The drift

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Note that in nonlinear expansion $x' \neq a$ so that the drift does not have a linear transport map even though $x(s) = x_0 + x_0' s$ is completely linear.

$$\begin{pmatrix} x \\ a \\ y \\ b \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} x_0 + s a_0 \\ a \\ y_0 + s b_0 \\ b_0 \\ \tau_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \underline{0} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \vec{z}_0$$

The quadrupole

$$x'' = -x\,k$$

$$y'' = \;\; y\,k$$

$$\underline{M}_4 = \begin{pmatrix} \cos(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}\,s) & & \\ -\sqrt{k}\sin(\sqrt{k}\,s) & \cos(\sqrt{k}\,s) & \multicolumn{2}{c}{\underline{0}} \\ & & \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ \multicolumn{2}{c}{\underline{0}} & \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix}$$

As for a drift, the energy does not change, i.e. $\delta = \delta_0$.
The time of flight only depends on energy, i.e. $\tau = \tau_0 + M56\,\delta$.

For k<0 one has to take into account that

$$\cos(\sqrt{k}\,s) = \cosh(\sqrt{|k|}\,s)\,, \quad \sin(\sqrt{k}\,s) = i\sinh(\sqrt{|k|}\,s)$$

$$\cosh(\sqrt{k}\,s) = \cos(\sqrt{|k|}\,s)\,, \quad \sinh(\sqrt{k}\,s) = i\sin(\sqrt{|k|}\,s)$$

Variation of constants

$$\vec{z}\,' = \vec{f}(\vec{z},s)$$

$$\vec{z}\,' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z},s)$$

Field errors, nonlinear fields, etc can lead to $\Delta\vec{f}(\vec{z},s)$

$$\vec{z}\,'_H = \underline{L}(s)\vec{z}_H \quad \Rightarrow \quad \vec{z}_H(s) = \underline{M}(s)\vec{z}_{H0} \quad \text{with} \quad \underline{M}\,'(s)\vec{a} = \underline{L}(s)\underline{M}(s)\vec{a}$$

$$\vec{z}(s) = \underline{M}(s)\vec{a}(s) \quad \Rightarrow \quad \vec{z}\,'(s) = \underline{M}\,'(s)\vec{a} + \underline{M}(s)\vec{a}\,'(s) = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z},s)$$

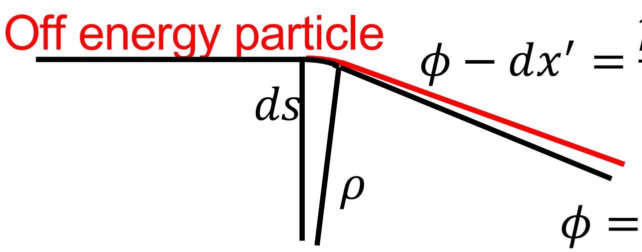
$$\vec{a}(s) = \vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vec{z}(s) = \underline{M}(s)\left\{\vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}\right\}$$

$$= \vec{z}_H(s) + \int_0^s \underline{M}(s-\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}$$

Perturbations are propagated from s to s'

The dipole equation of motion



$$\phi - dx' = \frac{p_0}{p} ds/\rho = \phi(1 - \frac{dp}{p})$$

$$\phi = ds/\rho$$

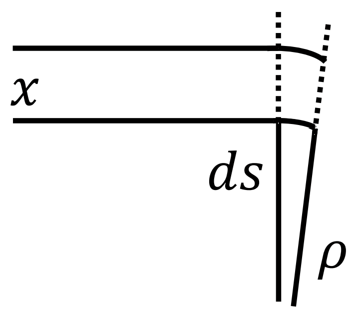
$$\rightarrow x'' = \frac{1}{\rho}\frac{dp}{p} = \frac{1}{\rho}\frac{dp}{dE}\frac{E}{p}\delta = \frac{1}{\rho}\frac{1}{\beta^2}\delta$$

$$x'' + x\,\kappa^2 = \frac{\kappa}{\beta^2}\delta \quad \text{with} \quad \kappa = \frac{1}{\rho} \quad \text{or} \quad \begin{aligned} x' &= a \\ a' &= -\kappa^2 x + \frac{\kappa}{\beta^2}\delta \end{aligned}$$

$$x_H'' + x_H\,\kappa^2 = 0 \;\Rightarrow\; \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) \\ -\kappa\sin(\kappa s) & \cos(\kappa s) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \underline{M}(s) \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} - \underline{M}(s)\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \int_0^s \underline{M}(s-\zeta)\begin{pmatrix} 0 \\ \frac{\kappa}{\beta^2}\delta \end{pmatrix} d\zeta = \int_0^s \begin{pmatrix} \frac{1}{\kappa}\sin(\kappa s) \\ \cos(\kappa s) \end{pmatrix} d\zeta \, \frac{\kappa}{\beta^2}\delta = \begin{pmatrix} \frac{1}{\kappa}(1-\cos(\kappa s)) \\ \sin(\kappa s) \end{pmatrix} \frac{1}{\beta^2}\delta$$

The dipole transport matrix



$$\frac{ds}{\rho}(\rho + x)$$

$$\rightarrow \tau' = -\frac{x}{\rho}\frac{1}{v}\frac{E_0}{p_0} = -\frac{x}{\rho}\frac{1}{\beta^2} = -\frac{\kappa}{\beta^2}x$$

$$\underline{M} = \begin{pmatrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) & \underline{0} & & 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ -\kappa\sin(\kappa s) & \cos(\kappa s) & & & 0 & \sin(\kappa s) \\ & & 1 & s & & \\ \underline{0} & & 0 & 1 & \underline{0} & \\ -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] & \underline{0} & & 1 & \kappa^{-1}[\sin(\kappa s) - s\kappa] \\ 0 & 0 & & & 0 & 1 \end{pmatrix}$$

(for $\beta = 1$)

$$M_{56} = -\kappa \int_0^s M_{16} ds$$

The combined function bend

$$x'' = -x\,(\underbrace{\kappa^2 + k}_{K}) + \delta\kappa$$

$$y'' = y\,k \quad , \quad \tau' = -\kappa\,x$$

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0}\,\vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{T} & \underline{0} & \underline{M}_\tau \end{pmatrix}$$

$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K}\,s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\,s) \\ -\sqrt{K}\sin(\sqrt{K}\,s) & \cos(\sqrt{K}\,s) \end{pmatrix}$$

$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \frac{\kappa}{K}[1-\cos(\sqrt{K}s)] \\ \frac{\kappa}{\sqrt{K}}\sin(\sqrt{K}s) \end{pmatrix}$$

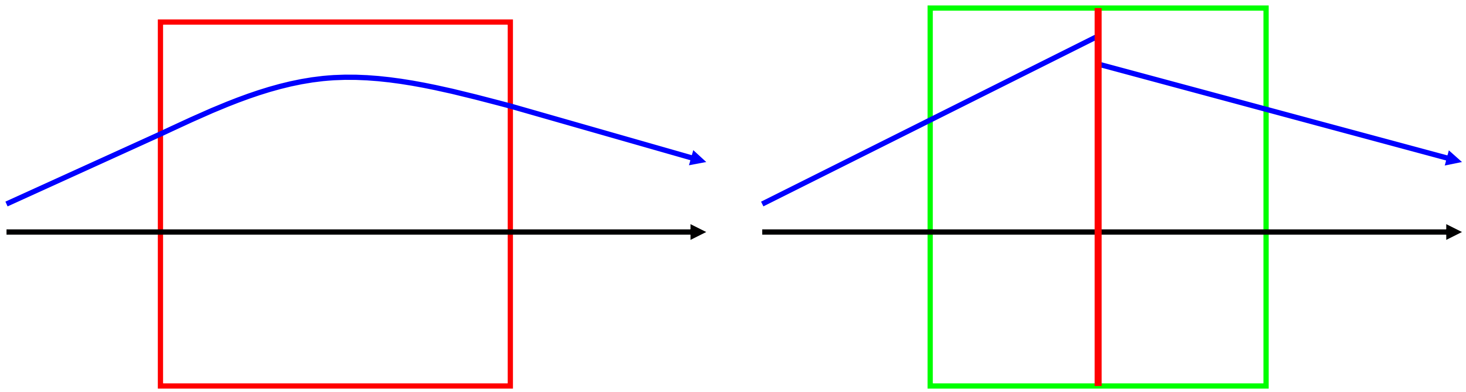
Options:

- For k>0:
  focusing in x, defocusing in y.
- For k<0, K<0:
  defocusing in x, focusing in y.
- For k<0, K>0:
  weak focusing in both planes.

$$\underline{M}_\tau = \begin{pmatrix} 1 & -\kappa\int_0^s M_{16}\,ds \\ 0 & 1 \end{pmatrix}$$

$\underline{T}$ from symplecticity

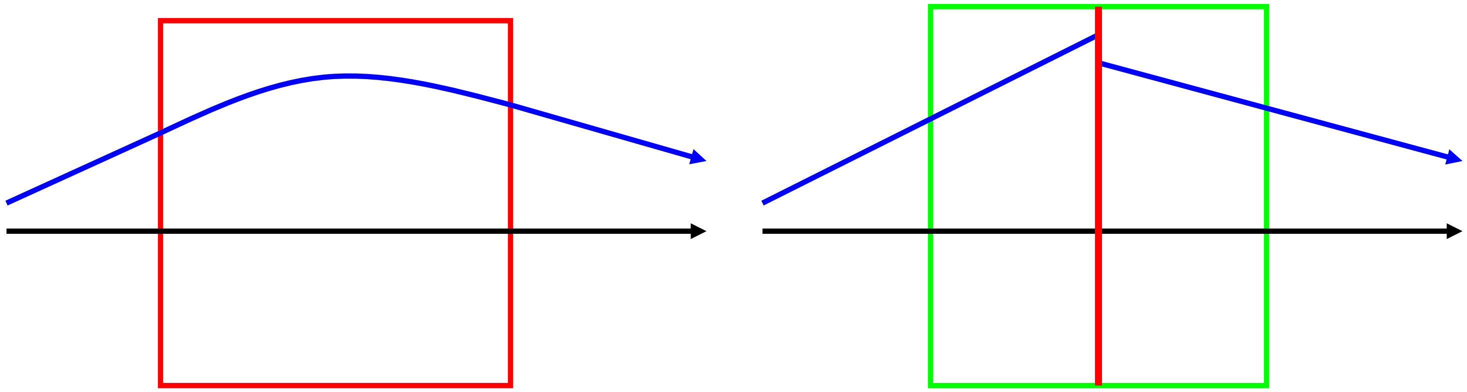
Thin lens approximation



$$\vec{z}(s) = \underline{M}(s)\,\vec{z}_0 = \underline{D}(\tfrac{s}{2})\underline{D}^{-1}(\tfrac{s}{2})\underline{M}(s)\underline{D}^{-1}(\tfrac{s}{2})\underline{D}(\tfrac{s}{2})\vec{z}_0$$

Drift: $\underline{M}^{\text{thin}}_{\text{drift}}(s) = \underline{D}^{-1}(\tfrac{s}{2})\underline{M}(s)\underline{D}^{-1}(\tfrac{s}{2}) = \underline{1}$

The thin lens quadrupole



$$\underline{M}^{\text{thin}}_{\text{quad},x}(s) = \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\sqrt{k}s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}s) \\ -\sqrt{k}\sin(\sqrt{k}s) & \cos(\sqrt{k}s) \end{pmatrix} \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & s \\ -ks & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{s}{2} \\ -ks & 1+\frac{ks^2}{2} \end{pmatrix}$$

Weak magnet limit: $\sqrt{k}s << 1$

$$\underline{M}^{\text{thin}}_{\text{quad},x}(s) \approx \begin{pmatrix} 1 & 0 \\ -ks & 1 \end{pmatrix}$$

The thin lens dipole

$$\underline{M} = \begin{pmatrix} \begin{matrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) \\ -\kappa\sin(\kappa s) & \cos(\kappa s) \end{matrix} & \underline{0} & \begin{matrix} 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ 0 & \sin(\kappa s) \end{matrix} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \begin{matrix} -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] \\ 0 & 0 \end{matrix} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}$$

Weak magnet limit: $\kappa s << 1$

$$\underline{M}^{\text{thin}}_{\text{bend},x\tau}(s) = \underline{D}(-\tfrac{s}{2})\underline{M}_{\text{bend},x\tau}\underline{D}(-\tfrac{s}{2}) \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\kappa^2 s & 1 & 0 & \kappa s \\ -\kappa s & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The thin lens combined function bend

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0}\,\vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{T} & \underline{0} & \underline{1} \end{pmatrix}$$

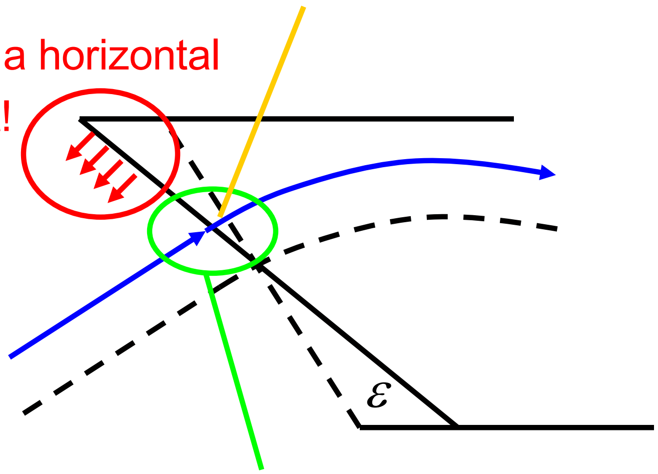
Weak magnet limit: $\kappa s << 1$

$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K}\,s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\,s) \\ -\sqrt{K}\sin(\sqrt{K}\,s) & \cos(\sqrt{K}\,s) \end{pmatrix} \quad \rightarrow \quad \underline{M}_x^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ -K\,s & 1 \end{pmatrix}$$

$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix} \quad \rightarrow \quad \underline{M}_y^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ k\,s & 1 \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \frac{\kappa}{K}[1-\cos(\sqrt{K}s)] \\ \frac{\kappa}{\sqrt{K}}\sin(\sqrt{K}s) \end{pmatrix} \quad \rightarrow \quad \vec{D} = \begin{pmatrix} 0 \\ \kappa\,s \end{pmatrix}$$

Edge focusing



Horizontal focusing with $\Delta x' = -x\frac{\tan(\varepsilon)}{\rho}$

$$B_x = \partial_y B_s\big|_{y=0}\, y\tan(\varepsilon) = \partial_s B_y\big|_{y=0}\, y\tan(\varepsilon)$$

$$y'' = \tfrac{q}{p}\partial_s B_y\big|_{y=0}\, y\tan(\varepsilon)$$

$$\Delta y' = \int y''\,ds = \tfrac{q}{p}B_y\, y\tan(\varepsilon) = y\frac{\tan(\varepsilon)}{\rho}$$

Quadrupole effect with

$$kl = \frac{\tan(\varepsilon)}{\rho}$$

$$\vec{z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\tan(\varepsilon)}{\rho} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\tan(\varepsilon)}{\rho} & 1 \end{pmatrix}\vec{z}_0$$

Orbit distortions for a one-pass accelerator

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \Delta f$$

The extra force can for example come from an erroneous dipole field or from a correction coil: $\Delta f = \frac{q}{p}\Delta B_y = \Delta\kappa$

Variation of constants: $\vec{z} = \underline{M}\vec{z}_0 + \Delta\vec{z}$ with $\Delta\vec{z} = \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \Delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$

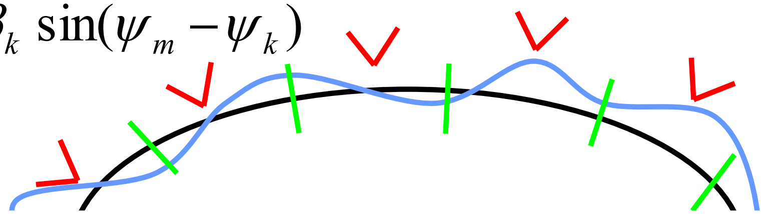
$$\Delta x(s) = \sum_k \Delta\vartheta_k \sqrt{\beta(s)\beta_k}\,\sin(\psi(s) - \psi_k)$$

Orbit correction for a one-pass accelerator

When the closed orbit $x_{\text{co}}^{\text{old}}(s_m)$ is measured at beam position monitors (BPMs, index m) and is influenced by corrector magnets (index k), then the monitor readings before and after changing the kick angles created in the correctors by $\Delta\vartheta_k$ are related by

$$x_{\text{co}}^{\text{new}}(s_m) = x_{\text{co}}^{\text{old}}(s_m) + \sum_k \Delta\vartheta_k \sqrt{\beta_m \beta_k} \sin(\psi_m - \psi_k)$$

$$= x_{\text{co}}^{\text{old}}(s_m) + \sum_k O_{mk} \Delta\vartheta_k$$



$$\vec{x}_{\text{co}}^{\text{new}} = \vec{x}_{\text{co}}^{\text{old}} + \underline{O}\Delta\vec{\vartheta}$$

$$\Delta\vec{\vartheta} = -\underline{O}^{-1}\vec{x}_{\text{co}}^{\text{old}} \quad \Rightarrow \quad \vec{x}_{\text{co}}^{\text{new}} = 0$$

It is often better not to try to correct the closed orbit at the the BPMs to zero in this way since

1. computation of the inverse can be numerically unstable, so that small errors in the old closed orbit measurement lead to a large error in the corrector coil settings.
2. A zero orbit at all BPMs can be a bad orbit in-between BPMs

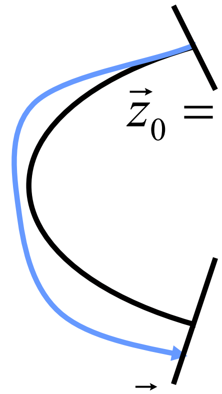
Dispersion of one-pass accelerators

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \kappa\delta$$

$$\vec{z} = \underline{M}\vec{z}_0 + \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix}0\\ \delta\kappa(\hat{s})\end{pmatrix}d\hat{s}$$

$$\Rightarrow \quad \vec{D}(s) = \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix}0\\ \kappa(\hat{s})\end{pmatrix}ds'$$



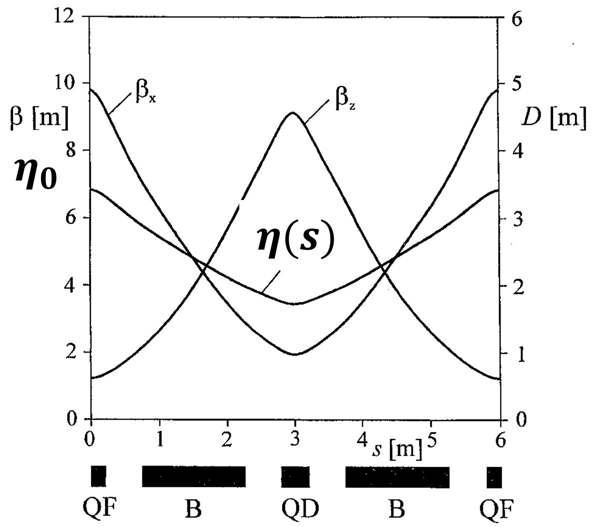
$$\Delta\kappa = \delta\kappa$$

$$D(s) = \sqrt{\beta(s)}\int_0^s \kappa(\hat{s})\sqrt{\beta(\hat{s})}\sin(\psi(s)-\psi(\hat{s}))d\hat{s}$$

Alternatively, one can multiply the 6x6 matrices and take $\boldsymbol{D(s) = M_{16}(s)}$

Fodo Cells and periodic dispersion

Alternating gradients allow focusing in both transverse plains. Therefore, focusing and defocusing quadrupoles are usually alternated and interleaved with bending magnets.



The dispersion that starts with 0 is called $\boldsymbol{D(s)}$, the dispersion that is periodic in a section is called $\boldsymbol{\eta(s)}$.

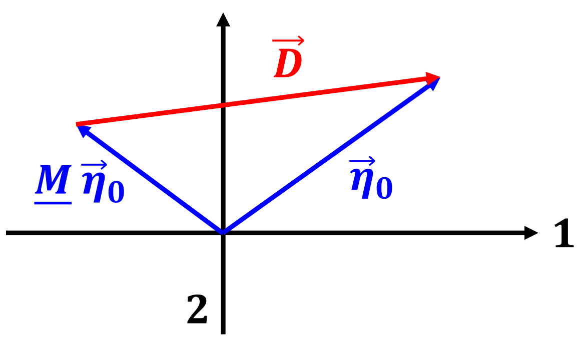
$$\underline{M}_0 = \underline{M}_{FoDo}^N$$

The periodic beta function and dispersion for each FODO is also periodic for an accelerator section that consists of many FODO cells. Often large sections of an accelerator consist of FODOs.

$$\vec{\eta}(s) = \begin{pmatrix} \eta(s) \\ \eta'(s) \end{pmatrix}, \vec{D}(s) = \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}$$

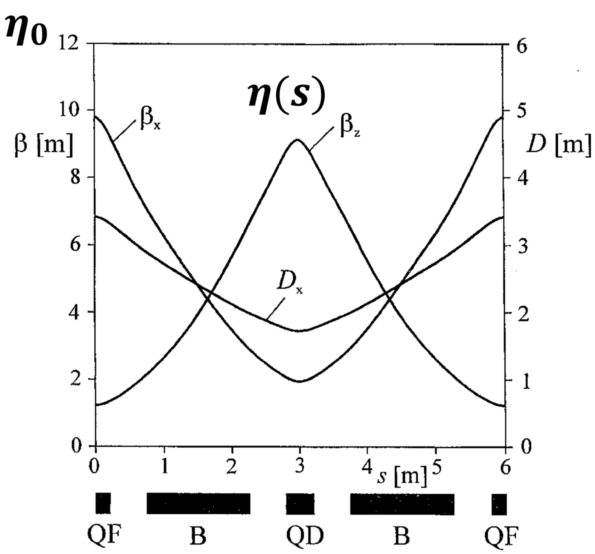
$$\vec{\eta}(s) = \underline{M}(s)\,\vec{\eta}_0 + \vec{D}(s)$$

After the FoDo: $\vec{\eta}_0 = \underline{M}(s)\,\vec{\eta}_0 + \vec{D} \Rightarrow \vec{\eta}_0 = (\underline{1} - \underline{M})^{-1}\,\vec{D}$

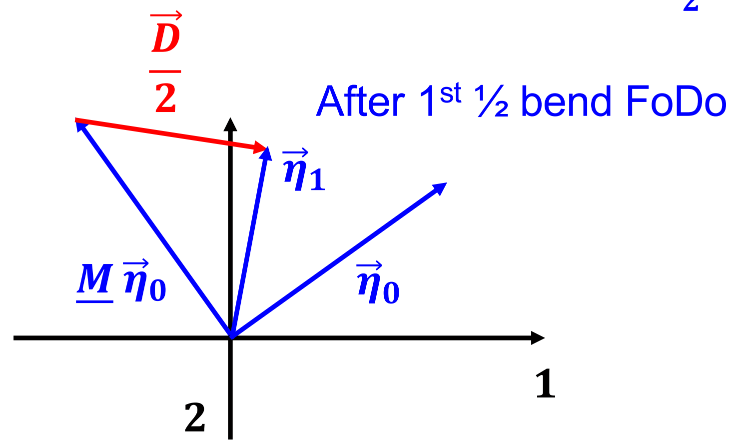


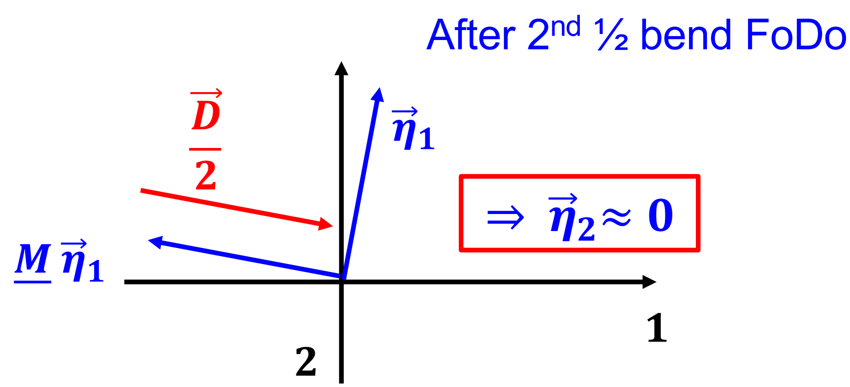
Dispersion suppression by missing bends

After an arc of periodic FoDo cells one would often like to suppress the dispersion to 0 while not changing the betas much from the periodic cells. This can be done by having one FoDo with bends reduced by a factor α followed by one reduced by (1- α).



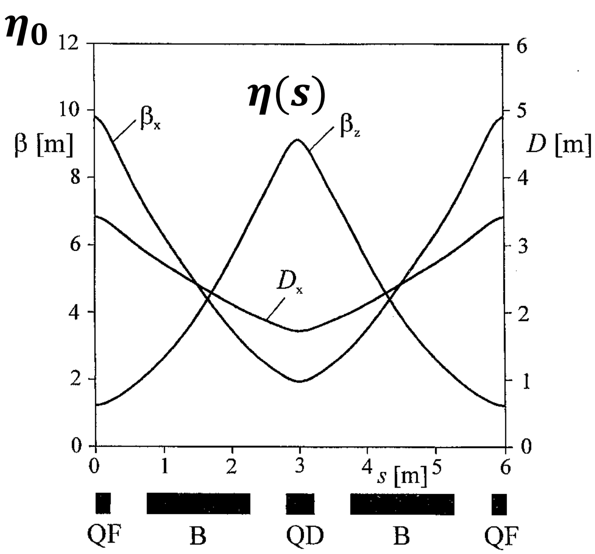
Example: **90 degrees** FoDo cell and $\boldsymbol{\alpha} = \frac{1}{2}$:



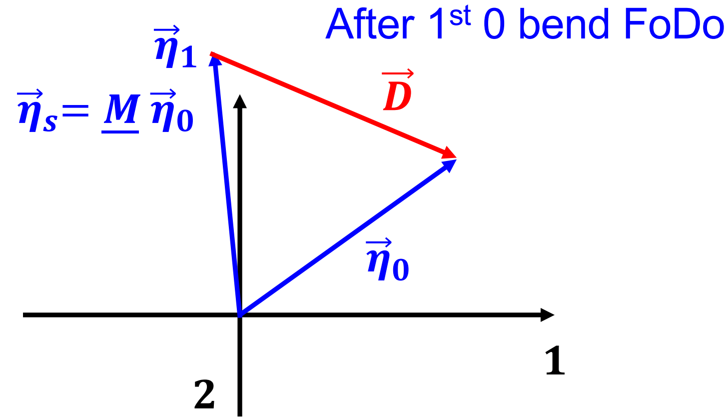


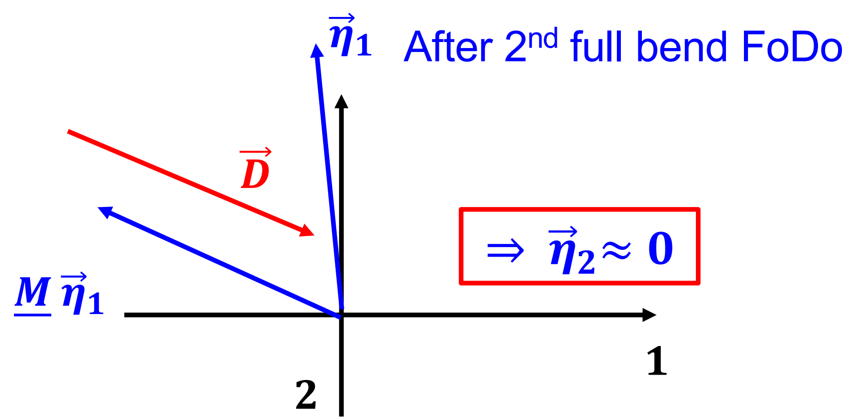
Dispersion suppression by missing bends

After an arc of periodic FoDo cells one would often like to suppress the dispersion to 0 while not changing the betas much from the periodic cells. This can be done by having one FoDo with bends reduced by a factor α followed by one reduced by (1- α).



Example: **60 degrees** FoDo cell and $\boldsymbol{\alpha} = \mathbf{0}$:

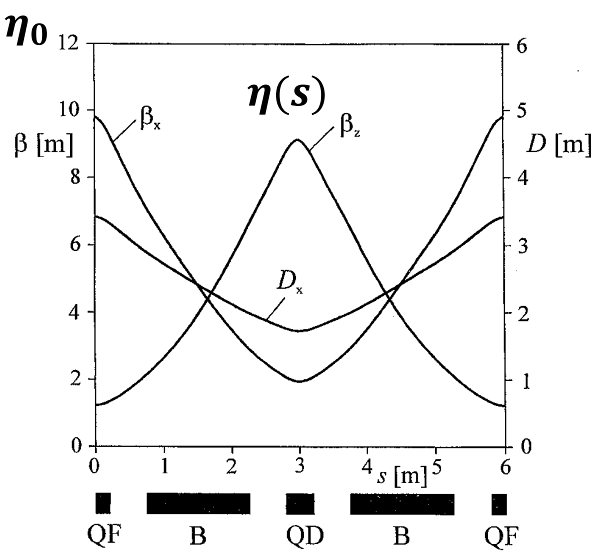




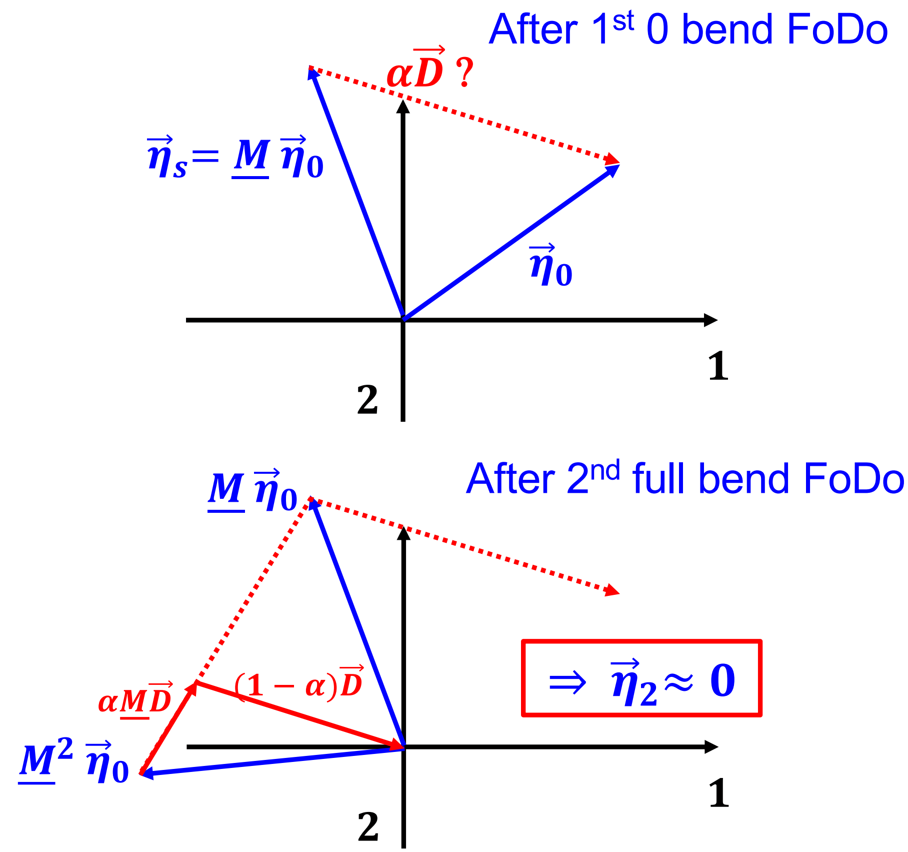
For every FoDo phase advance there is an α to make η 0.

Dispersion suppression by missing bends

After an arc of periodic FoDo cells one would often like to suppress the dispersion to 0 while not changing the betas much from the periodic cells. This can be done by having one FoDo with bends reduced by a factor α followed by one reduced by (1- α).



For any other FoDo phase advance: is there an α ?



For every FoDo phase advance there is an α to make η 0.

Closed orbit in periodic accelerators

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \Delta f$$

The extra force can for example come from an erroneous dipole field or from a correction coil: $\Delta f = \frac{q}{p}\Delta B_y = \Delta\kappa$

Variation of constants: $\vec{z} = \underline{M}\vec{z}_0 + \Delta\vec{z}$ with $\Delta\vec{z} = \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \Delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$

For the periodic or closed orbit: $\vec{z}_{\text{co}} = \underline{M}_0\vec{z}_{\text{co}} + \underline{M}_0\int_0^L \underline{M}^{-1}(\hat{s})\begin{pmatrix} 0 \\ \Delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$

$$\vec{z}_{\text{co}} = [\underline{M}_0^{-1} - \underline{1}]^{-1}\int_0^L \underline{M}^{-1}(\hat{s})\begin{pmatrix} 0 \\ \Delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$$

$$= \frac{(\cos\mu - 1)\underline{1} + \sin\mu\,\underline{\beta}}{(\cos\mu - 1)^2 + \sin^2\mu}\int_0^L \begin{pmatrix} -\sqrt{\beta\hat{\beta}}\sin\hat{\psi} \\ \sqrt{\frac{\hat{\beta}}{\beta}}[\cos\hat{\psi} + \alpha\sin\hat{\psi}] \end{pmatrix} \Delta\kappa(\hat{s})\, d\hat{s}$$

Closed orbit in periodic accelerators

$$\vec{z}_{co}(L) = \frac{\cos\mu \underline{1} + \sin\mu \underline{\beta} - \underline{1}}{2 - 2\cos\mu} \int_0^L \begin{pmatrix} \sqrt{\beta\hat{\beta}}\sin(-\hat{\psi}) \\ \sqrt{\frac{\hat{\beta}}{\beta}}[\cos(-\hat{\psi}) - \alpha\sin(-\hat{\psi})] \end{pmatrix} \Delta\kappa(\hat{s})\, d\hat{s}$$

$$x_{co}(L) = \frac{1}{4\sin^2\frac{\mu}{2}} \int_0^L \sqrt{\beta\hat{\beta}}[\sin(\mu - \hat{\psi}) + \sin\hat{\psi}]\Delta\kappa(\hat{s})\, d\hat{s}$$

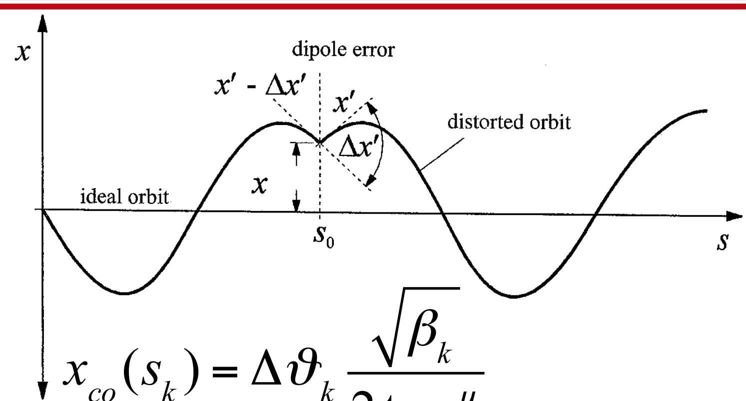
$$\sin\left(\left(\psi - \frac{\mu}{2}\right) + \frac{\mu}{2}\right) - \sin\left(\left(\psi - \frac{\mu}{2}\right) - \frac{\mu}{2}\right)$$

$$= \frac{1}{2\sin\frac{\mu}{2}} \int_0^L \sqrt{\beta\hat{\beta}}\cos(\hat{\psi} - \tfrac{\mu}{2})\Delta\kappa(\hat{s})\, d\hat{s}$$

$$x_{co}(s) = \frac{1}{2\sin\frac{\mu}{2}}\left[\int_s^L \sqrt{\beta\hat{\beta}}\cos(\hat{\psi} - \psi - \tfrac{\mu}{2})\Delta\hat{\kappa}\, d\hat{s} + \int_0^s \sqrt{\beta\hat{\beta}}\cos(\hat{\psi} - \psi + \tfrac{\mu}{2})\Delta\hat{\kappa}\, d\hat{s}\right.$$

$$= \frac{\sqrt{\beta}}{2\sin\frac{\mu}{2}} \int_0^L \sqrt{\hat{\beta}}\cos(|\hat{\psi} - \psi| - \tfrac{\mu}{2})\Delta\hat{\kappa}\, d\hat{s}$$

Closed orbit for one kick



$$x_{co}(s_k) = \Delta\vartheta_k \frac{\sqrt{\beta_k}}{2\tan\frac{\mu}{2}}$$

$$x_{co}(s) = \Delta\vartheta_k \frac{\sqrt{\beta\beta_k}}{2\sin\frac{\mu}{2}} \cos(|\psi - \psi_k| - \tfrac{\mu}{2})$$

Free betatron oscillation

$$x'_{co}(s_k) - x'_{co}(s_k + L) = \Delta\vartheta_k \frac{-\sin(-\frac{\mu}{2}) + \sin(\frac{\mu}{2})}{2\sin\frac{\mu}{2}} = \Delta\vartheta_k$$

$$x_{max} = \sqrt{2J\beta}$$

$$x_{co}(s) = \sqrt{2J\beta}\sin(\psi + \varphi_0)\,, \quad J = \frac{\Delta\vartheta_k^2 \beta_k}{8\sin^2\frac{\mu}{2}}$$

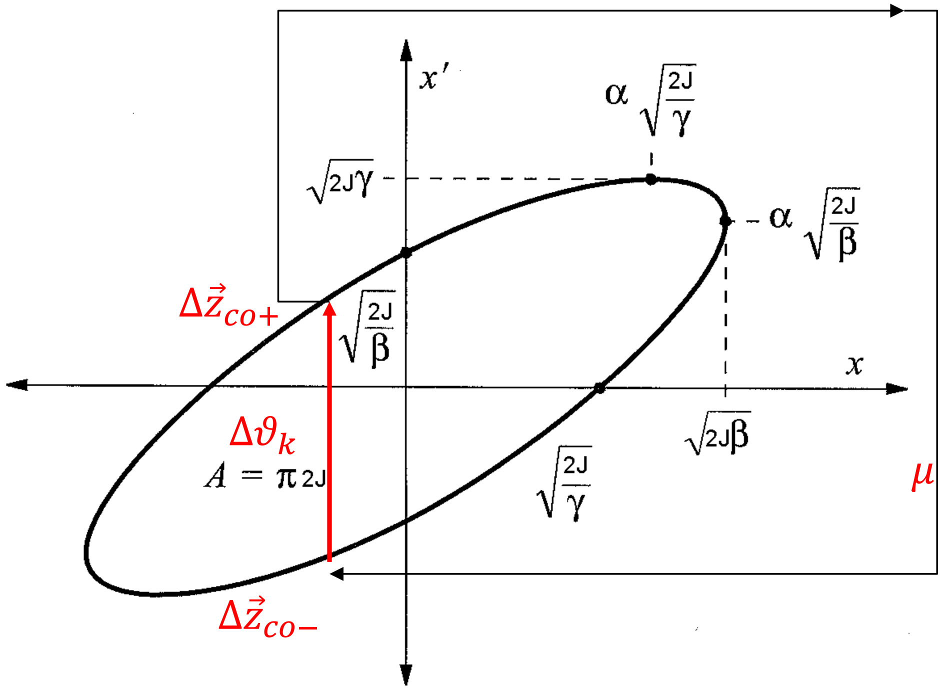
The oscillation amplitude J diverges when the tune ν is close to an integer.

$$s < s_k : \varphi_0 = \tfrac{\pi}{2} - \psi_k + \tfrac{\mu}{2} \quad , \quad s > s_k : \varphi_0 = \tfrac{\pi}{2} - \psi_k - \tfrac{\mu}{2}$$

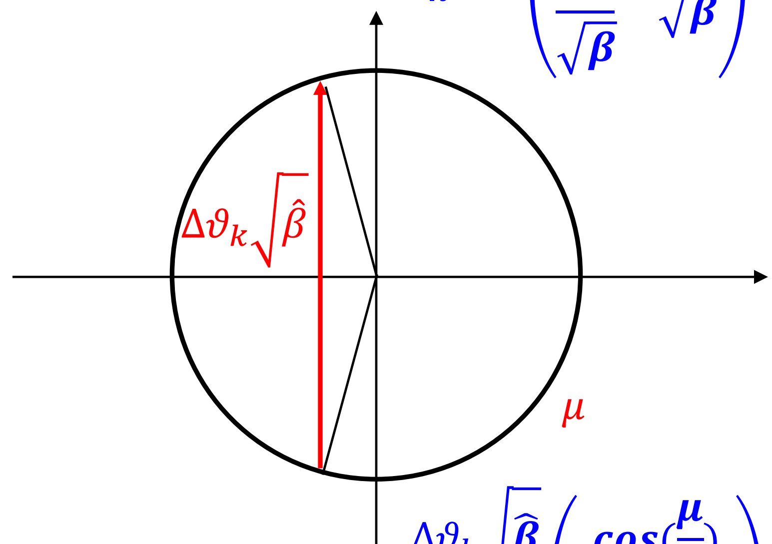
Phase jump by μ

Closed orbit for one kick

Free betatron oscillation



$$\begin{pmatrix}x_n\\a_n\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{\beta}} & 0\\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta}\end{pmatrix}\begin{pmatrix}x\\x'\end{pmatrix}$$



$$\begin{pmatrix}x_{n\pm}\\a_{n\pm}\end{pmatrix} = \frac{\Delta\vartheta_k\sqrt{\hat{\beta}}}{\sin(\frac{\mu}{2})}\begin{pmatrix}\cos(\frac{\mu}{2})\\ \pm\sin(\frac{\mu}{2})\end{pmatrix}$$

$$x_n(s) = \frac{\Delta\vartheta_k\sqrt{\beta\hat{\beta}}}{2\sin(\frac{\mu}{2})}\cos(\frac{\mu}{2} - |\psi - \psi_k|)$$

$$\begin{pmatrix}x_{n\pm}\\a_{n\pm}\end{pmatrix} = \frac{\Delta\vartheta_k\sqrt{\hat{\beta}}}{\sin(\frac{\mu}{2})}\begin{pmatrix}\frac{\mu}{2} - |\psi - \psi_k|)\\ \ldots\end{pmatrix}$$

Closed orbit correction in periodic accelerators

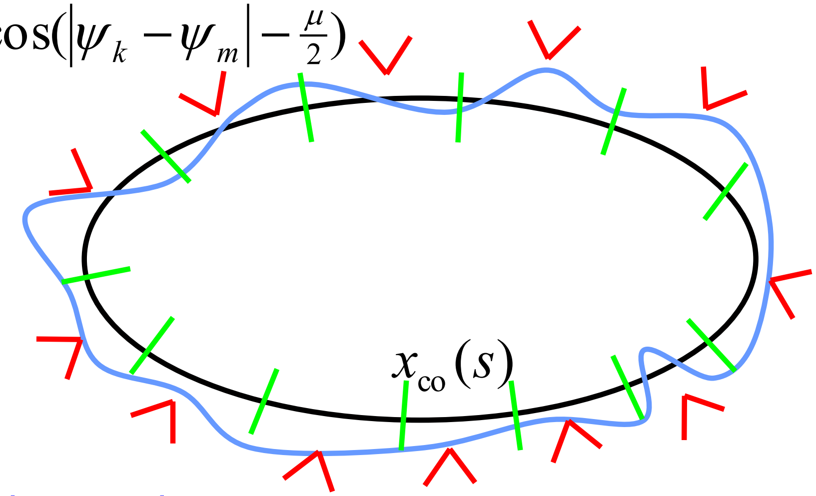
When the closed orbit $x_{co}^{old}(s_m)$ is measured at beam position monitors (BPMs, index m) and is influenced by corrector magnets (index k), then the monitor readings before and after changing the kick angles created in the correctors by $\Delta\vartheta_k$ are related by

$$x_{co}^{new}(s_m) = x_{co}^{old}(s_m) + \sum_k \Delta\vartheta_k \frac{\sqrt{\beta_m\beta_k}}{2\sin\frac{\mu}{2}} \cos(|\psi_k - \psi_m| - \tfrac{\mu}{2})$$

$$= x_{co}^{old}(s_m) + \sum_k O_{mk}\Delta\vartheta_k$$

$$\vec{x}_{co}^{new} = \vec{x}_{co}^{old} + \underline{O}\Delta\vec{\vartheta}$$

$$\Delta\vec{\vartheta} = -\underline{O}^{-1}\vec{x}_{co}^{old} \quad \Rightarrow \quad \vec{x}_{co}^{new} = 0$$

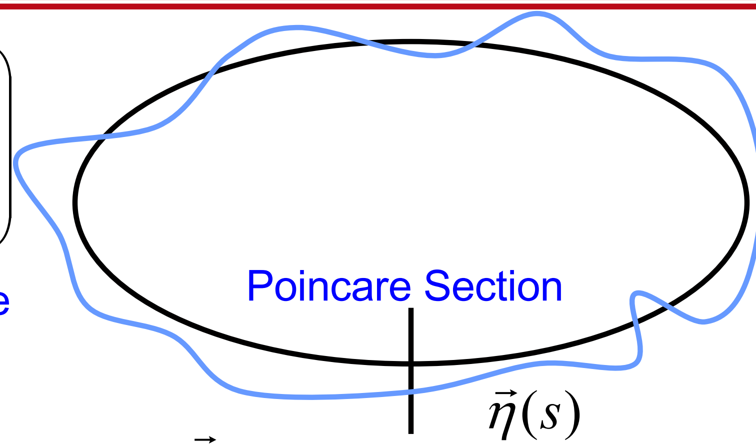


It is often better not to try to correct the closed orbit at the the BPMs to zero in this way since

1. computation of the inverse can be numerically unstable, so that small errors in the old closed orbit measurement lead to a large error in the corrector coil settings.
2. A zero orbit at all BPMs can be a bad orbit inbetween BPMs

Periodic dispersion

$$\begin{pmatrix} \underline{M}_{0x}\vec{z}_0 + \vec{D}(L)\delta \\ M_{56}\delta \\ \delta \end{pmatrix} = \begin{pmatrix} \underline{M}_{0x} & \vec{0} & \vec{D}(L) \\ \vec{T}^T & 1 & M_{56} \\ \vec{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{z}_0 \\ 0 \\ \delta \end{pmatrix}$$



The periodic orbit for particles with relative energy deviation δ is

$$\vec{\eta}(0) = \underline{M}_0\vec{\eta}(0) + \vec{D}(L) \qquad \vec{\eta}(L) = \underline{M}_0\vec{\eta}(0) + \vec{D}(L) \quad \text{with} \quad \vec{\eta}(L) = \vec{\eta}(0)$$

$$\Downarrow$$

$$\boxed{\vec{\eta}(0) = [\underline{1} - \underline{M}_0(0)]^{-1}\vec{D}(L)}$$

Particles with energy deviation δ oscillates around this periodic orbit.

$$\vec{z} = \vec{z}_\beta + \delta\vec{\eta}$$

$$\underline{\vec{z}_\beta(L) + \delta\vec{\eta}(L)} = \vec{z}(L) = \underline{M}_0\vec{z}(0) + \vec{D}(L)\delta = \underline{M}_0[\vec{z}_\beta(0) + \delta\vec{\eta}(0)] + \vec{D}(L)\delta$$

$$= \underline{M}_0\vec{z}_\beta(0) + \delta\vec{\eta}(L)$$

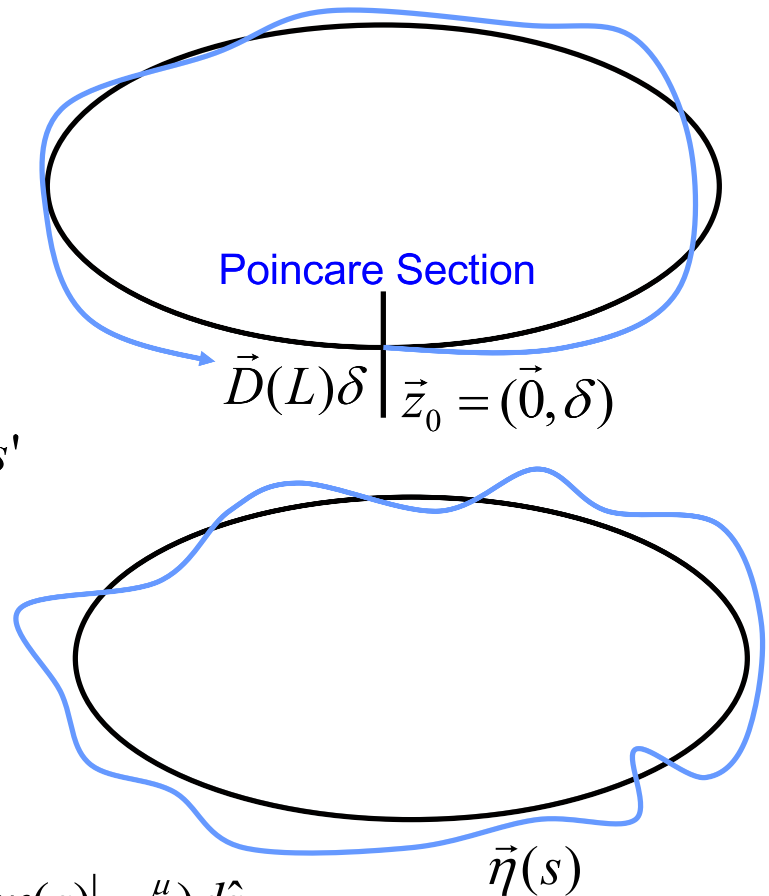
Periodic dispersion integral

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \kappa\delta$$

$$\vec{z} = \underline{M}\vec{z}_0 + \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$$

$$\Rightarrow \quad \vec{D}(L) = \int_0^L \underline{M}(L-\hat{s})\begin{pmatrix} 0 \\ \kappa(\hat{s}) \end{pmatrix} ds'$$



$$\Delta\kappa = \delta\kappa$$

$$\eta(s) = \frac{\sqrt{\beta(s)}}{2\sin\frac{\mu}{2}} \oint \kappa(\hat{s})\sqrt{\beta(\hat{s})}\cos\left(\left|\psi(\hat{s}) - \psi(s)\right| - \tfrac{\mu}{2}\right) d\hat{s}$$

Qadrupole errors in one-pass accelerators

$$\vec{z}' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z},s)$$

$$\vec{z}(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z},\hat{s})\,d\hat{s} \approx \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_H,\hat{s})\,d\hat{s}$$

$$x'' = -(\kappa^2 + k)x - \Delta k(s)x \quad \Rightarrow \quad \begin{pmatrix} x' \\ a' \end{pmatrix} = \begin{pmatrix} a \\ -(\kappa^2 + k)x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \Delta k(s) & 0 \end{pmatrix}\begin{pmatrix} x \\ a \end{pmatrix}$$

$$\vec{z}(s) = \left\{ \underline{M}(s) - \int_0^s \underline{M}(s,\hat{s}) \begin{pmatrix} 0 & 0 \\ \Delta k(\hat{s}) & 0 \end{pmatrix} \underline{M}(\hat{s},0)\,d\hat{s} \right\} \vec{z}_0$$

One quadrupole error:

$$\Delta\underline{M}(s,\hat{s}) = -\underline{M}(s,\hat{s}) \begin{pmatrix} 0 & 0 \\ \Delta kl(\hat{s}) & 0 \end{pmatrix}$$

Qadrupole errors and Twiss in one pass accelerators

$$\Delta \underline{M}(s,\hat{s}) = -\underline{M}(s,\hat{s})\begin{pmatrix} 0 & 0 \\ \Delta kl(\hat{s}) & 0 \end{pmatrix} \qquad \underline{M}(s) = \begin{pmatrix} \cdots & \sqrt{\beta_0\beta}\sin\tilde{\psi} \\ \cdots & \sqrt{\frac{\beta_0}{\beta}}[\cos\tilde{\psi} - \alpha\sin\tilde{\psi}] \end{pmatrix}$$

$$\Delta \underline{M}(s,\hat{s}) = -\Delta kl(\hat{s})\begin{pmatrix} \sqrt{\hat{\beta}\beta}\sin\psi & 0 \\ \sqrt{\frac{\hat{\beta}}{\beta}}[\cos\psi - \alpha\sin\psi] & 0 \end{pmatrix} , \qquad \psi = \psi(s) - \psi(\hat{s})$$

$$= \begin{pmatrix} \frac{\frac{1}{2}\Delta\beta[\cos\psi + \hat{\alpha}\sin\psi] + \Delta\psi\beta[\hat{\alpha}\cos\psi - \sin\psi]}{\sqrt{\hat{\beta}\beta}} & \sqrt{\hat{\beta}}\left(\frac{\frac{\Delta\beta}{2}\sin\psi + \Delta\psi\beta\cos\psi}{\sqrt{\beta}}\right) \\ \cdots & \cdots \end{pmatrix}$$

$$\tfrac{1}{2}\Delta\beta\cos\psi + \tfrac{1}{2}\Delta\beta\frac{\sin^2\psi}{\cos\psi} = \tfrac{1}{2}\Delta\beta\frac{1}{\cos\psi} = -\Delta kl(\hat{s})\beta\hat{\beta}\sin\psi \qquad \Delta\psi = -\frac{\Delta\beta}{2\beta}\tan\psi$$

$$\boxed{\Delta\beta = -\Delta kl(\hat{s})\beta\hat{\beta}\sin 2\psi} \qquad \boxed{\Delta\psi = \Delta kl(\hat{s})\hat{\beta}\tfrac{1}{2}(1 - \cos 2\psi)}$$

Twiss changes in one-pass accelerators

$$\Delta\psi = \Delta k l_j \beta_j \sin^2(\psi - \psi_j)$$

→ More focusing always increases the tune

$$\frac{\Delta\beta}{\beta} = -\Delta k l_j \beta_j \sin(2[\psi - \psi_j])$$

→ Beta beat oscillates twice as fast as orbit.

Notice the self consistency: $\Delta\psi = \int_0^s \left(\frac{1}{\hat{\beta}+\Delta\hat{\beta}} - \frac{1}{\hat{\beta}}\right) d\hat{s} = -\int_0^s \frac{\Delta\hat{\beta}}{\hat{\beta}^2} d\hat{s} = -\int_0^s \frac{\Delta\hat{\beta}}{\hat{\beta}} d\hat{\psi}$

$$\Delta\alpha = -\frac{\Delta\beta'}{2} = \Delta k l_j \beta_j [\cos(2[\psi - \psi_j]) - \alpha \sin(2[\psi - \psi_j])]$$

$$\begin{pmatrix} \frac{\Delta\beta}{\beta} \\ \frac{\beta\Delta\alpha - \alpha\Delta\beta}{\beta} \end{pmatrix} = \Delta k l_j \beta_j \begin{pmatrix} \sin(2[\psi - \psi_j]) \\ \cos(2[\psi - \psi_j]) \end{pmatrix}$$

$$\left(\Delta k l_j \beta_j\right)^2 = [\left(\frac{\Delta\beta}{\beta}\right)^2 + \left(\Delta\alpha - \alpha\frac{\Delta\beta}{\beta}\right)^2]$$

Twiss correction in one-pass accelerators

$$\frac{\Delta\beta}{\beta} = -\sum_j \Delta k l_j \beta_j \sin(2[\psi - \psi_j]) \qquad \Delta\psi = \sum_j \Delta k l_j \beta_j \tfrac{1}{2}[1 - \cos(2[\psi - \psi_j])]$$

When beta functions and betatron phases have been measured at many places, quadrupoles can be changed with these formulas to correct the Twiss errors.

Quadruple errors and the tune

$$\cos(\mu+\Delta\mu)=\tfrac{1}{2}\mathrm{Tr}[M_0(s_j)+\Delta M_0(s_j)]\approx\cos\mu-\Delta\mu\sin\mu$$

$$=\tfrac{1}{2}\mathrm{Tr}[\begin{pmatrix}1 & 0\\ -\Delta kl_j & 1\end{pmatrix}\begin{pmatrix}\cos\mu+\alpha_j\sin\mu & \beta_j\sin\mu\\ -\gamma_j\sin\mu & \cos\mu-\alpha_j\sin\mu\end{pmatrix}]$$

$$=\cos\mu-\tfrac{1}{2}\Delta kl_j\beta_j\sin\mu$$

$$\boxed{\Delta\mu=\tfrac{1}{2}\Delta kl_j\beta_j}$$

Oscillation frequencies can be measured relatively easily and accurately.

Measurement of beta function: Change k and measure tune.

Quadruple errors and periodic beta function

One pass accelerators:

$$\Delta x(s) = \Delta\vartheta\sqrt{\beta\hat{\beta}}\sin(\psi - \hat{\psi})$$

$$\frac{\Delta\beta}{\beta} = -\Delta k l \hat{\beta}\sin(2(\psi - \hat{\psi}))$$

Periodic accelerators:

$$\Delta x_{co}(s) = \frac{\Delta\vartheta\sqrt{\beta\hat{\beta}}}{\sin(\frac{\mu}{2})}\cos(|\psi - \hat{\psi} - \frac{\mu}{2})$$

$$\frac{\Delta\beta}{\beta} = -\frac{\Delta k l \hat{\beta}}{2\sin(\mu)}\cos(2|\psi - \hat{\psi}| - \mu)$$