Simplest example: motion through an empty drift

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\ddot{x} = 0 \quad \Rightarrow \quad x'' = 0 \quad \Rightarrow \quad a = x',\, a' = 0$$

Linear solution:

$$x(s) = x_0 + x_0' s$$

$$\begin{pmatrix} x \\ a \\ y \\ b \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} x_0 + s a_0 \\ a \\ y_0 + s b_0 \\ b_0 \\ \tau_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \underline{0} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \vec{z}_0$$

Significance of the Hamiltonian

The equations of motion can be determined by one function:

$$\tfrac{d}{ds}x = \partial_{p_x} H(\vec{z},s)\,, \quad \tfrac{d}{ds}p_x = -\partial_x H(\vec{z},s)\,, \quad \dots$$

$$\tfrac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial} H(\vec{z},s) = \vec{F}(\vec{z},s) \quad \text{with} \quad \underline{J} = \text{diag}(\underline{J}_2)\,, \quad \underline{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The force has a Hamiltonian Jacobi Matrix:

A linear force: $\vec{F}(\vec{z},s) = \underline{F}(s)\cdot\vec{z}$

The Jacobi Matrix of a linear force: $\underline{F}(s)$

The general Jacobi Matrix : $F_{ij} = \partial_{z_j} F_i$ or $\underline{F} = \left(\vec{\partial}\,\vec{F}^T\right)^T$

Hamiltonian Matrices: $\underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = 0$

Prove : $F_{ij} = \partial_{z_j} F_i = \partial_{z_j} J_{ik} \partial_{z_k} H = J_{ik}\,\partial_k \partial_j H \quad \Rightarrow \quad \underline{F} = \underline{J}\,\underline{D}\underline{H}$

$$\underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = \underline{J}\,\underline{D}\underline{J}H + \underline{J}\,\underline{D}^T\,\underline{J}^T H = 0$$

Hamiltonian → Symplectic Flow

The flow of a Hamiltonian equation of motion has a symplectic Jacobi Matrix

The flow or transport map: $\vec{z}(s)=\vec{M}(s,\vec{z}_0)$

A linear flow: $\vec{z}(s)=\underline{M}(s)\cdot\vec{z}_0$

The Jacobi Matrix of a linear flow: $\underline{M}(s)$

The general Jacobi Matrix : $M_{ij}=\partial_{z_{0j}}M_i$ or $\underline{M}=\left(\vec{\partial}_0\vec{M}^T\right)^T$

The Symplectic Group SP(2N) : $\underline{M}\,\underline{J}\,\underline{M}^T=\underline{J}$

$$\tfrac{d}{ds}\vec{z}=\tfrac{d}{ds}\vec{M}(s,\vec{z}_0)=\underline{J}\vec{\nabla}H=\vec{F}$$

$$\tfrac{d}{ds}M_{ij}=\partial_{z_{0j}}F_i(\vec{z},s)=\partial_{z_{0j}}M_k\partial_{z_k}F_i(\vec{z},s)$$

$$\tfrac{d}{ds}\underline{M}(s,\vec{z}_0)=\underline{F}(\vec{z},s)\underline{M}(s,\vec{z}_0)$$

$$K=\underline{M}\,\underline{J}\,\underline{M}^T$$

$$\tfrac{d}{ds}\underline{K}=\tfrac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T+\underline{M}\,\underline{J}\tfrac{d}{ds}\underline{M}^T=\underline{F}\,\underline{M}\,\underline{J}\,\underline{M}^T+\underline{M}\,\underline{J}\,\underline{M}^T\underline{F}^T=\underline{F}\,\underline{K}+\underline{K}\,\underline{F}^T$$

K = J is a solution. Since this is a linear ODE , K = J is the unique solution.

Symplectic Flow → Hamiltonian

For every symplectic transport map there is a Hamilton function

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

Force vector: $\vec{h}(\vec{z}, s) = -\underline{J}\left[\frac{d}{ds}\vec{M}(s, \vec{z}_0)\right]_{\vec{z}_0 = \vec{M}^{-1}(\vec{z}, s)}$

Since then: $\frac{d}{ds}\vec{z} = \underline{J}\vec{h}(\vec{z}, s)$

There is a Hamilton function H with: $\vec{h} = \vec{\partial} H$

If and only if: $\partial_{z_j} h_i = \partial_{z_i} h_j \quad \Rightarrow \quad \underline{h} = \underline{h}^T$

$$\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J} \quad \Rightarrow \quad \begin{cases} \frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T = -\underline{M}\,\underline{J}\,\frac{d}{ds}\underline{M}^T \\ \underline{M}^{-1} = -\underline{J}\,\underline{M}^T\underline{J} \end{cases}$$

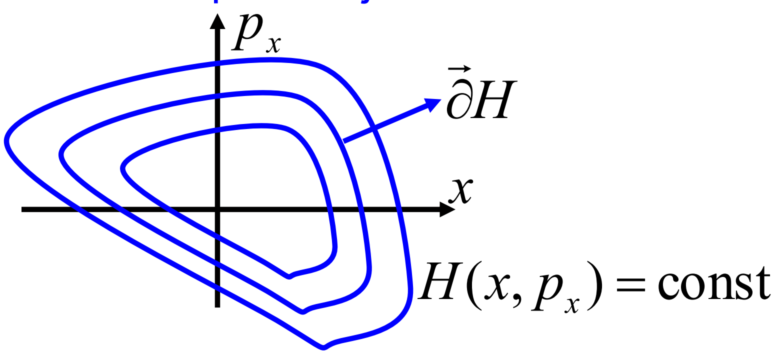
$$\vec{h} \circ \vec{M} = -\underline{J}\frac{d}{ds}\vec{M}$$

$$\underline{h}(\vec{M})\underline{M} = -\underline{J}\frac{d}{ds}\underline{M}$$

$$\underline{h}(\vec{M}) = -\underline{J}\frac{d}{ds}\underline{M}\,\underline{M}^{-1} = \underline{J}\frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T\underline{J} = -\underline{J}\,\underline{M}\,\underline{J}\frac{d}{ds}\underline{M}^T\underline{J} = \underline{M}^{-T}\frac{d}{ds}\underline{M}^T\underline{J} = \underline{h}^T$$

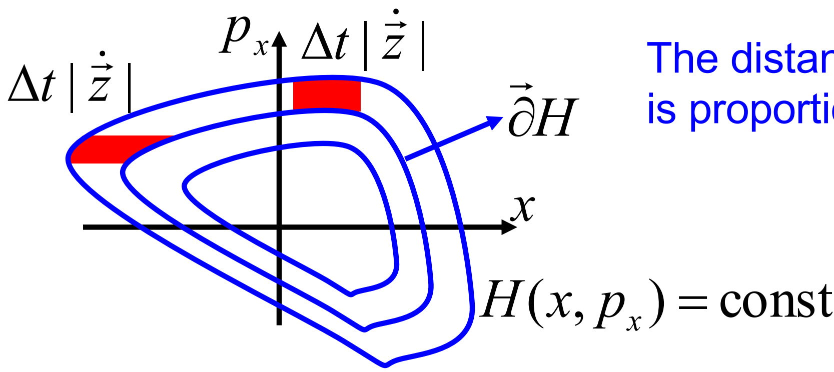
Phase space densities in 2D

- Phase space trajectories move on surfaces of constant energy



$$\frac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial} H \quad \Rightarrow \quad \frac{d}{ds}\vec{z} \perp \vec{\partial} H$$

- A phase space volume does not change when it is transported by Hamiltonian motion.

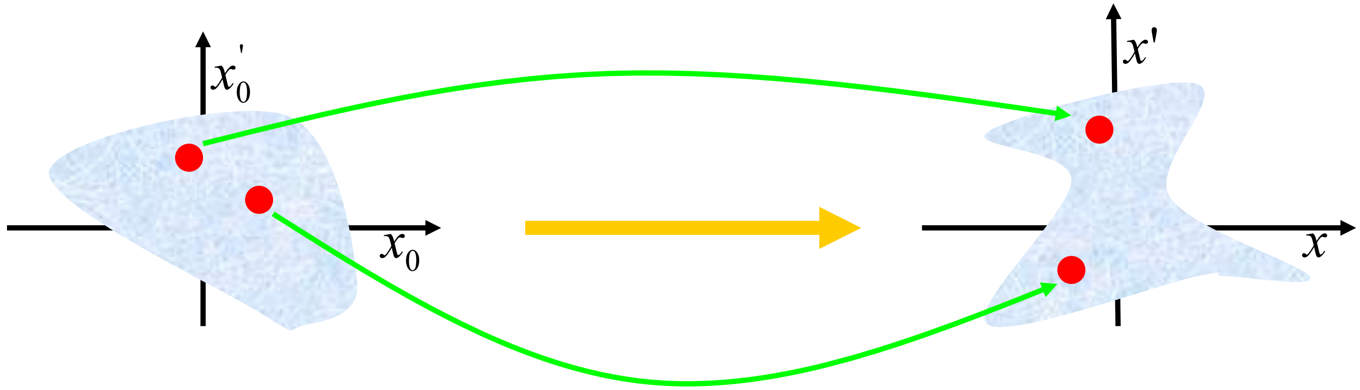


The distance d of lines with equal energy is proportional to $1 / |\vec{\partial} H| \propto |\dot{\vec{z}}|^{-1}$

$$d * \Delta t \,|\, \dot{\vec{z}} \,| = \text{const}$$

Liouville's Theorem

- A phase space volume does not change when it is transported by Hamiltonian motion. $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0 \quad \text{with} \quad \det[\underline{M}(s)] = +1$

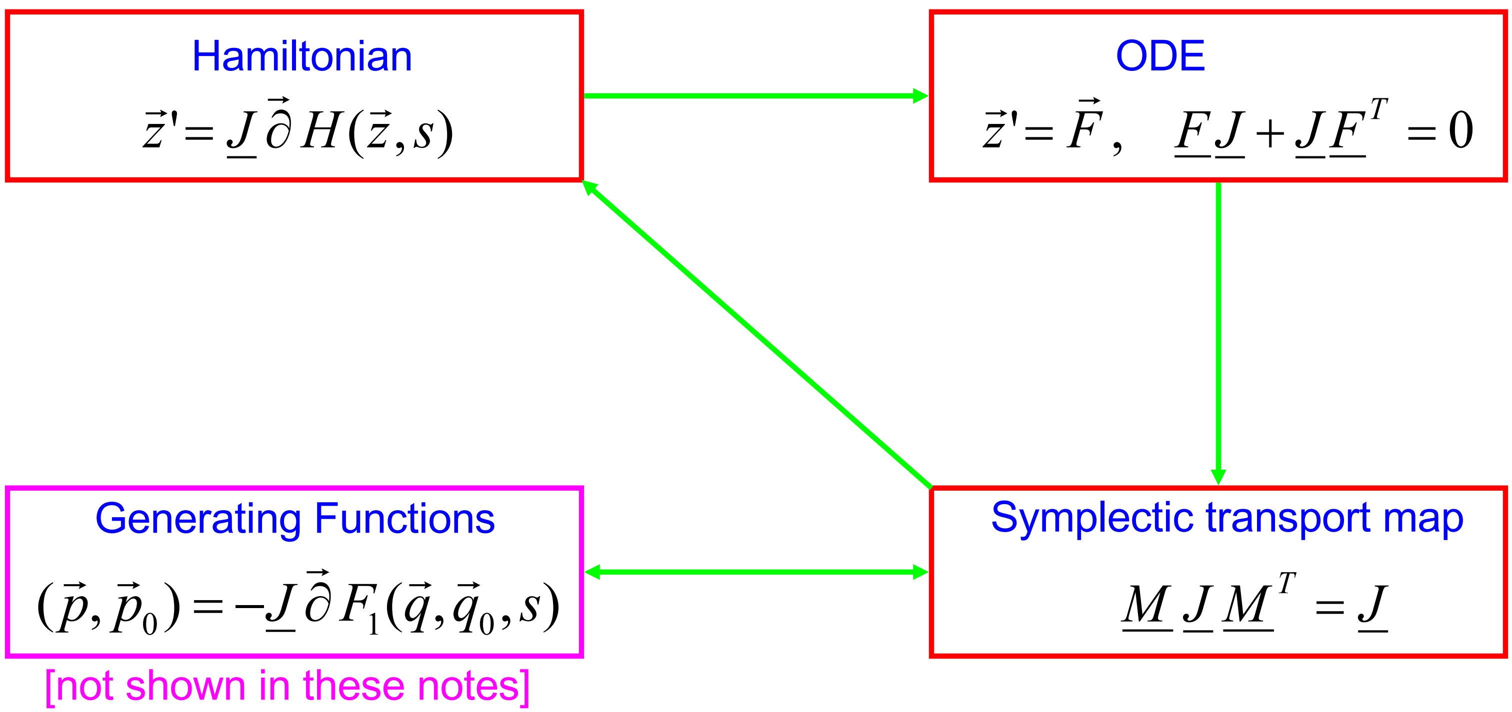


Volume = $V = \iint_V d^n\vec{z} = \iint_{V_0} \left| \frac{\partial \vec{z}}{\partial \vec{z}_0} \right| d^n\vec{z}_0 = \iint_{V_0} |\underline{M}| d^n\vec{z}_0 = \iint_{V_0} d^n\vec{z}_0 = V_0$

Hamiltonian Motion → $V = V_0$

But Hamiltonian requires symplecticity, which is much more than just $\det[\underline{M}(s)] = +1$

Symplectic representations



Eigenvalues of symplectic matrices

For matrices with real coefficients:

If there is an eigenvector and eigenvalue: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$

then the complex conjugates are also eigenvector and eigenvalue: $\underline{M}\vec{v}_i^{\,*} = \lambda_i^* \vec{v}_i^{\,*}$

For symplectic matrices:

If there are eigenvectors and eigenvalues: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$ with $\underline{J} = \underline{M}^T \underline{J}\,\underline{M}$

then $\vec{v}_i^{\,T} \underline{J}\vec{v}_j = \vec{v}_i^{\,T} \underline{M}^T \underline{J}\,\underline{M}\vec{v}_j = \lambda_i \lambda_j \vec{v}_i^{\,T} \underline{J}\vec{v}_j \quad \Rightarrow \quad \vec{v}_i^{\,T} \underline{J}\vec{v}_j (\lambda_i \lambda_j - 1) = 0$

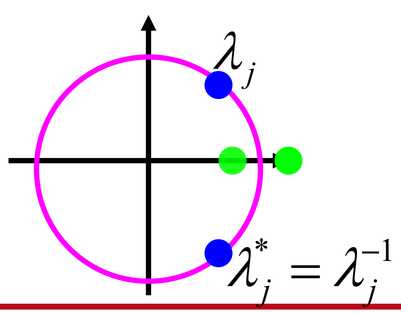
Therefore $\underline{J}\vec{v}_j$ is orthogonal to all eigenvectors with eigenvalues that are not $1/\lambda_j$. Since it cannot be orthogonal to all eigenvectors, there is at least one eigenvector with eigenvalue $1/\lambda_j$

Two dimensions: λ_j is eigenvalue

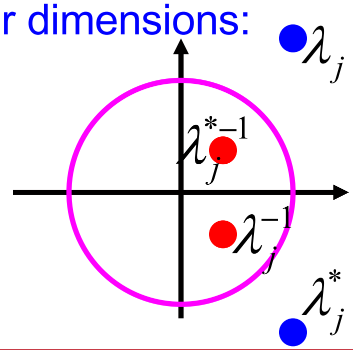
Then $1/\lambda_j$ and λ_j^* are eigenvalues

$\underline{\lambda_2 = 1/\lambda_1 = \lambda_1^*} \quad \Rightarrow \quad |\lambda_j| = 1$

$\lambda_2 = 1/\lambda_1 = \lambda_2^*$



Four dimensions:

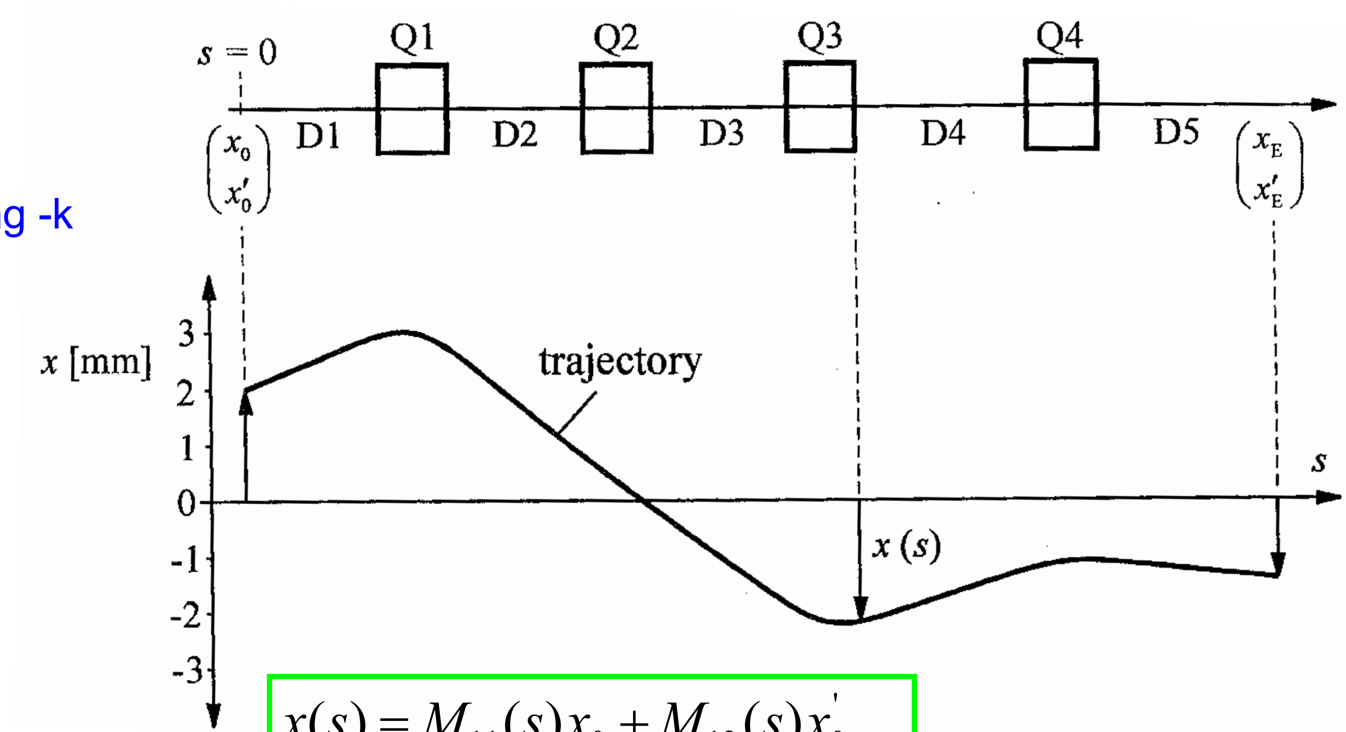


Betatron formalism for linear motion

$$x'' = -x\,K$$
$$y'' = \;\; y\,k$$

In y: quadrupole defocusing -k

In x: $K = k + \frac{1}{\rho^2}$



$$x(s) = M_{11}(s)x_0 + M_{12}(s)x_0'$$
$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s)+\phi_0)$$

Twiss parameters

$$x'' = -k\,x$$

$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s)+\phi_0)$$

$$x'(s) = \sqrt{\tfrac{2J}{\beta}}[\beta\psi'\cos(\psi(s)+\phi_0) - \alpha\sin(\psi(s)+\phi_0)] \quad \text{with} \quad \alpha = -\tfrac{1}{2}\beta'$$

$$x''(s) = \sqrt{\tfrac{2J}{\beta}}[(\beta\psi'' - 2\alpha\psi')\cos(\psi(s)+\phi_0) - (\alpha' + \tfrac{\alpha^2}{\beta} + \beta\psi'^2)\sin(\psi(s)+\phi_0)]$$

$$= \sqrt{\tfrac{2J}{\beta}}[-k\beta\sin(\psi(s)+\phi_0)]$$

$$\beta\psi'' - 2\alpha\psi' = \beta\psi'' + \beta'\psi' = (\beta\psi')' = 0 \quad \Rightarrow \quad \psi' = \tfrac{I}{\beta}$$

$$\alpha' + \gamma = k\beta \quad \text{with} \quad \gamma = \tfrac{I^2+\alpha^2}{\beta}$$

Universal choice: I=1!

$$\beta' = -2\alpha$$

$$\alpha' = k\beta - \gamma$$

$$\psi = \int_0^s \frac{I}{\beta(s')}\,ds'$$

$\alpha, \beta, \gamma, \psi$ are called Twiss parameters.

What are the initial conditions?

The phase ellipse

Particles with a common J and different ϕ all lie on an ellipse in phase space:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{I}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

(Linear transform of a circle)

$$x_{\max} = \sqrt{2J\beta} \text{ at } x' = -\alpha\sqrt{\frac{2J}{\beta}}$$

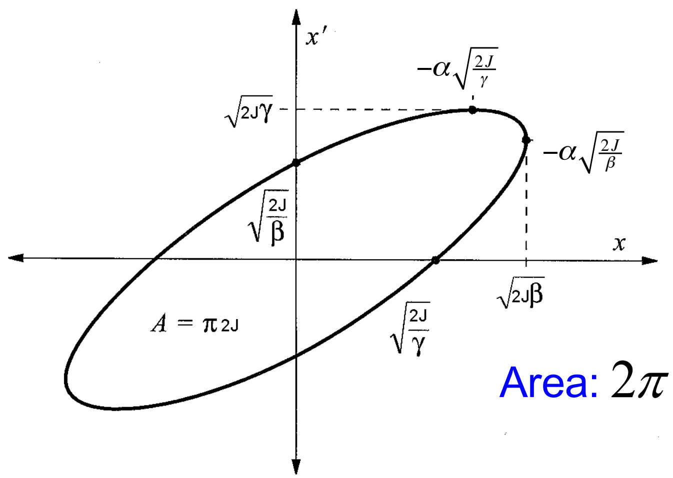
$$(x,x')\begin{pmatrix} \frac{I}{\sqrt{\beta}} & \frac{\alpha}{\sqrt{\beta}} \\ 0 & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} \frac{I}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = (x,x')\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = 2J$$

(Quadratic form)

$$\beta\gamma - \alpha^2 = I^2$$

Area: $2\pi J / I$

I=1 is therefore a useful choice!



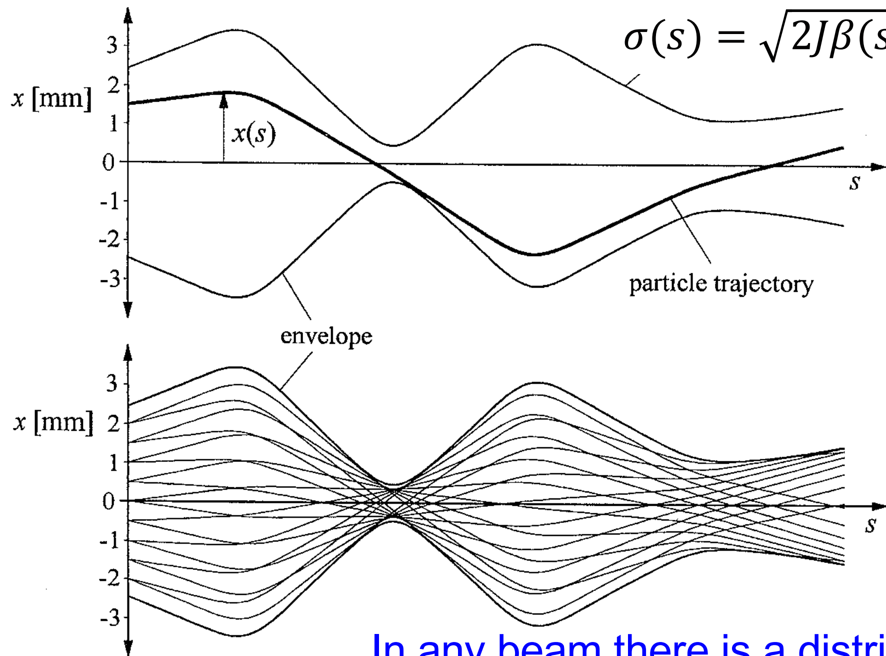
What β is for x, γ is for x'

$$x'_{\max} = \sqrt{2J\gamma} \text{ at } x = -\alpha\sqrt{\frac{2J}{\gamma}}$$

Area: $2\pi J$ → $\int_0^{2\pi}\int_0^{J} dJ\,d\phi = 2\pi J = \iint dx\,dx'$

The beam envelope

$\sigma(s) = \sqrt{2J\beta(s)}$



$$x(s) = \sqrt{2J\beta(s)}\sin(\psi(s)+\phi_0)$$

$$\sigma' = -\alpha\sqrt{\frac{2J}{\beta}}$$

$$\sigma'' = -(k\beta-\gamma)\sqrt{\frac{2J}{\beta}} - \alpha^2\sqrt{\frac{2J}{\beta^3}}$$

$$\sigma'' = -(k\beta^2-1)\sqrt{\frac{2J}{\beta^3}}$$

In any beam there is a distribution of initial parameters. If the particles with the largest J are distributed in ϕ over all angles, then the envelope of the beam is described by $\sigma = \sqrt{2J_{max}\beta(s)}$.

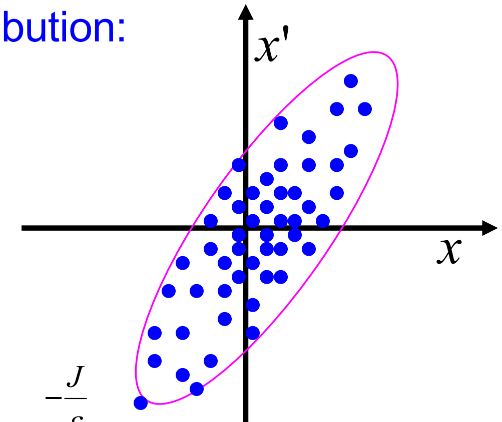
The initial conditions of β and α are chosen so that this is approximately the case.

The envelope equation: $\sigma'' = -k\sigma + \frac{(2J)^2}{\sigma^3}$

The phase space distribution

Often one can fit a Gauss distribution to the particle distribution:

$$\rho(x,x') = \tfrac{1}{2\pi\varepsilon} e^{-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{2\varepsilon}}$$



The equi-density lines are then ellipses. And one chooses the starting conditions for β and α according to these ellipses!

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix} \qquad \rho(J,\phi_0) = \tfrac{1}{2\pi\varepsilon} e^{-\frac{J}{\varepsilon}}$$

$$\langle 1 \rangle = \tfrac{1}{2\pi\varepsilon} \int_0^{2\pi}\int_0^{\infty} e^{-J/\varepsilon} dJ d\phi_0 = 1$$

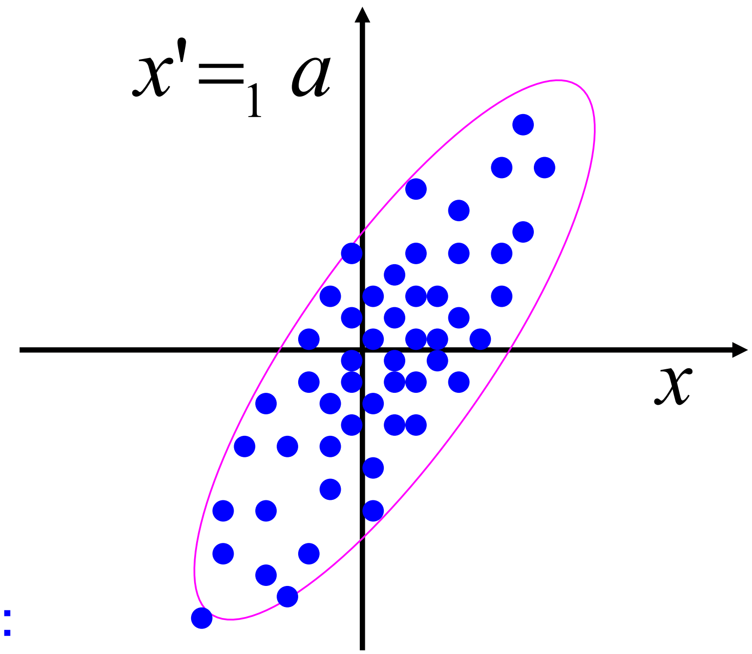
Initial beam distribution ⟶ initial α, β, γ

$$\left\langle x^2 \right\rangle = \tfrac{1}{2\pi\varepsilon} \iint 2J\beta \sin\phi_0^2 e^{-J/\varepsilon} dJ d\phi_0 = \varepsilon\beta \longrightarrow \left\langle x'^2 \right\rangle = \varepsilon\gamma$$

$$\left\langle x x' \right\rangle = -\tfrac{1}{2\pi\varepsilon} \iint 2J\alpha \sin\varphi_0^2 e^{-J/\varepsilon} dJ d\varphi_0 = -\varepsilon\alpha$$

$$\varepsilon = \sqrt{\left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle - \left\langle x x' \right\rangle^2}$$ is called the emittance.

The normalized emittance



with $a = p_x/p_0$
and the beam's reference momentum p_0.

Remarks:

(1) The phase space area that a beam fills in (x , a) phase space shrinks during acceleration by the factor p_0/p. This area is the emittance ε.

(2) The phase space area that a beam fills in (x , p_x) phase space is conserved. This area (divided by mc) is the normalized emittance ε_n.

$$\varepsilon = \varepsilon_n / \beta_r \gamma_r$$

Invariant of motion

$$x(s)=\sqrt{2J\beta(s)}\sin(\psi(s)+\phi_0)$$

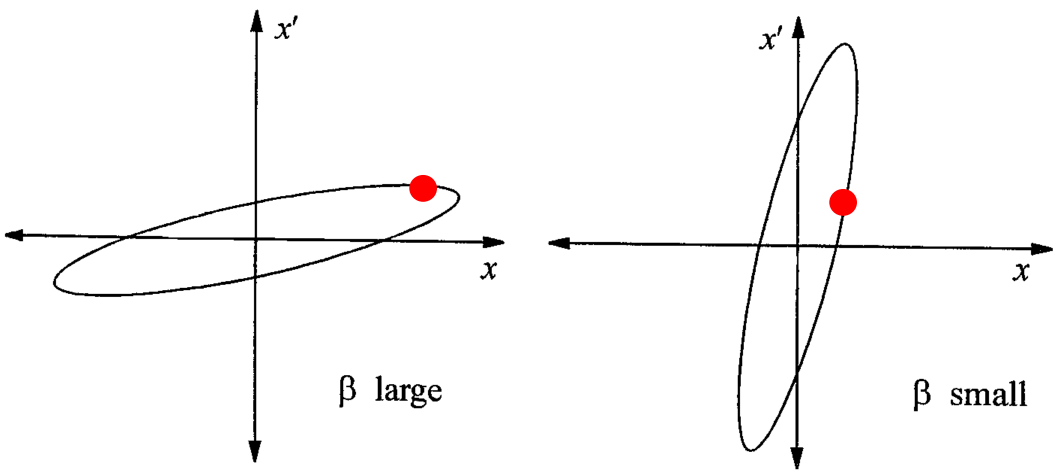
Where J and ϕ are given by the starting conditions x_0 and x'_0.

$$\gamma x^2+2\alpha xx'+\beta x'^2=2J$$

Leads to the invariant of motion:

$$f(x,x',s)=\gamma(s)x^2+2\alpha(s)xx'+\beta(s)x'^2 \quad\Rightarrow\quad \tfrac{d}{ds}f=0$$

It is called the Courant-Snyder invariant.



Twiss differential equation ➔ usually too hard

$$\gamma = \frac{1+\alpha^2}{\beta}$$

$$\beta' = -2\alpha$$

$$\alpha' = k\beta - \gamma$$

$$\beta'' = 2\gamma = 2\frac{1+\frac{1}{4}\beta'^2}{\beta} = \frac{d\beta'}{d\beta}\frac{d\beta}{ds}$$

$$\frac{\beta'}{1+\frac{1}{4}\beta'^2}d\beta' = 2\frac{d\beta}{\beta}$$

$$\log(1+\tfrac{1}{4}\beta'^2) = \log(\beta/\beta_0)$$

$$\beta' = 2\sqrt{\beta/\beta_0 - 1}$$

$$\frac{d\beta}{2\sqrt{\beta/\beta_0 - 1}} = ds$$

$$\beta_0\sqrt{\beta/\beta_0 - 1} = s - s_0$$

$$\beta(s) = \beta_0\left(1+\left(\frac{s-s_0}{\beta_0}\right)^2\right)$$

Propagation of Twiss parameters

$$(x_0, x_0')\begin{pmatrix}\gamma_0 & \alpha_0\\ \alpha_0 & \beta_0\end{pmatrix}\begin{pmatrix}x_0\\ x_0'\end{pmatrix} = 2J$$

$$(x, x')\begin{pmatrix}\gamma & \alpha\\ \alpha & \beta\end{pmatrix}\begin{pmatrix}x\\ x'\end{pmatrix} = 2J = (x_0, x_0')\underline{M}^T\begin{pmatrix}\gamma & \alpha\\ \alpha & \beta\end{pmatrix}\underline{M}\begin{pmatrix}x_0\\ x_0'\end{pmatrix}$$

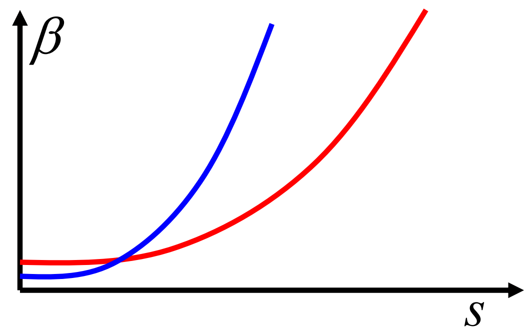
$$\begin{pmatrix}\gamma & \alpha\\ \alpha & \beta\end{pmatrix} = \underline{M}^{-T}\begin{pmatrix}\gamma_0 & \alpha_0\\ \alpha_0 & \beta_0\end{pmatrix}\underline{M}^{-1}$$

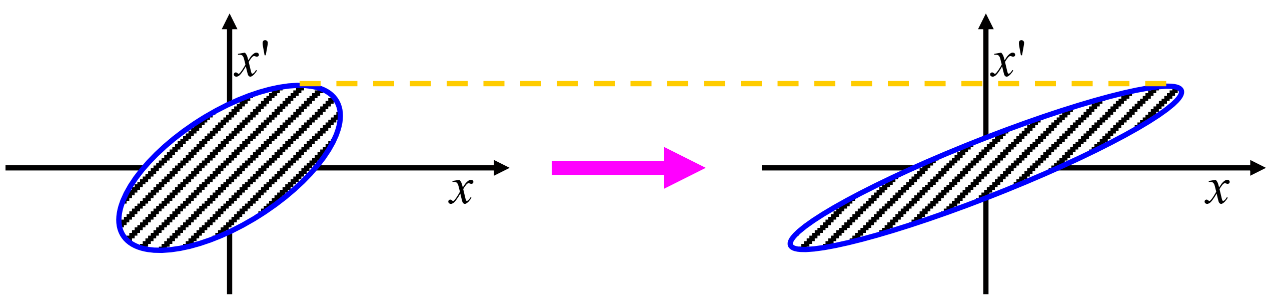
$$\begin{pmatrix}\beta & -\alpha\\ -\alpha & \gamma\end{pmatrix} = \underline{M}\begin{pmatrix}\beta_0 & -\alpha_0\\ -\alpha_0 & \gamma_0\end{pmatrix}\underline{M}^T$$

Twiss parameters in a drift

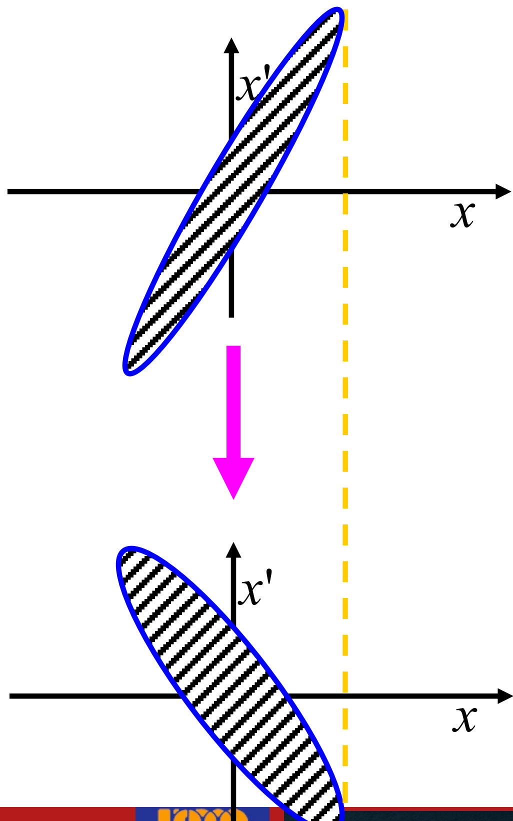
$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} \beta_0 - 2\alpha_0 s + \gamma_0 s^2 & \gamma_0 s - \alpha_0 \\ \gamma_0 s - \alpha_0 & \gamma_0 \end{pmatrix}$$

$$\beta = \beta_0^* [1 + (\tfrac{s}{\beta_0^*})^2] \quad \text{for} \quad \alpha_0^* = 0$$



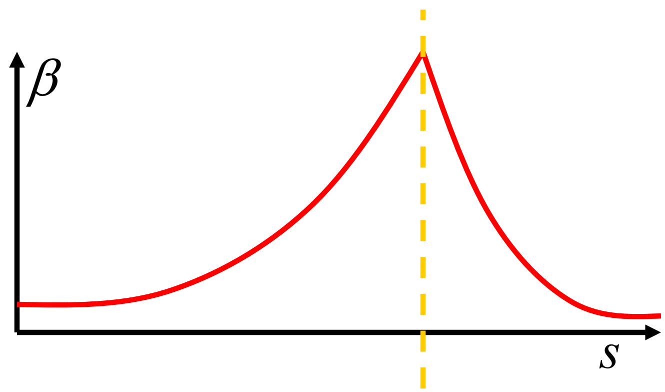


Twiss parameters in a thin quadrupole



$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\alpha = \alpha_0 + k\beta_0}$$



From Twiss parameter to Transfer Marix

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta_0} & 0 \\ -\frac{\alpha_0}{\sqrt{\beta_0}} & \frac{1}{\sqrt{\beta_0}} \end{pmatrix}\begin{pmatrix} \sin(\phi_0) \\ \cos(\phi_0) \end{pmatrix}$$

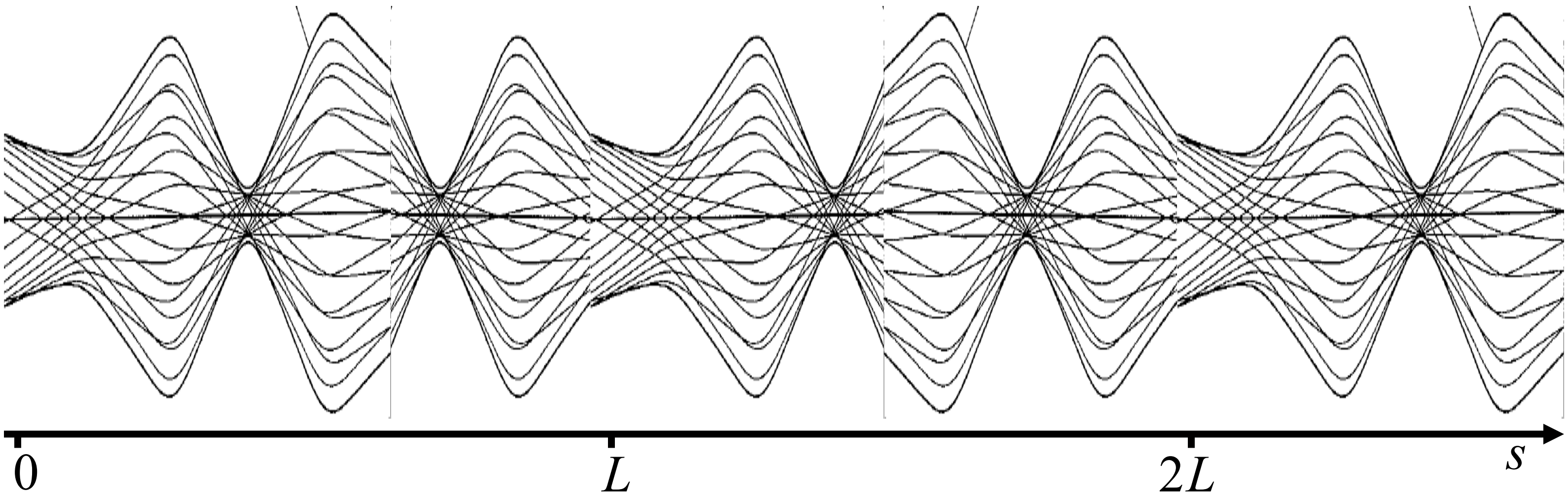
$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

$$= \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \cos\psi(s) & \sin\psi(s) \\ -\sin\psi(s) & \cos\psi(s) \end{pmatrix}\begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix}$$

$$\underline{M}(s) = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \cos\psi(s) & \sin\psi(s) \\ -\sin\psi(s) & \cos\psi(s) \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}[\cos\psi + \alpha_0 \sin\psi] & \sqrt{\beta_0\beta}\sin\psi \\ \sqrt{\frac{1}{\beta_0\beta}}[(\alpha_0 - \alpha)\cos\psi - (1+\alpha_0\alpha)\sin\psi] & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi - \alpha\sin\psi] \end{pmatrix}$$

Periodic solutions in a periodic accelerator



$$\vec{z}(s) = \underline{M}(s,0)\vec{z}(0)$$

$$\vec{z}(L) = \underline{M}(L,0)\vec{z}(0)$$

$$\vec{z}(s+L) = \underline{M}_0(s)\vec{z}(s) \quad , \quad \underline{M}_0 = \underline{M}(s+L,s)$$

$$\vec{z}(s+nL) = \underline{M}_0^n(s)\vec{z}(s)$$

Periodic beta functions

If the particle distribution in a ring or any other periodic structure is stable, it is periodic from turn to turn.

$$\rho(x,x',s+L)=\rho(x,x',s)$$

To be matched to such a beam, the Twiss parameters α, β, γ must be the same after every turn.

$$\underline{M}(s,0)=\begin{pmatrix}\sqrt{\frac{\beta}{\beta_0}}[\cos\psi+\alpha_0\sin\psi] & \sqrt{\beta_0\beta}\sin\psi\\ \sqrt{\frac{1}{\beta_0\beta}}[(\alpha_0-\alpha)\cos\psi-(1+\alpha_0\alpha)\sin\psi] & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi-\alpha\sin\psi]\end{pmatrix}$$

$$\underline{M}_p(s)=\begin{pmatrix}\cos\mu+\alpha\sin\mu & \beta\sin\mu\\ -\gamma\sin\mu & \cos\mu-\alpha\sin\mu\end{pmatrix}=\underline{1}\cos\mu+\begin{pmatrix}\alpha & \beta\\ -\gamma & -\alpha\end{pmatrix}\sin\mu$$

$$\mu=\psi(s+L)-\psi(s)$$

One turn matrix to periodic Twiss parameters

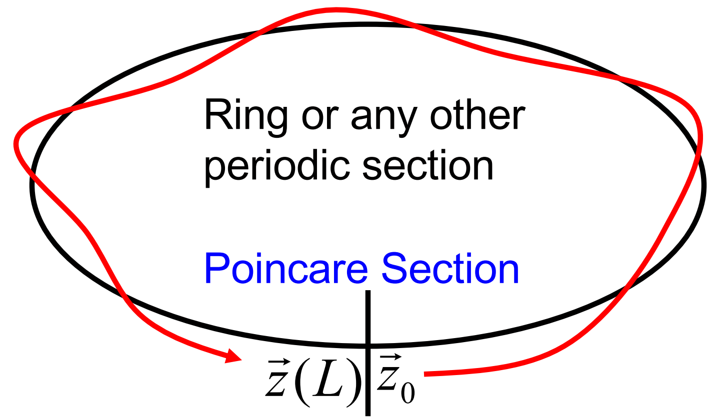
The periodic Twiss parameters are the solution of a nonlinear differential equation with periodic boundary conditions:

$$\beta' = -2\alpha \qquad \text{with} \qquad \beta(L) = \beta(0)$$

$$\alpha' = k\beta - \tfrac{1+\alpha^2}{\beta} \qquad \text{with} \qquad \alpha(L) = \alpha(0)$$

$$\mu = \int_0^L \tfrac{1}{\beta(\hat{s})} d\hat{s}$$

Note: $\beta(s) > 0$



$$\underline{M}_0(s) = \underline{1}\cos\mu + \underline{\beta}\sin\mu \,;\; \underline{\beta} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\cos\mu = \tfrac{1}{2}\mathrm{Tr}[\underline{M}_0(s)]$$

$$\beta = \underline{M}_{0,12}\,\tfrac{1}{\sin\mu}$$

$$\alpha = (\underline{M}_{0,11} - \underline{M}_{0,22})\,\tfrac{1}{2\sin\mu}$$

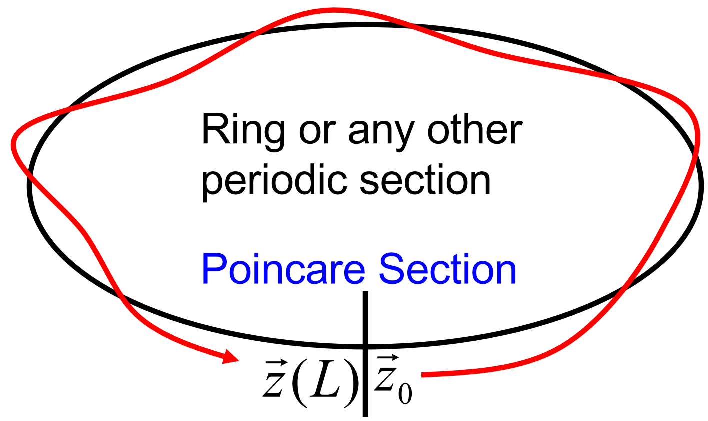
$$\gamma = \tfrac{1+\alpha^2}{\beta}$$

Stable beam motion and thus a periodic beta function can only exist when |Tr[M]| < 2.

The tune of a particle accelerator

The betatron phase advance per turn devided by 2π is called the TUNE.

$$\mu = 2\pi\nu = \psi(s+L) - \psi(s)$$

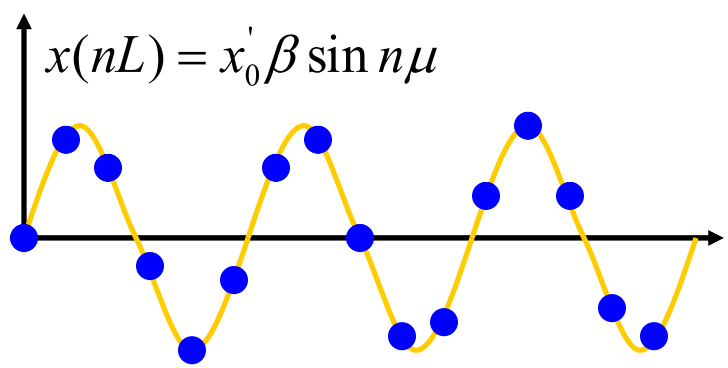


It is a property of the ring and does not depend on the azimuth s.

$$\underline{M}_0(s) = \underline{1}\cos\mu + \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} \sin\mu$$

$$2\cos\mu(s) = \mathrm{Tr}[\underline{M}_0(s)] = \mathrm{Tr}[\underline{M}(s,0)\underline{M}_0(0)\underline{M}^{-1}(s,0)]$$

$$= \mathrm{Tr}[\underline{M}_0(0)] = 2\cos\mu(0)$$



$$\underline{M}_0^n = \underline{1}\cos n\mu + \underline{\beta}\sin n\mu$$