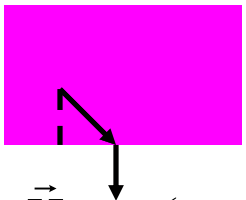
$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \vec{\nabla}^2\psi(\vec{r}) = 0$$



(x=0,y=0) is the beam's design curve

For finite fields on the design curve,
Ψ can be power expanded in x and y: $\psi(x,y,z) = \sum_{n,m=0}^{\infty} b_{nm}(z) x^n y^m$

Surfaces of equal scalar magnetic potential



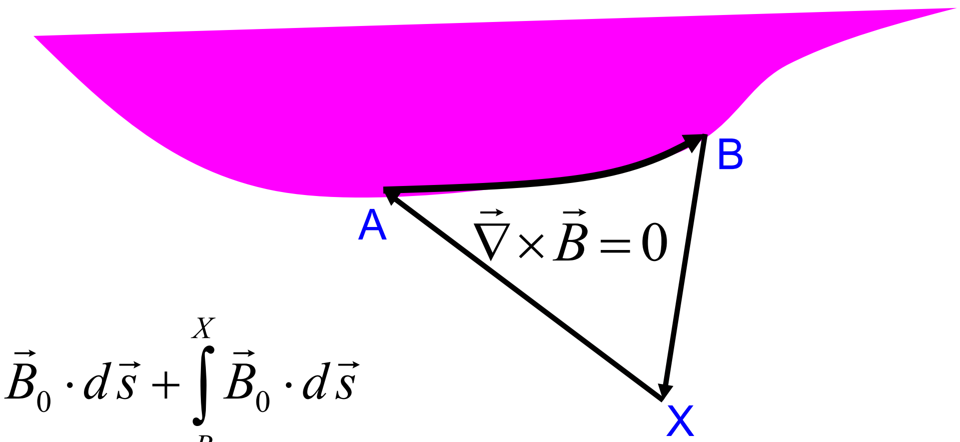
$$\vec{B}_\perp(\text{out}) = \vec{B}_\perp(\text{in})$$

$$\vec{H}_{\text{parallel}}(\text{out}) = \vec{H}_{\text{parallel}}(\text{in})$$

$$\vec{B}_{\text{parallel}}(\text{out}) = \frac{1}{\mu_r}\vec{B}_{\text{parallel}}(\text{in})$$

$$\vec{B}(\vec{r}) = -\vec{\nabla}\Psi(\vec{r})$$

For large permeability, H(out) is perpendicular to the surface.



$$0 = \oint \vec{B}\cdot d\vec{s} = \int_X^A \vec{B}_0\cdot d\vec{s} + \int_A^B \vec{B}_0\cdot d\vec{s} + \int_B^X \vec{B}_0\cdot d\vec{s}$$

$$= \int_X^A \vec{B}_0\cdot d\vec{s} + \frac{1}{\mu_r}\int_A^B \vec{B}_0\cdot d\vec{s} + \int_B^X \vec{B}_0\cdot d\vec{s}$$

$$\approx \int_X^A \vec{B}_0\cdot d\vec{s} + \int_B^X \vec{B}_0\cdot d\vec{s} = \Psi(A) - \Psi(B)$$

For highly permeable materials (like iron) surfaces have a constant potential.

Green's Theorem

$$\vec{\nabla}^2\psi = 0$$

Green function:

$$\vec{\nabla}_0^2 G(\vec{r},\vec{r}_0) = \delta(\vec{r}-\vec{r}_0)$$

$$\begin{aligned}\psi(\vec{r}) &= \int_V \psi(\vec{r}_0)\delta(\vec{r}-\vec{r}_0)\, d^3\vec{r}_0 \\ &= \int_V \left[\psi(\vec{r}_0)\vec{\nabla}_0^2 G - G\vec{\nabla}_0^2\psi(\vec{r}_0)\right] d^3\vec{r}_0 \\ &= \int_V \vec{\nabla}_0\left[\psi(\vec{r}_0)\vec{\nabla}_0 G - G\vec{\nabla}_0\psi(\vec{r}_0)\right] d^3\vec{r}_0 \\ &= \int_{\partial V} \left[\psi(\vec{r}_0)\vec{\nabla}_0 G - G\vec{\nabla}_0\psi(\vec{r}_0)\right]\cdot d^2\vec{r}_0 \\ &= \int_{\partial V} \left[\psi(\vec{r}_0)\vec{\nabla}_0 G + \vec{B}(\vec{r}_0)G\right]\cdot d^2\vec{r}_0\end{aligned}$$

Knowledge of the field and the scalar magnetic potential on a closed surface inside a magnet determines the magnetic field for the complete volume which is enclosed.

Expansion of the scalar potential

If field data in a plane (for example the midplane of a cyclotron or of a beam line magnet) is known, the complete filed is determined:

$$\psi(x,y,z)=\sum_{n=0}^{\infty} b_n(x,z)y^n \quad \Rightarrow \quad \vec{B}(x,0,z)=-\begin{pmatrix}\partial_x b_0(x,z)\\ b_1(x,z)\\ \partial_z b_0(x,z)\end{pmatrix}$$

$$0=\vec{\nabla}^2\psi=\sum_{n=0}^{\infty}(\partial_x^2+\partial_z^2)\, b_n\, y^n+\sum_{n=2}^{\infty} n(n-1)\, b_n\, y^{n-2}$$

$$=\sum_{n=0}^{\infty}\left[(\partial_x^2+\partial_z^2)\, b_n+(n+2)(n+1)\, b_{n+2}\right]y^n$$

$$b_{n+2}(x,z)=-\frac{1}{(n+2)(n+1)}(\partial_x^2+\partial_y^2)\, b_n(x,z)$$

Data of the magnetic field in the plane y=0 is used to determine $b_0(x,z)$ and $b_1(x,z)$.

Complex expansion of the potential

$$w \;= x + iy \quad , \quad \overline{w} \;= x - iy$$

$$\partial_x \;= \partial_w + \partial_{\overline{w}} \;, \quad \partial_y = i\partial_w - i\partial_{\overline{w}} = i(\partial_w - \partial_{\overline{w}})$$

$$\vec{\nabla}^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 = (\partial_w + \partial_{\overline{w}})^2 - (\partial_w - \partial_{\overline{w}})^2 + \partial_z^2 = 4\partial_w\partial_{\overline{w}} + \partial_z^2$$

$$\psi \quad = \mathrm{Im}\{\sum_{\nu,\lambda=0}^{\infty} a_{\nu\lambda}(z)\cdot(w\overline{w})^{\lambda}\,\overline{w}^{\nu}\}$$

$$\vec{\nabla}^2\psi = \mathrm{Im}\{\sum_{\nu=0,\lambda=1}^{\infty} 4a_{\nu\lambda}(\lambda+\nu)\lambda(w\overline{w})^{\lambda-1}\,\overline{w}^{\nu} + \sum_{\nu=0,\lambda=0}^{\infty} a''_{\nu\lambda}(w\overline{w})^{\lambda}\,\overline{w}^{\nu}\}$$

$$= \mathrm{Im}\{\sum_{\nu,\lambda=0}^{\infty}[4(\lambda+1+\nu)(\lambda+1)a_{\nu\lambda+1} + a''_{\nu\lambda}](w\overline{w})^{\lambda}\,\overline{w}^{\nu}\} = 0$$

Iteration equation: $a_{\nu\lambda+1} = \dfrac{-1}{4(\lambda+1+\nu)(\lambda+1)}a''_{\nu\lambda} \quad , \quad a_{\nu 0} = \Psi_{\nu}(z)$

The functions $\Psi_\nu(z)$ along a line determine the complete field inside a magnet.

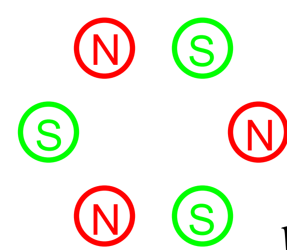
Multipole coefficients

$\Psi_\nu(z)$ are called the z-dependent multipole coefficients

$$\psi(x,y,z) = \mathrm{Im}\{\sum_{\nu,\lambda=0}^{\infty} \frac{(-1)^\lambda \nu!}{(\lambda+\nu)!\lambda!}\left(\frac{w\bar{w}}{4}\right)^\lambda \bar{w}^\nu \Psi_\nu^{[2\lambda]}(z)\}$$

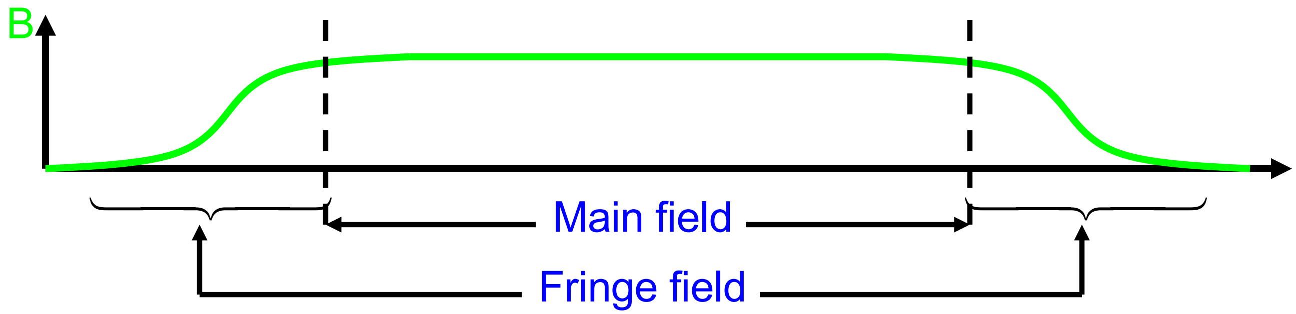
$$\psi(r,\varphi,z) = \sum_{\nu,\lambda=0}^{\infty} \frac{(-1)^\lambda \nu!}{(\lambda+\nu)!\lambda!}\left(\frac{r}{2}\right)^{2\lambda} r^\nu \,\mathrm{Im}\{\Psi_\nu^{[2\lambda]}(z) e^{-i\nu\varphi}\}$$

The index ν describes C_ν Symmetry
around the z-axis $\vec{e}_z$
due to a sign change after $\Delta\varphi = \frac{\pi}{\nu}$



$\nu = 3$

Fringe Fields and Main fields



Only the fringe field region has terms with $\lambda \neq 0$ and $\partial_z^2 \psi \neq 0$

Main fields in accelerator physics: $\lambda = 0\,, \quad \partial_z^2 \psi = 0$

$$\Psi_\nu = \begin{cases} e^{i\nu\vartheta_\nu} |\Psi_\nu| & \text{for} \quad \nu \neq 0 \\ i \; |\Psi_0| & \text{for} \quad \nu = 0 \end{cases}$$

$$\psi(r,\varphi) = \sum_{\nu=1}^{\infty} r^\nu |\Psi_\nu| \operatorname{Im}\{e^{-i\nu(\varphi-\vartheta_\nu)}\} + |\Psi_0|$$

Main-Field Potential

Main field potential: $$\psi = |\Psi_0| - \sum_{\nu=1}^{\infty} r^\nu |\Psi_\nu| \sin[\nu(\varphi - \vartheta_\nu)]$$

The isolated multipole: $$\psi = -r^\nu |\Psi_\nu| \sin(\nu\varphi)$$

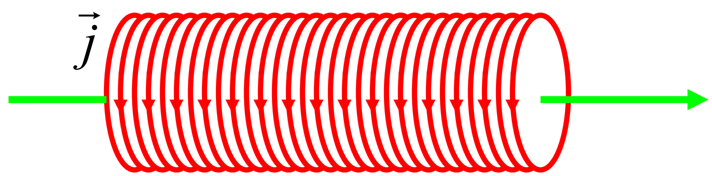
Where the rotation ϑ_ν of the coordinate system is set to 0

The potentials produced by different multipole components Ψ_ν have

a) Different rotation symmetry C_ν

b) Different radial dependence r^ν

Multipoles in Accelerators: ν=0, Solenoids



$$m\gamma\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = q\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}\times\begin{pmatrix}-\frac{x}{2}B_z'\\ -\frac{y}{2}B_z'\\ B_z\end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \frac{qB_z}{m\gamma}\begin{pmatrix}\dot{y}\\ -\dot{x}\end{pmatrix} + \frac{qB_z'\dot{z}}{2m\gamma}\begin{pmatrix}y\\ -x\end{pmatrix}$$

$$\Downarrow$$

$$\ddot{w} = -i\frac{qB_z}{m\gamma}\dot{w} - i\frac{q\dot{B}_z}{2m\gamma}w$$

$$\psi = \Psi_0(z) - \frac{w\bar{w}}{4}\Psi_0''(z) \pm \ldots$$

$$\vec{B} = \begin{pmatrix}\frac{x}{2}\Psi_0''\\ \frac{y}{2}\Psi_0''\\ -\Psi_0'\end{pmatrix} \Rightarrow \vec{\nabla}\cdot\vec{B} = 0$$

$$g = \frac{qB_z}{2m\gamma}\ ,\quad w_0 = w\,e^{i\int_0^t g\,dt}$$

$$\ddot{w}_0 = (\ddot{w} + i2g\dot{w} + i\dot{g}w - g^2 w)\,e^{i\int_0^t g\,dt}$$
$$= -g^2 w_0$$

$$\ddot{x}_0 = -g^2 x_0$$
$$\ddot{y}_0 = -g^2 y_0$$

Focusing in a rotating coordinate system

Strong vs. Solenoid Focusing

If the solenoids field was perpendicular to the particle's motion,

its bending radius would be $\rho_z = \dfrac{p}{qB_z}$

$$\ddot{r} = -\left(\frac{qB_z}{2m\gamma}\right)^2 r = -\frac{qv_z}{m\gamma} B_z \frac{r}{4\rho_z}$$

Solenoid focusing is weak compared to the deflections created by a transverse magnetic field.

Transverse fields: $\vec{B} = B_x\,\vec{e}_x + B_y\,\vec{e}_y$

$$m\gamma\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = q\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}\times\begin{pmatrix}B_x\\ B_y\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \frac{qv_z}{m\gamma}\begin{pmatrix}-B_y\\ B_x\end{pmatrix}$$

Strong focusing

$$\ddot{x} = -\frac{q\,v_z}{m\,\gamma}\frac{\partial B_y}{\partial x}x$$

Weak focusing < Strong focusing by about magnet aperture / bending radius

Natural Ring vs. Solenoid Focusing

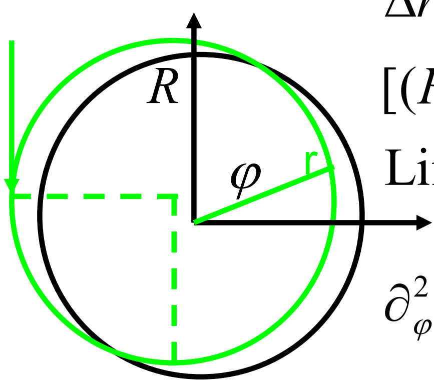
Solenoid magnets are used in detectors for particle identification via $\rho = \frac{p}{qB}$

The solenoid's rotation $\dot{\varphi} = -\frac{qB_z}{2m\gamma}$ of the beam is often compensated by a reversed solenoid called compensator.

Solenoid or Weak Focusing:

Solenoids are also used to focus low γ beams: $\ddot{w} = -\left(\frac{qB_z}{2m\gamma}\right)^2 w$

Weak focusing from natural ring focusing:



$$\Delta r = r - R$$

$$[(R+\Delta r)\cos\varphi - \Delta x_0]^2 + [(R+\Delta r)\sin\varphi - \Delta y_0]^2 = R^2$$

$$\text{Linearization in } \Delta: \quad \Delta r = (\cos\varphi\,\Delta x_0 + \sin\varphi\,\Delta y_0)$$

$$\partial_\varphi^2 \Delta r = -\Delta r \quad \Rightarrow \quad \Delta\ddot{r} = -\dot{\varphi}^2 \Delta r = -\left(\frac{v}{\rho}\right)^2 \Delta r = -\left(\frac{qB}{m\gamma}\right)^2 \Delta r$$

$$\frac{d^2\Delta r}{ds^2} = \frac{d^2\Delta r}{dt^2}\frac{1}{v^2} = \frac{1}{\rho^2}\Delta r$$

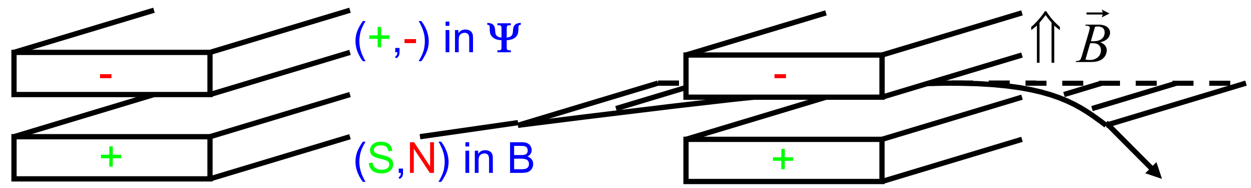
Focusing strength: $\frac{1}{\rho^2}$

Multipoles in Accelerators: ν=1, Dipoles

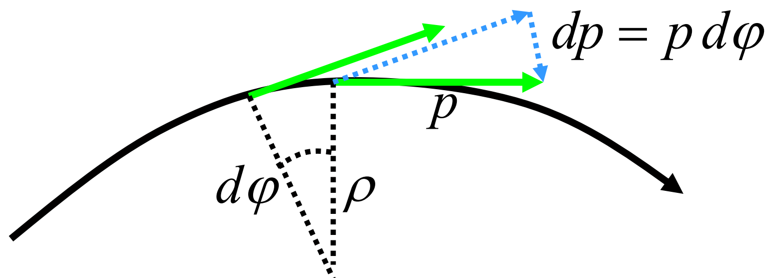
$$\psi = \Psi_1 \,\mathrm{Im}\{x - iy\} = -\Psi_1 \cdot y \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_1 \,\vec{e}_y$$

Equipotential
$y = \text{const.}$

C_1 Symmetry



Dipole magnets are used for steering the beams direction

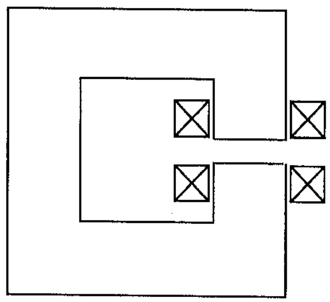


$$\frac{d\vec{p}}{dt} = q\,\vec{v} \times \vec{B} \quad \Rightarrow \quad \frac{dp}{dt} = qvB_\perp \quad \Rightarrow \quad \rho = \frac{dl}{d\varphi} = \frac{vdt}{dp/p} = \frac{p}{qB_\perp}$$

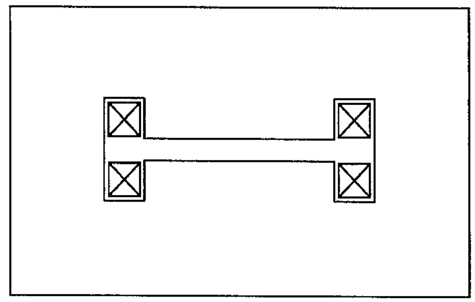
Bending radius: $\rho = \dfrac{p}{qB}$

Types of iron-dominated Dipoles

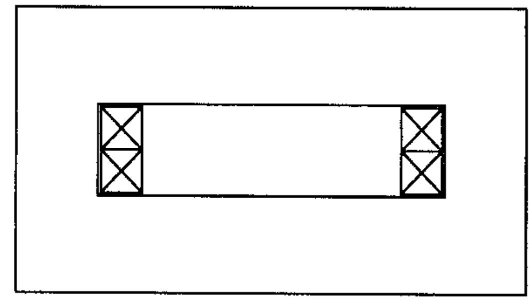
C-shape magnet:

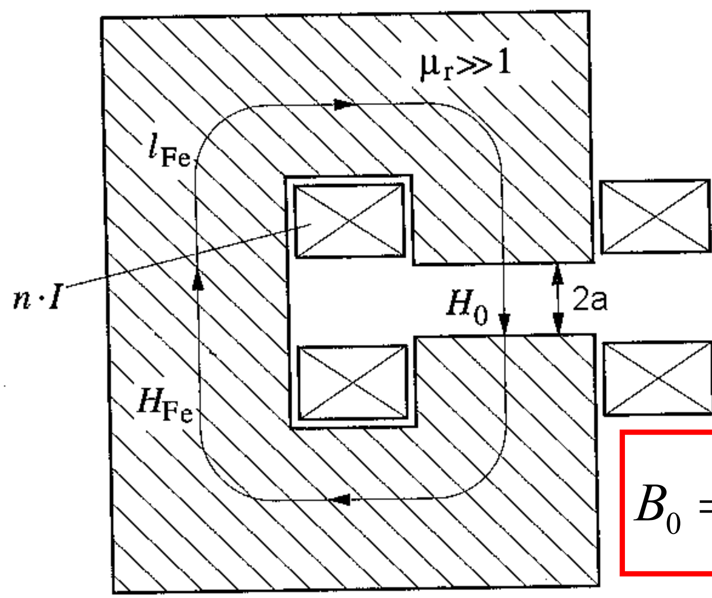


H-shape magnet:



Window frame magnet:





$$\vec{B}_\perp(\text{out}) = \vec{B}_\perp(\text{in})$$

$$\vec{H}_\perp(\text{out}) = \mu_r \vec{H}_\perp(\text{in})$$

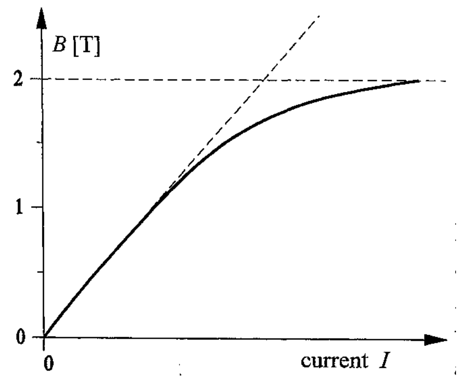
$$2nI = \oint \vec{H} \cdot d\vec{s} = H_{Fe} l_{Fe} + H_0 2a$$

$$= \tfrac{1}{\mu_r} H_0 l_{Fe} + H_0 2a \approx H_0 2a$$

$$B_0 = \mu_0 \frac{nI}{a}$$

Dipole strength: $\frac{1}{\rho} = \frac{q\mu_0}{p} \frac{nI}{a}$

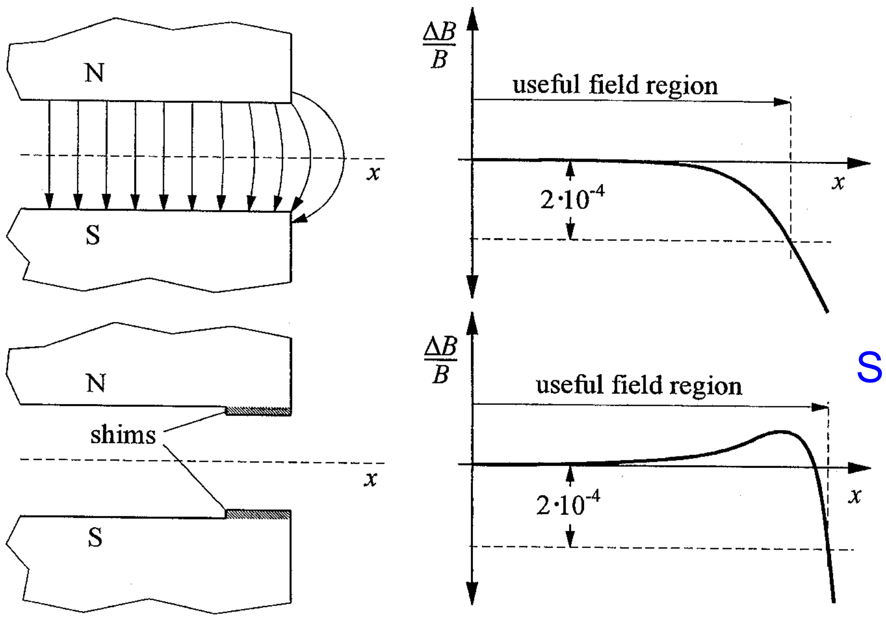
Iron-dominated Dipoles Fields



B = 2 T: Typical limit, since the field becomes dominated by the coils, not the iron. Limiting j for Cu is about 100A/mm^2

B < 1.5 T: Typically used region

B < 1 T: Region in which $B_0 = \mu_0 \frac{nI}{a}$

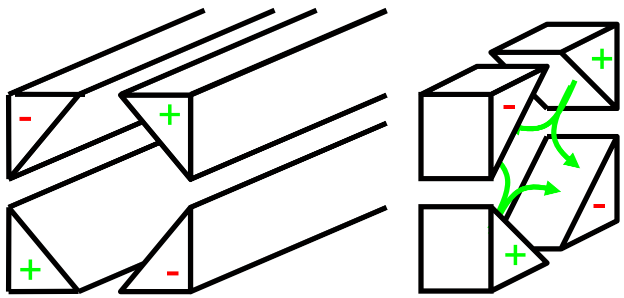


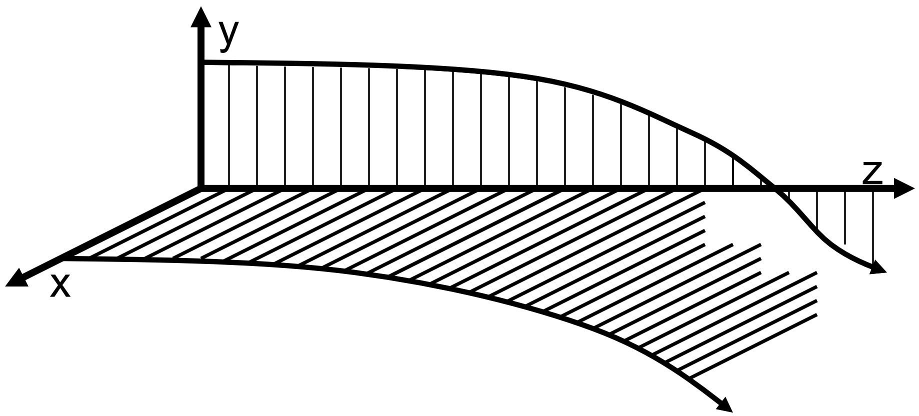
Shims reduce the space that is open to the beam, but they also reduce the fringe field region.

Multipoles in Accelerators: ν=2, Quadrupoles

$$\psi = \Psi_2 \,\mathrm{Im}\{(x-iy)^2\} = -\Psi_2 \cdot 2xy \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_2\, 2\begin{pmatrix} y \\ x \end{pmatrix}$$

C_2 Symmetry

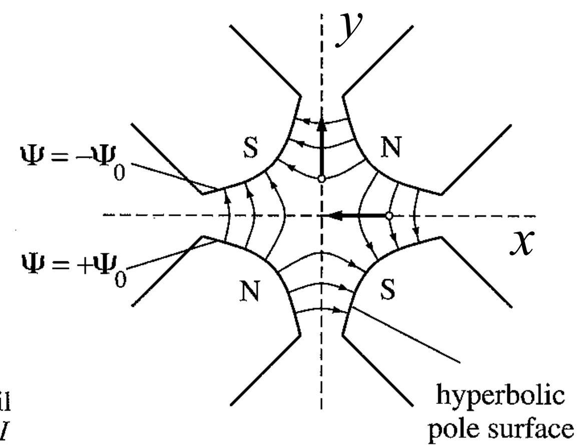


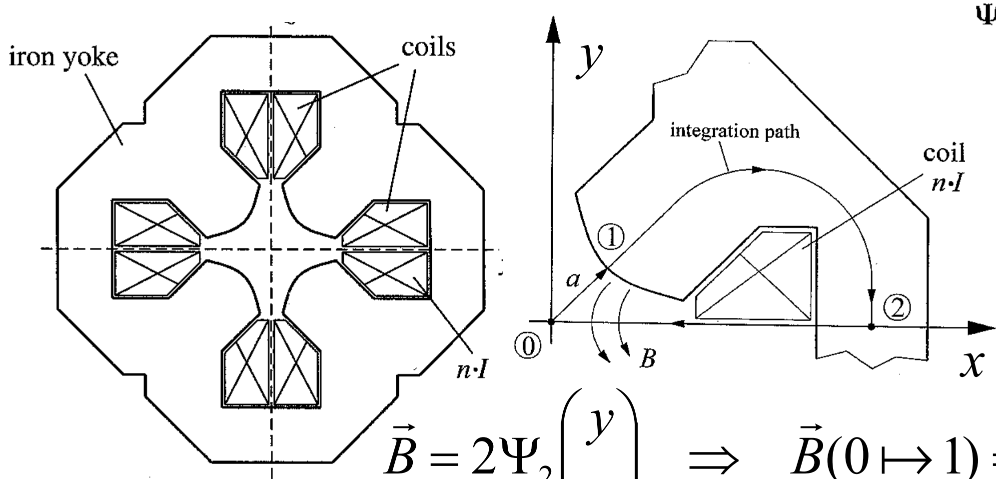


In a quadrupole particles are focused in one plane and defocused in the other plane. Other modes of strong focusing are not possible.

Iron-dominated quadrupoles fields

$$\psi = -\Psi_2 \cdot 2xy \quad \Rightarrow \quad \text{Equipotential: } x = \frac{\text{const.}}{y}$$





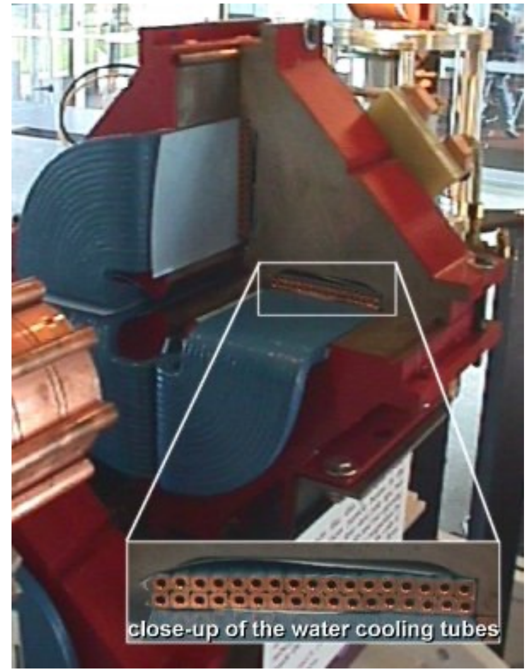
$$\vec{B} = 2\Psi_2 \begin{pmatrix} y \\ x \end{pmatrix} \quad \Rightarrow \quad \vec{B}(0 \mapsto 1) = 2\Psi_2 r\,\vec{e}_r$$

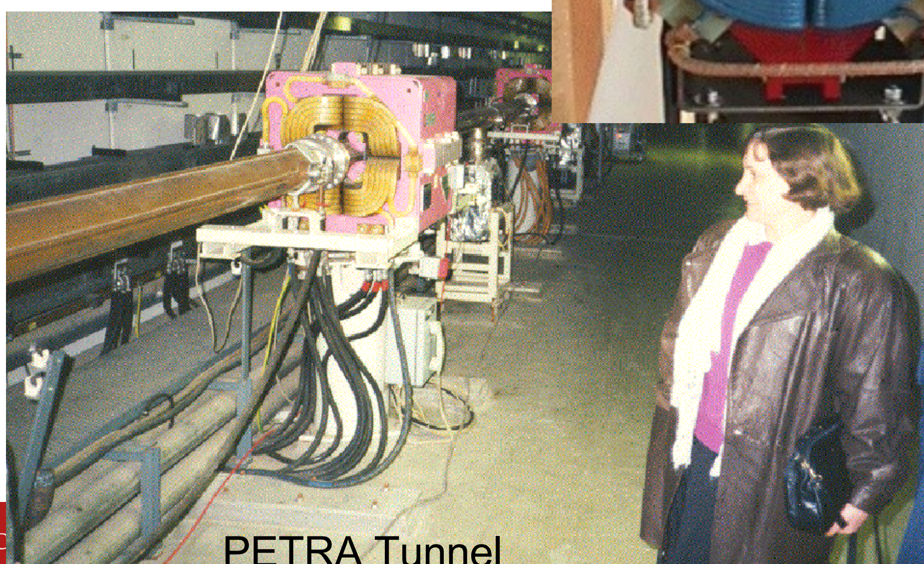
Quadrupole strength:

$$nI = \oint \vec{H} \cdot d\vec{s} \approx \int_0^a H_r dr = \Psi_2 \frac{a^2}{\mu_0}$$

$$k_1 = \frac{q}{p} \partial_x B_y \Big|_0 = \frac{q\mu_0}{p} \frac{2nI}{a^2}$$

Real Quadrupoles



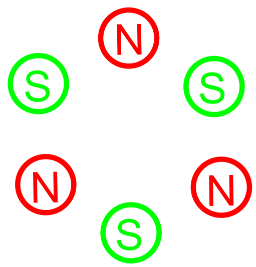


The coils show that this is an upright quadrupole not a rotated or skew quadrupole.

Multipoles in Accelerators: ν=3, Sextupoles

$$\psi = \Psi_3 \,\mathrm{Im}\{(x-iy)^3\} = \Psi_3 \cdot (y^3 - 3x^2 y) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

C_3 Symmetry



i) Sextupole fields hardly influence the particles close to the center, where one can linearize in x and y.

ii) In linear approximation a by Δx shifted sextupole has a quadrupole field.

iii) When Δx depends on the energy, one can build an energy dependent quadrupole.

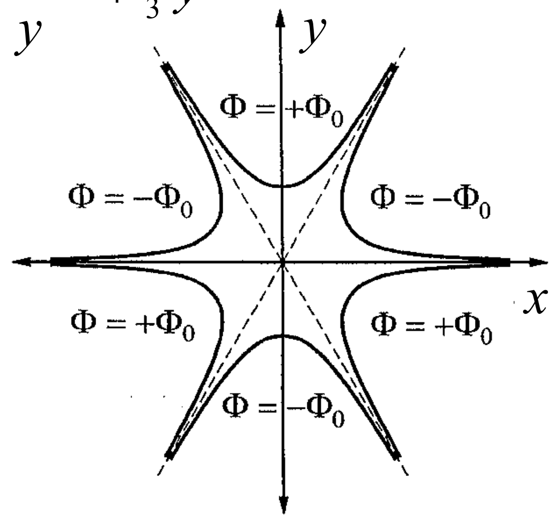
$$\vec{B} = -\vec{\nabla}\psi = \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

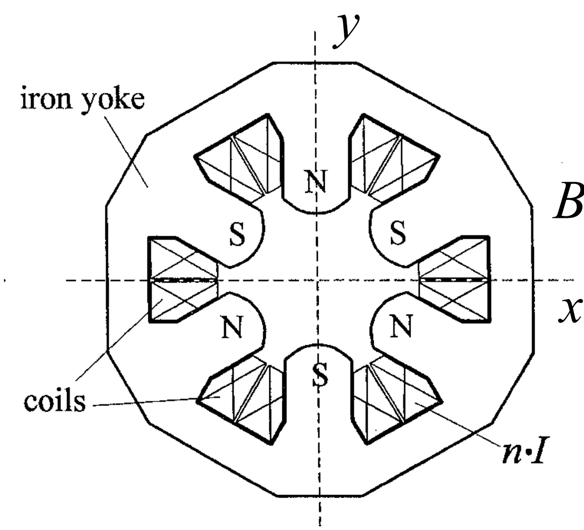
$$x \mapsto \Delta x + x$$

$$\vec{B} \approx \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix} + 6\Psi_3 \Delta x \begin{pmatrix} y \\ x \end{pmatrix} + O(\Delta x^2)$$

Iron-dominated sextupole fields

$\psi = \Psi_2 \cdot (y^3 - 3x^2 y) \quad \Rightarrow$ Equipotential: $x = \sqrt{\frac{\text{const.}}{y} + \frac{1}{3} y^2}$





$$B_y\big|_{x=0} = -\Psi_3 3y^2$$

$$nI = \oint \vec{H} \cdot d\vec{s} \approx \int_0^a H_r dr = \Psi_3 \frac{a^3}{\mu_0}$$

Quadrupole strength:

$$k_2 = \frac{q}{p} \partial_x^2 B_y\big|_0 = \frac{q\mu_0}{p} \frac{6nI}{a^3}$$

Real Sextupoles

Higher-order multipoles

$$\psi = \Psi_n \,\mathrm{Im}\{(x - iy)^n\} = \Psi_n \cdot (\ldots - i\,n\,x^{n-1}y) \quad \Rightarrow \quad \vec{B}(y=0) = \Psi_n\, n \begin{pmatrix} 0 \\ x^{n-1} \end{pmatrix}$$

Multipole strength: $k_n = \frac{q}{p} \partial_x^n B_y \Big|_{x,y=0} = \frac{q}{p} \Psi_{n+1}\,(n+1)!$ units: $\frac{1}{\mathrm{m}^{n+1}}$

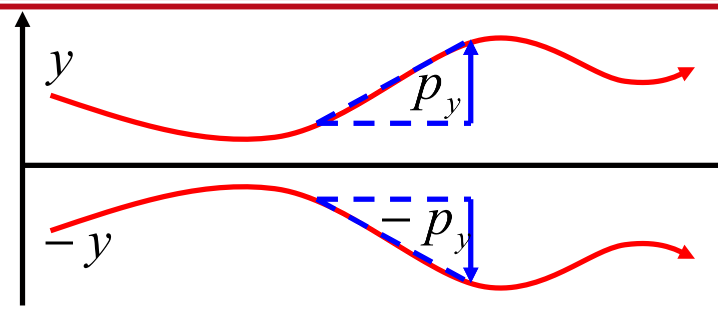
p/q is also called Bρ and used to describe the energy of multiply charge ions

Names: dipole, quadrupole, sextupole, octupole, decapole, duodecapole, …

Higher order multipoles come from

- Field errors in magnets
- Magnetized materials
- From multipole magnets that compensate such erroneous fields
- To compensate nonlinear effects of other magnets
- To stabilize the motion of many particle systems
- To stabilize the nonlinear motion of individual particles

Midplane-symmetric motion



$$\vec{r}^{\oplus} = (x, -y, z)$$

$$\vec{p}^{\oplus} = (p_x, -p_y, p_z)$$

$$\tfrac{d}{dt}\vec{p} = \vec{F}(\vec{r}, \vec{p}) \quad \Rightarrow \quad \tfrac{d}{dt}\vec{p}^{\oplus} = \vec{F}(\vec{r}^{\oplus}, \vec{p}^{\oplus})$$

$$\begin{aligned} v_y B_z - v_z B_y &= -v_y B_z(x,-y,z) - v_z B_y(x,-y,z) & & B_x(x,-y,z) = -B_x(x,y,z) \\ v_z B_x - v_x B_z &= -v_z B_x(x,-y,z) + v_x B_z(x,-y,z) \Rightarrow & & B_y(x,-y,z) = \ \ B_y(x,y,z) \\ v_x B_y - v_y B_x &= \ \ v_x B_y(x,-y,z) + v_y B_x(x,-y,z) & & B_z(x,-y,z) = -B_z(x,y,z) \end{aligned}$$

$$\boxed{\psi(x,-y,z) = -\psi(x,y,z)}$$

$$\Psi_n \,\mathrm{Im}\left\{e^{in\vartheta_n}(x+iy)^n\right\} = -\Psi_n \,\mathrm{Im}\left\{e^{in\vartheta_n}(x-iy)^n\right\}$$

$$\Rightarrow \quad \Psi_n \,\mathrm{Im}\left[e^{in\vartheta_n}\, 2\,\mathrm{Re}\left\{(x+iy)^n\right\}\right] = 0 \quad \Rightarrow \quad \boxed{\vartheta_n = 0}$$

The discussed multipoles

produce midplane symmetric motion. When the field is rotated by $\pi/2$, i.e $\vartheta_n = \pi/2n$, one speaks of a skew multipole.

Superconducting magnets

Above 2T the field from the bare coils dominate over the magnetization of the iron.

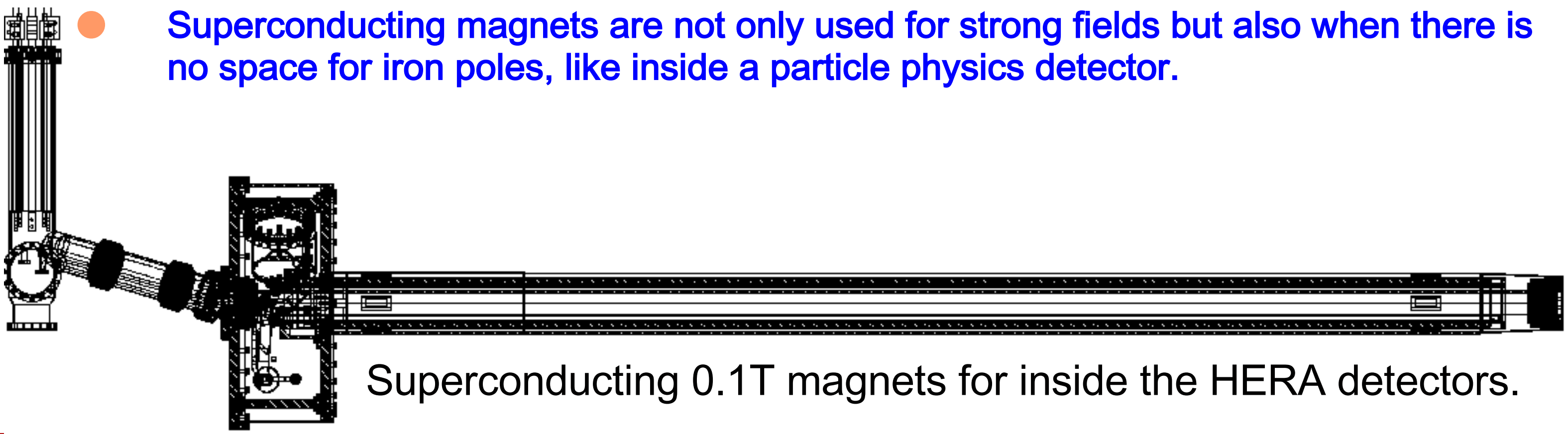
But Cu wires cannot create much filed without iron poles:

5T at 5cm distance from a 3cm wire would require a current density of

$$j = \frac{I}{d^2} = \frac{1}{d^2}\frac{2\pi\, r\, B}{\mu_0} = 1389\,\frac{\text{A}}{\text{mm}^2}$$

Cu can only support about 100A/mm^2.

- Superconducting cables routinely allow current densities of 1500A/mm^2 at 4.6 K and 6T. Materials used are usually Nb aloys, e.g. NbTi, Nb_3Ti or Nb_3Sn.
- Superconducting magnets are not only used for strong fields but also when there is no space for iron poles, like inside a particle physics detector.

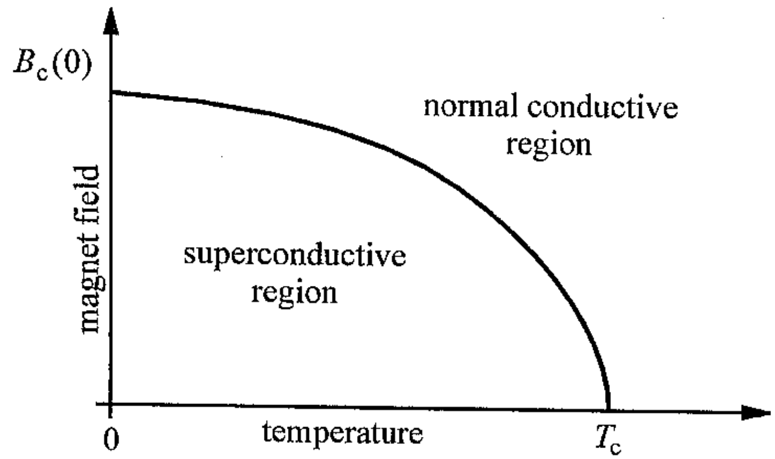


Superconducting 0.1T magnets for inside the HERA detectors.

Superconducting cables

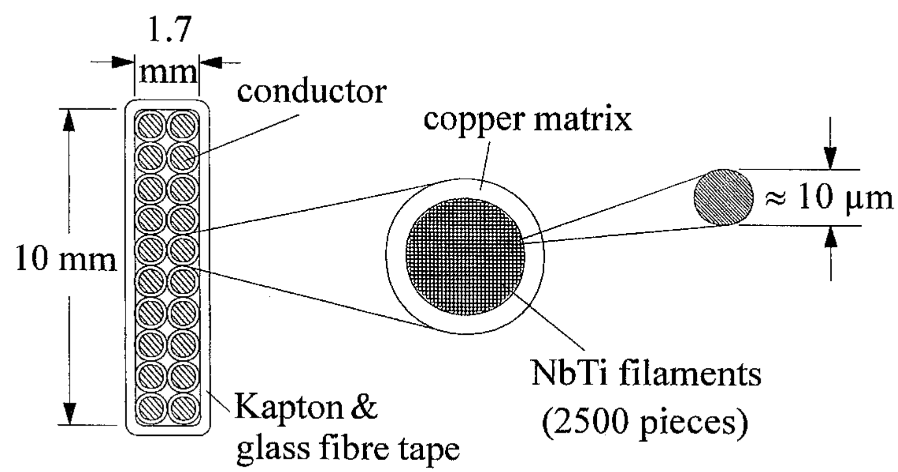
Problems:

- Superconductivity brakes down for too large fields
- Due to the Meissner-Ochsenfeld effect superconductivity current only flows on a thin surface layer.



Remedy:

- Superconducting cable consists of many very thin filaments (about 10μm).



Complex scalar magnetic potential of a wire

Straight wire at the origin: $\vec{\nabla}\times\vec{B}=\mu_0\,\vec{j}\quad\Rightarrow\quad\vec{B}(r)=\frac{\mu_0\,I}{2\pi\,r}\vec{e}_\varphi=\frac{\mu_0\,I}{2\pi\,r}\begin{pmatrix}-y\\ x\end{pmatrix}$

Wire at $\vec{a}$:

$$\vec{B}(x,y)=\frac{\mu_0\,I}{2\pi\,(\vec{r}-\vec{a})^2}\begin{pmatrix}-[y-a_y]\\ x-a_x\end{pmatrix}$$

This can be represented by complex multipole coefficients Ψ_ν

$$\vec{B}(x,y)=-\vec{\nabla}\Psi\quad\Rightarrow\quad B_x+iB_y=-\left(\partial_x+i\partial_y\right)\psi=-2\partial_{\overline{w}}\psi$$

$$B_x+iB_y=\frac{\mu_0 I}{2\pi}\frac{-i(w_a-w)}{(w_a-w)(\overline{w}_a-\overline{w})}=i\,\frac{\mu_0 I}{2\pi}\frac{-\frac{w_a}{a^2}}{1-\frac{\overline{w}w_a}{a^2}}$$

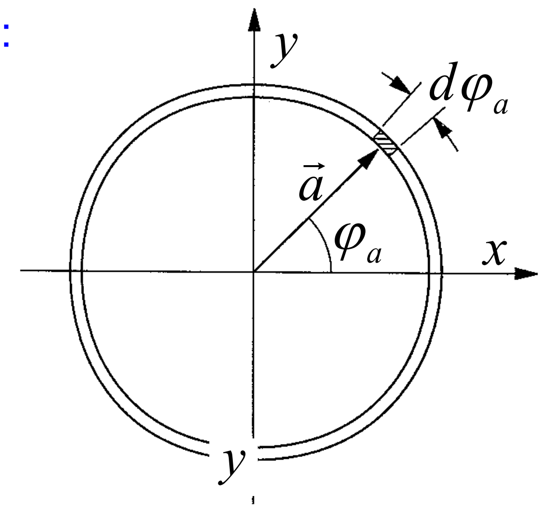
$$=i\,\frac{\mu_0 I}{2\pi}\partial_{\overline{w}}\ln(1-\tfrac{\overline{w}w_a}{a^2})=-2\partial_{\overline{w}}\,\mathrm{Im}\left\{\tfrac{\mu_0 I}{2\pi}\ln(1-\tfrac{\overline{w}w_a}{a^2})\right\}$$

$$\psi=\mathrm{Im}\left\{\tfrac{\mu_0 I}{2\pi}\ln(1-\tfrac{\overline{w}w_a}{a^2})\right\}=-\mathrm{Im}\left\{\tfrac{\mu_0 I}{2\pi}\sum_{\nu=1}^{\infty}\tfrac{1}{\nu}\left(\tfrac{w_a}{a^2}\right)^\nu\overline{w}^\nu\right\}\quad\Rightarrow\quad\Psi_\nu=\tfrac{\mu_0 I}{2\pi}\tfrac{1}{\nu}\tfrac{1}{a^\nu}e^{i\nu\varphi_a}$$

Air-coil multipoles

Creating a multipole be created by an arrangement of wires:

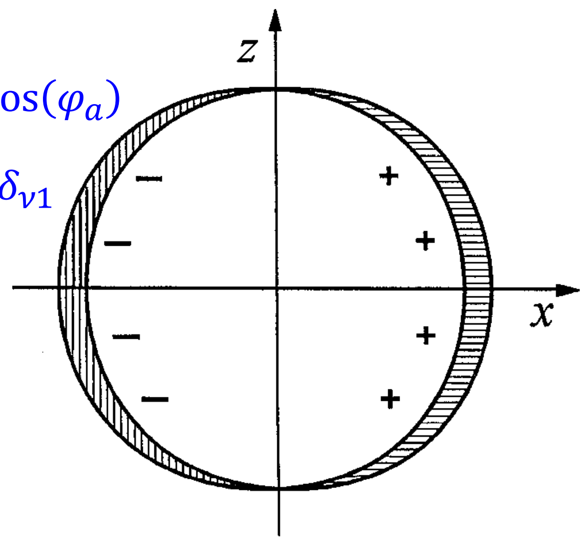
$$\Psi_\nu = \int_0^{2\pi} \frac{\mu_0}{2\pi} \frac{1}{\nu} \frac{1}{a^\nu} e^{i\nu\varphi_a} \frac{dI}{d\varphi_a} d\varphi_a$$



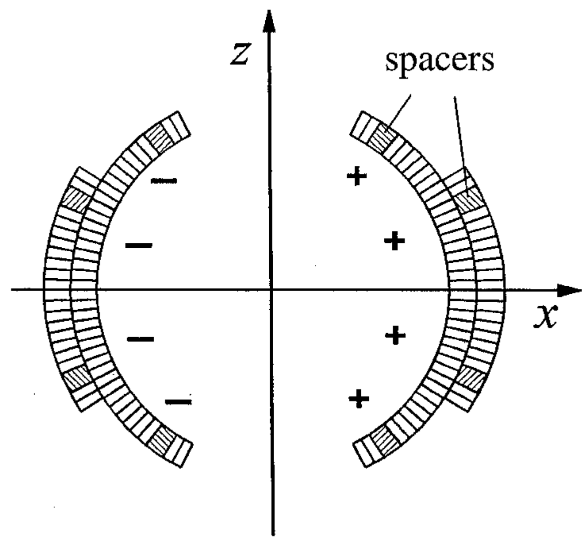
The Multipole coefficient Ψ_ν is the **Fourier coefficient** of the angular charge distribution $\frac{\mu_0}{\nu a^\nu}\frac{dI}{d\varphi_a}$.

$$\frac{dI}{d\varphi_a} \propto \cos(\varphi_a)$$

➔ $\Psi_\nu \propto \delta_{\nu 1}$

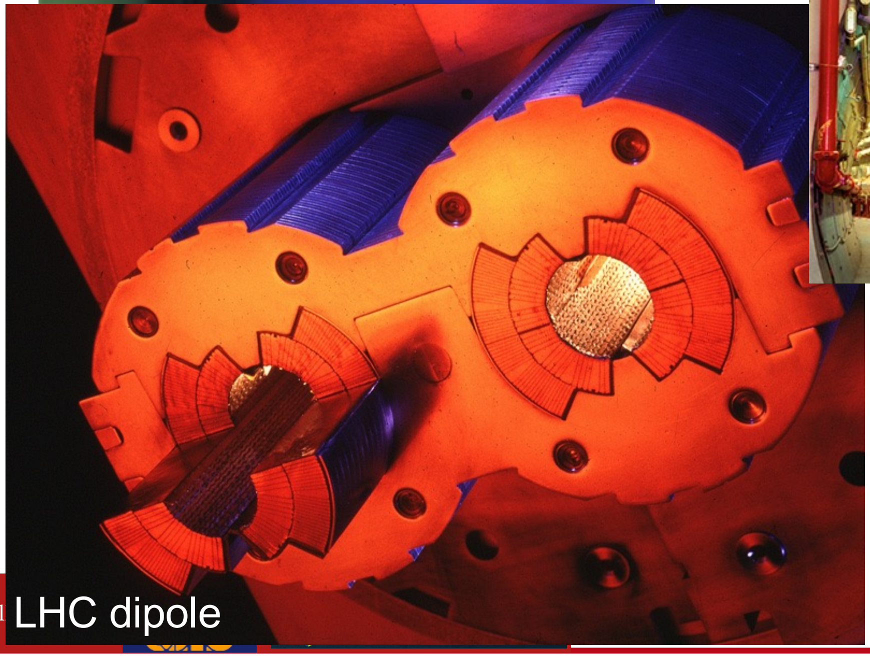


Ideal multipole



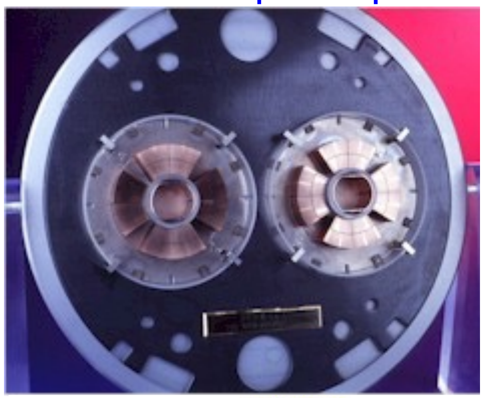
Approximate multipole

Real Air-coil multipoles

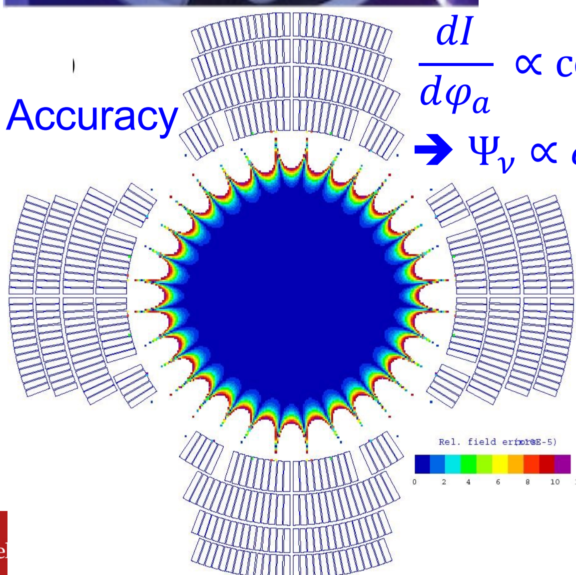


Special super-conducting Air-coil magnets

LHC double quadrupole

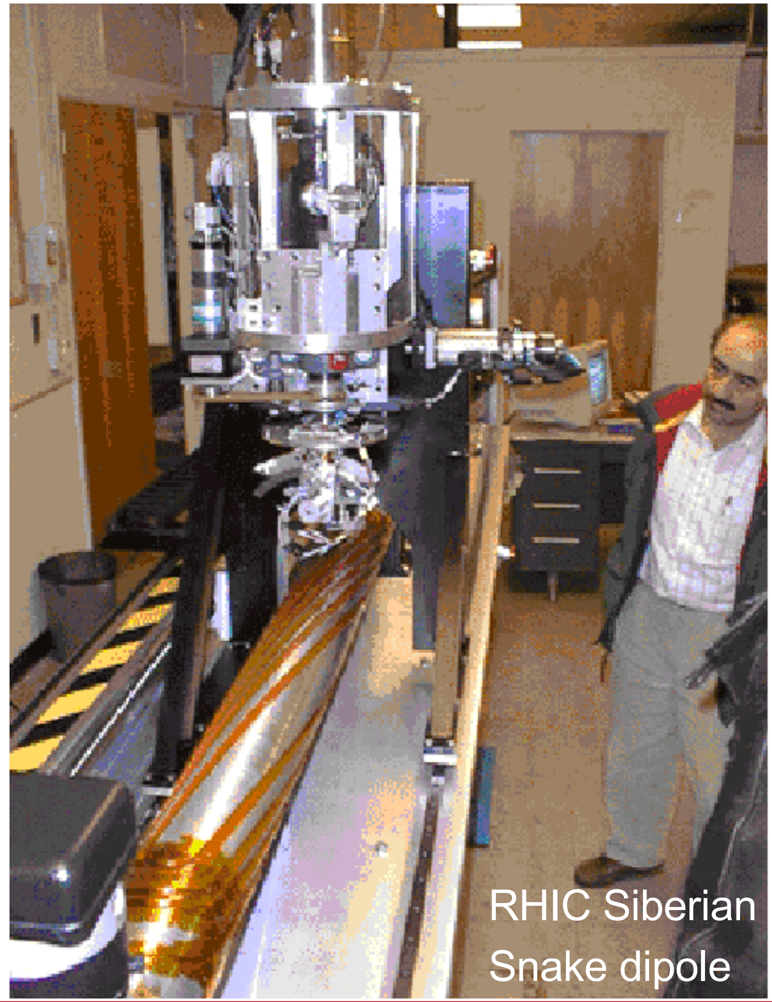


Accuracy



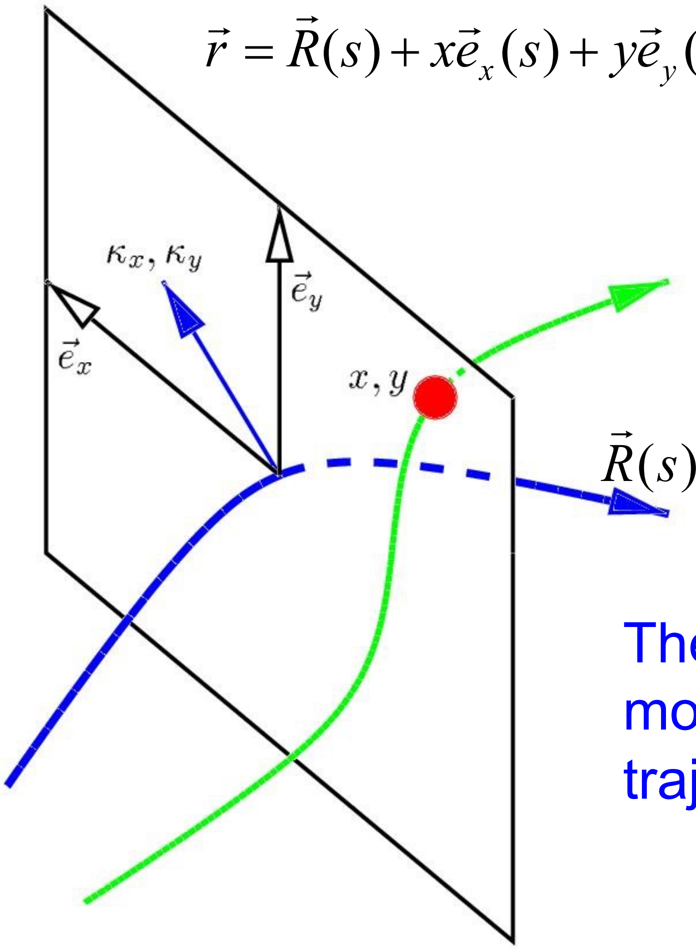
$$\frac{dI}{d\varphi_a} \propto \cos(2\varphi_a)$$

$$\rightarrow \Psi_\nu \propto \delta_{\nu 2}$$



The comoving coordinate system

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$



$$|d\vec{R}| = ds$$

$$\vec{e}_s \equiv \frac{d}{ds}\vec{R}(s)$$

The time dependence of a particle's motion is often not as interesting as the trajectory along the accelerator length "s".

The 4-dimensional equation of motion

$$\frac{d^2}{dt^2}\vec{r} = \vec{f}_r(\vec{r}, \tfrac{d}{dt}\vec{r}, t)$$

3 dimensional ODE of 2nd order can be changed to a

6 dimensional ODE of 1st order:

$$\left.\begin{aligned} &\frac{d}{dt}\vec{r} = \frac{1}{m\gamma}\vec{p} = \frac{c}{\sqrt{p^2-(mc)^2}}\vec{p} \\ &\frac{d}{dt}\vec{p} = \vec{F}(\vec{r},\vec{p},t) \end{aligned}\right\} \quad \frac{d}{dt}\vec{Z} = \vec{f}_Z(\vec{Z},t)\,, \quad \vec{Z} = (\vec{r},\vec{p})$$

If the force does not depend on time, as in a typical beam line magnet, the energy is conserved so that one can reduce the dimension to 5. The equation of motion is then autonomous.

Furthermore, the time dependence is often not as interesting as the trajectory along the accelerator length "s". Using "s" as the independent variable reduces the dimensions to 4. The equation of motion is then no longer autonomous.

$$\frac{d}{ds}\vec{z} = \vec{f}_z(\vec{z},s)\,, \quad \vec{z} = (x, y, p_x, p_y)$$

6D equation of motion

Usually one prefers to compute the trajectory as a function of “s” along the accelerator even when the energy is not conserved, as when accelerating cavities are in the accelerator.

Then the energy “E” and the time “t” at which a particle arrives at the cavities are important. And the equations become 6 dimensional again:

$$\tfrac{d}{ds}\vec{z} = \vec{f}_z(\vec{z},s)\,, \quad \vec{z} = (x, y, p_x, p_y, -t, E)$$

But: $\vec{z} = (\vec{r}, \vec{p})$ is an especially suitable variable, since it is a phase space vector so that its equation of motion comes from a Hamiltonian, or by variation principle from a Lagrangian.

$$\delta \int \left[p_x \dot{x} + p_y \dot{y} + p_s \dot{s} - H(\vec{r}, \vec{p}, t) \right] dt = 0 \quad \Rightarrow \quad \text{Hamiltonian motion}$$

$$\delta \int \left[p_x x' + p_y y' - H t' + p_s(x, y, p_x, p_y, t, H) \right] ds = 0 \quad \Rightarrow \quad \text{Hamiltonian motion}$$

The new canonical coordinates are: $\vec{z} = (x, y, p_x, p_y, -t, E) \quad \text{with} \quad \mathrm{E} = \mathrm{H}$

The new Hamiltonian is: $K = -p_s(\vec{z}, s)$

6D phase space motion

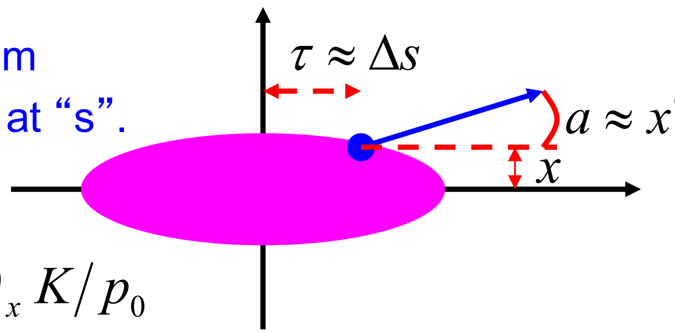
Using a reference momentum p_0 and a reference time t_0:

$$\vec{z} = (x, a, y, b, \tau, \delta)$$

$$a = \frac{p_x}{p_0}, \quad b = \frac{p_y}{p_0}, \quad \delta = \frac{E - E_0}{E_0}, \quad \tau = (t_0 - t)\frac{c^2}{v_0} = (t_0 - t)\frac{E_0}{p_0}$$

Usually p_0 is the design momentum of the beam

And t_0 is the time at which the bunch center is at "s".



$$\left.\begin{aligned} x' &= \partial_{p_x} K \\ p_x' &= -\partial_x K \end{aligned}\right\} \Rightarrow \begin{cases} x' = \partial_a K/p_0 \,, & a' = -\partial_x K/p_0 \\ y' = \partial_b K/p_0 \,, & b' = -\partial_y K/p_0 \end{cases}$$

$$-t' = \partial_E K \quad \Rightarrow \quad \tau' = \tfrac{c^2}{v_0}\partial_\delta K/E_0 = \partial_\delta K/p_0$$

$$E' = -\partial_{-t} K \quad \Rightarrow \quad \delta' = -\tfrac{1}{E_0}\partial_\tau K \tfrac{c^2}{v_0} = -\partial_\tau K/p_0$$

New Hamiltonian:

$$\widetilde{H} = K/p_0$$

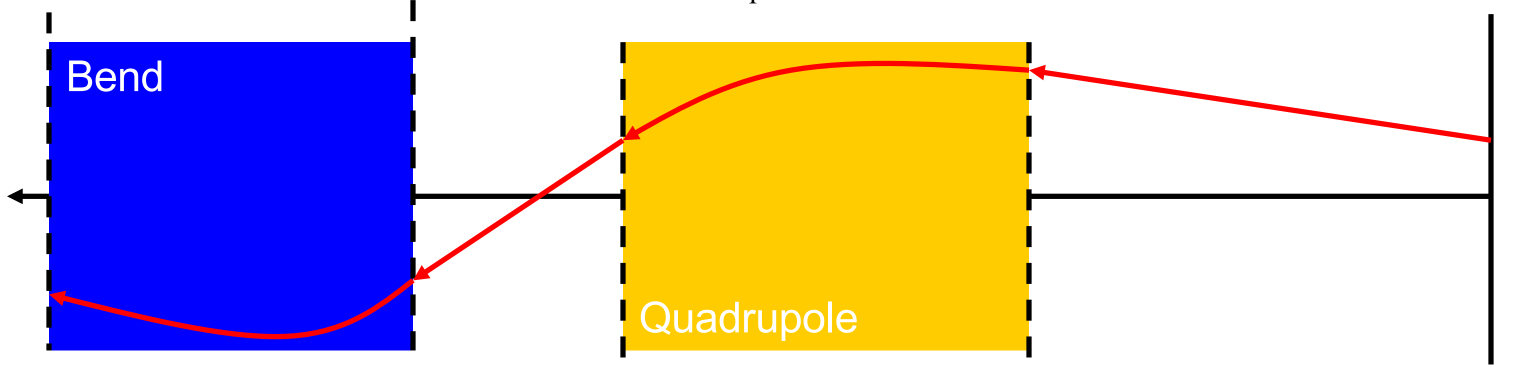
The matrix solution of linear equations of motions

Linear equation of motion: $\vec{z}' = \underline{F}(s)\vec{z}$ $\rightarrow$ $\vec{z}(s) = \underline{M}(s)\,\vec{z}_0$

Matrix solution of the starting condition $\vec{z}(0) = \vec{z}_0$

$$\vec{z} = \underline{M}_{\text{bend}}(L_4)\underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$



$$\vec{z} = \underline{M}_{\text{drift}}(L_1)\vec{z}_0 \qquad \vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

Simplest example: motion through an empty drift

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\ddot{x} = 0 \quad \Rightarrow \quad x'' = 0 \quad \Rightarrow \quad a = x',\, a' = 0$$

Linear solution:

$$x(s) = x_0 + x_0' s$$

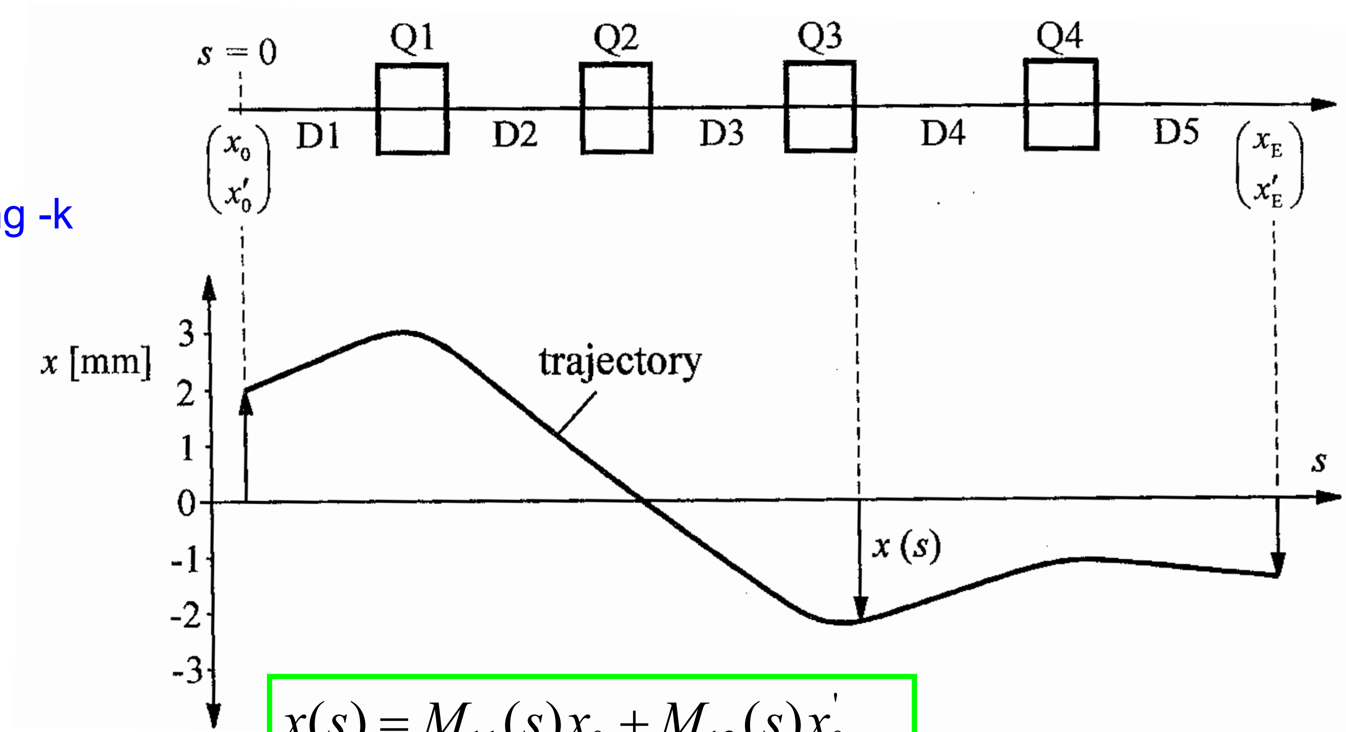
$$\begin{pmatrix} x \\ a \\ y \\ b \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} x_0 + s a_0 \\ a \\ y_0 + s b_0 \\ b_0 \\ \tau_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \underline{0} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \vec{z}_0$$

Betatron formalism for linear motion

$$x'' = -x\,K$$
$$y'' = \;\; y\,k$$

In y: quadrupole defocusing -k

In x: $K = k + \frac{1}{\rho^2}$



$$x(s) = M_{11}(s)x_0 + M_{12}(s)x_0'$$
$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s)+\phi_0)$$