Energy dependent Twiss parameters

Natural Chromaticity: $\xi_x = \frac{1}{2\pi}\partial\mu_x/\partial\delta\,,\ \xi_y = \frac{1}{2\pi}\partial\mu_y/\partial\delta$

$$\Delta\mu_x = \frac{1}{2}\Delta k l \hat{\beta}_x \longrightarrow \partial\mu_x/\partial\delta = -\frac{1}{2}\int_0^L k(s)\beta_x(s)\,ds$$

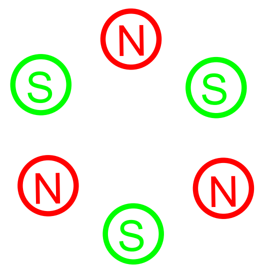
$$\Delta\mu_y = \frac{1}{2}\Delta k l \hat{\beta}_y \longrightarrow \partial\mu_y/\partial\delta = \frac{1}{2}\int_0^L k(s)\beta_y(s)\,ds$$

The periodic beta functions will similarly depend on energy.

Sextupoles (revisited)

$$\psi = \Psi_3 \,\mathrm{Im}\{(x - iy)^3\} = \Psi_3 \cdot (y^3 - 3x^2 y) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

C_3 Symmetry



i) Sextupole fields hardly influence the particles close to the center, where one can linearize in x and y.

ii) In linear approximation a by Δx shifted sextupole has a quadrupole field.

iii) When Δx depends on the energy, one can build an energy dependent quadrupole.

$$\vec{B} = -\vec{\nabla}\psi = \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

$$x \mapsto \Delta x + x$$

$$\vec{B} \approx \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix} + 6\Psi_3 \Delta x \begin{pmatrix} y \\ x \end{pmatrix} + O(\Delta x^2)$$

$$k_2 = \frac{q}{p} 3! \Psi_3 \Rightarrow k_1 = k_2 \Delta x$$

Chromaticity and its correction

Chromaticity ξ = energy dependence of the tune

$$\upsilon(\delta) = \upsilon + \tfrac{\partial \upsilon}{\partial \delta}\delta + \ldots$$

$$\xi = \tfrac{\partial \upsilon}{\partial \delta} \quad \text{with} \quad \upsilon = \tfrac{\mu}{2\pi}$$

Natural chromaticity ξ_0 = energy dependence of the tune due to quadrupoles only

$$\xi_{x0} = -\frac{1}{4\pi}\oint \beta_x(\hat{s})k_1(\hat{s})d\hat{s} \qquad \xi_{y0} = \frac{1}{4\pi}\oint \beta_y(\hat{s})k_1(\hat{s})d\hat{s}$$

Particles with energy difference oscillate around the periodic dispersion leading to a quadrupole effect in sextupoles that also shifts the tune:

$$\boxed{\xi_x = \frac{1}{4\pi}\oint \beta_x(-k_1 + \eta_x k_2)d\hat{s}} \qquad \boxed{\xi_y = \frac{1}{4\pi}\oint \beta_y(k_1 - \eta_x k_2)d\hat{s}}$$

Typically the the chormaticity ξ is chosen to be slightly positive, between 0 and 3.

Chromatic beta beat minimization

Chromatic beta beat is mostly created by the strongest quadrupoles, it can be influenced by sextupoles, under the provision that these on average still correct the chromaticity.

$$\frac{d\beta}{d\delta} = \beta(k_1 l - k_2 l \cdot \eta)\hat{\beta}\sin(2|\psi - \hat{\psi}| - \mu)$$

Periodic accelerators:

$$\frac{d\beta}{d\delta} = \frac{\beta}{2\sin\mu}(k_1 l - k_2 l \cdot \eta)\hat{\beta}\cos(2|\psi - \hat{\psi}| - \mu)$$

Sextupoles are used to compensate the chromaticity.

Several sextupoles are used to have their average compensate the chromaticity but have their regional variation compensate the chromatic beta beat on average and at critical sections, e.g. interaction points.

Single resonance model

$$x'' = -K\,x + \Delta f_x(x,y,s)$$

$$\frac{d}{d\vartheta} J = \sum_{n,m=-\infty}^{\infty} m H_{nm}(J)\sin(n\vartheta + m\varphi + \Psi_{nm}(J))$$

$$\frac{d}{d\vartheta}\varphi = \upsilon + \partial_J \sum_{n,m=-\infty}^{\infty} H_{nm}(J)\cos(n\vartheta + m\varphi + \Psi_{nm}(J))$$

Strong deviation from: $J = J_0\ ,\quad \varphi = \upsilon\vartheta + \varphi_0$

Occur when there is coherence between the perturbation and the phase space rotation: $n + m\frac{d}{ds}\varphi \approx 0$

Resonance condition: tune is rational $\boxed{n + m\ \upsilon = 0}$

On resonance the integral would increases indefinitely !

Neglecting all but the most important term

$$H(\varphi, J, \vartheta) \approx \upsilon J + H_{00}(J) + H_{nm}(J)\cos(n\vartheta + m\varphi + \Psi_{nm}(J))$$

Sum and difference resonances

$n+m_x\upsilon_x+m_y\upsilon_y\approx 0$ means that oscillations in y can drive oscillations in x in

$$x''=-K\,x+\Delta f_x(x,y,s)$$

The resonance term in the Hamiltonian then changes only slowly:

$$H(\vec{\varphi},\vec{J},\vartheta)=\vec{\upsilon}\cdot\vec{J}+H_{00}(\vec{J})+H_{n\vec{m}}(\vec{J})\cos(n\vartheta+\vec{m}\cdot\vec{\varphi}+\Psi_{n\vec{m}}(\vec{J}))$$

$$\tfrac{d}{d\vartheta}\vec{\varphi}=\vec{\partial}_J H \quad , \quad \tfrac{d}{d\vartheta}\vec{J}=-\vec{\partial}_\varphi H$$

$$J=\vec{m}\cdot\vec{J}$$

$$J_\perp=m_x J_y-m_y J_x=\vec{m}\times\vec{J} \quad\Rightarrow\quad \tfrac{d}{d\vartheta}J_\perp=0$$

Difference resonances lead to stable motion since:

$$n+|m_x|\upsilon_x-|m_y|\upsilon_y\approx 0\Rightarrow| \quad |m_y|\,J_x+|m_x|\,J_y=\text{const}$$

Sum resonances lead to unstable motion since:

$$n+|m_x|\upsilon_x+|m_y|\upsilon_y\approx 0\Rightarrow \quad |m_y|\,J_x-|m_x|\,J_y=\text{const}$$

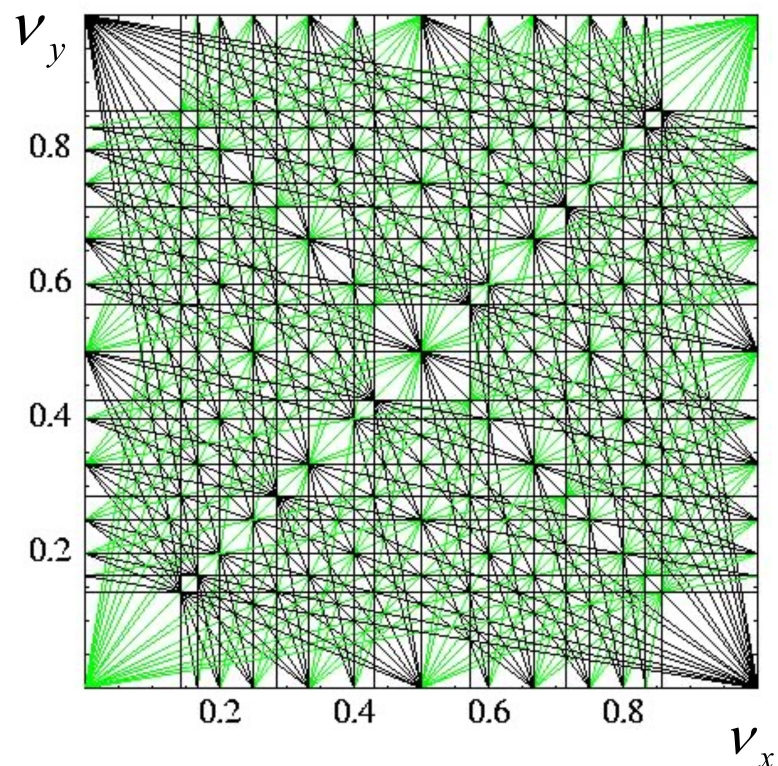
Common reasons for working points

(1) Avoid resonances $n + m_x \upsilon_x + m_y \upsilon_y \approx 0$

(2) +/- Colliders: be above a half integer to squeze the beam size $\frac{\Delta\beta}{\beta} = -\frac{\Delta k \beta}{2\tan\mu}$

(3) Polarized beams: close to integer

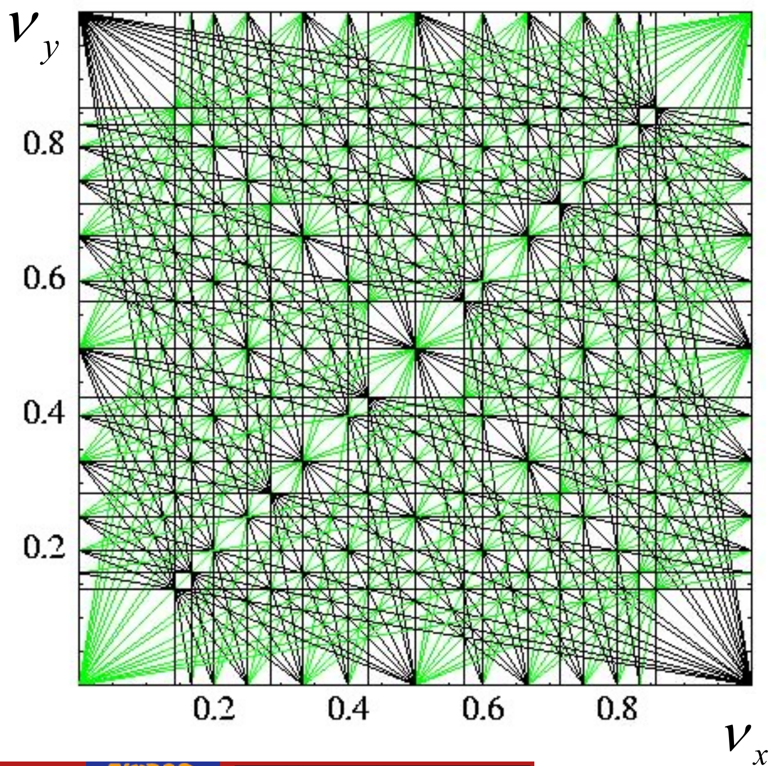
(4) Where the loss rate is smallest



Resonance diagram and choosing the tune

$n + m_x \upsilon_x + m_y \upsilon_y \approx 0$ means that oscillations in y can drive oscillations in x in

$$x'' = -K\,x + \Delta f_x(x, y, s)$$



All these resonances have to be avoided by their respective resonance width.

The position of an accelerator in the tune plane s called its Working Point.

Perturbations

$$\begin{pmatrix} x' \\ a' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix} \begin{pmatrix} x \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta f \end{pmatrix}$$

$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi + \phi_0) \\ \cos(\psi + \phi_0) \end{pmatrix} = \sqrt{2J}\, \underline{\beta}\, \vec{S}$$

This would be a solution with constant J and ϕ when $\Delta f = 0$.

Variation of constants:

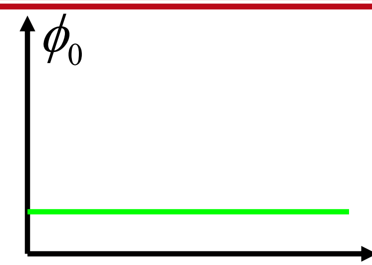
$$\frac{J'}{\sqrt{2J}} \underline{\beta}\, \vec{S} + \sqrt{2J}\, \phi_0' \begin{pmatrix} 0 & \sqrt{\beta} \\ -\frac{1}{\sqrt{\beta}} & -\frac{\alpha}{\sqrt{\beta}} \end{pmatrix} \vec{S} = \begin{pmatrix} 0 \\ \Delta f \end{pmatrix}$$

$$\frac{J'}{\sqrt{2J}} \vec{S} + \sqrt{2J}\, \phi_0' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{S} = \underline{\beta}^{-1} \begin{pmatrix} 0 \\ \Delta f \end{pmatrix} \quad \text{with} \quad \underline{\beta}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

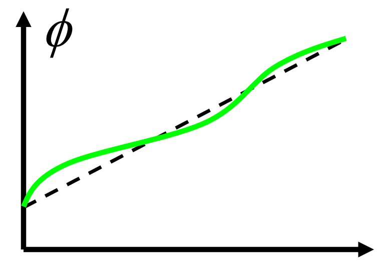
$$\frac{J'}{\sqrt{2J}} = \cos(\psi + \phi_0)\sqrt{\beta}\,\Delta f \quad , \quad \sqrt{2J}\, \phi_0' = -\sin(\psi + \phi_0)\sqrt{\beta}\,\Delta f$$

Simplification of linear motion

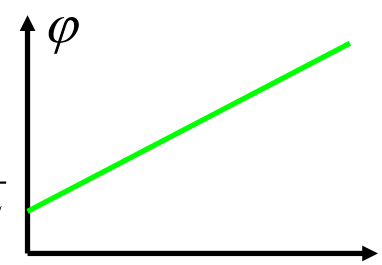
$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi+\phi_0) \\ \cos(\psi+\phi_0) \end{pmatrix} \Rightarrow \begin{matrix} J'=0 \\ \phi_0'=0 \end{matrix}$$



$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix} \Rightarrow \begin{matrix} J'=0 \\ \phi'=\frac{1}{\beta} \end{matrix}$$



$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi-\mu\frac{s}{L}+\varphi) \\ \cos(\psi-\mu\frac{s}{L}+\varphi) \end{pmatrix} \Rightarrow \begin{matrix} J'=0 \\ \varphi'=\mu\frac{1}{L} \end{matrix}$$



$$\tilde{\psi}=\psi-\mu\frac{s}{L} \Rightarrow \tilde{\psi}(s+L)=\tilde{\psi}(s)$$

Corresponds to Floquet's Theorem

Quasi-periodic perturbation

$$J' = \cos(\psi + \phi_0)\sqrt{2J\beta}\Delta f \quad , \quad \phi_0' = -\sin(\psi + \phi_0)\sqrt{\tfrac{\beta}{2J}}\Delta f$$

$$\tilde{\psi} = \psi - \mu \tfrac{s}{L} \ , \ \varphi = \mu \tfrac{s}{L} + \phi_0$$

$$J' = \cos(\tilde{\psi} + \varphi)\sqrt{2J\beta}\Delta f \quad , \quad \varphi' = \mu \tfrac{1}{L} - \sin(\tilde{\psi} + \varphi)\sqrt{\tfrac{\beta}{2J}}\Delta f$$

New independent variable $\vartheta = 2\pi \frac{s}{L}$

$$\tfrac{d}{d\vartheta} J = \cos(\tilde{\psi} + \varphi)\sqrt{2J\beta}\Delta f \, \tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi = \upsilon - \sin(\tilde{\psi} + \varphi)\sqrt{\tfrac{\beta}{2J}}\Delta f \, \tfrac{L}{2\pi}$$

$$\Delta f(x) = \Delta f(\sqrt{2J\beta}\sin(\tilde{\psi} + \varphi))$$

The perturbations are 2π periodic in ϑ and in φ

φ is approximately $\varphi \approx \upsilon \cdot \vartheta$

For irrational ν, the perturbations are quasi-periodic.

Tune shift with amplitude

$$\tfrac{d}{d\vartheta}J = \cos(\tilde{\psi}+\varphi)\sqrt{2J\beta}\,\Delta f\,\tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi = \upsilon - \sin(\tilde{\psi}+\varphi)\sqrt{\tfrac{\beta}{2J}}\,\Delta f\,\tfrac{L}{2\pi}$$

$$\tfrac{d}{d\vartheta}\varphi = \partial_J H \quad , \quad \tfrac{d}{d\vartheta}J = -\partial_\phi H \quad , \quad H(\varphi,J,\vartheta) = \upsilon\ J - \tfrac{L}{2\pi}\int_0^x \Delta f(\hat{x},s)\,d\hat{x}$$

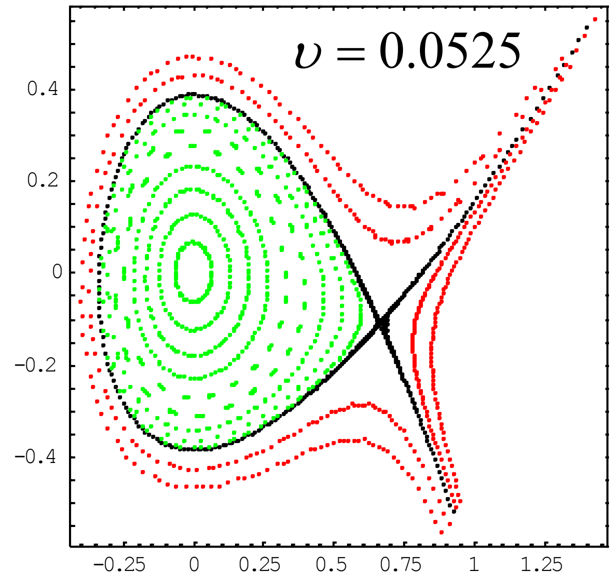
The motion remains Hamiltonian in the perturbed coordinates !

If there is a part in $\partial_J H$ that does not depend on φ, s $\Rightarrow$ Tune shift

The effect of other terms tends to average out.

$$\varphi(\vartheta) - \varphi_0 \approx \vartheta \cdot \partial_J \langle H \rangle_{\varphi,\vartheta}(J)$$

$$\upsilon(J) = \upsilon + \partial_J \langle \Delta H \rangle_{\varphi,\vartheta}(J)$$



Shift examples

$$H(\varphi,J)=\upsilon\cdot J-\tfrac{L}{2\pi}\int_0^x \Delta f(\hat{x},s)\,d\hat{x}\quad,\quad \Delta\upsilon(J)=\partial_J\langle\Delta H\rangle_{\varphi,\vartheta}$$

Quadrupole: $\Delta f=-\Delta k\,x$

$$\Delta H=\tfrac{L}{2\pi}\Delta k\,\tfrac{1}{2}x^2=\tfrac{L}{2\pi}\Delta k\,J\beta\sin^2(\tilde{\psi}+\varphi)$$

$$\langle\Delta H\rangle_{\varphi,\vartheta}=\tfrac{1}{2\pi}\int_0^{2\pi}\Delta k\beta\,d\vartheta\,L\frac{J}{4\pi}=\int_0^L\Delta k\beta\,ds\,\frac{J}{4\pi}\Rightarrow\Delta\upsilon=\tfrac{1}{4\pi}\oint\Delta k\beta\,ds$$

Sextupole: $\Delta f=-k_2\tfrac{1}{2}x^2$

$$\Delta H=\tfrac{L}{2\pi}k_2\tfrac{1}{3!}x^3=\tfrac{L}{2\pi}k_2\tfrac{1}{3!}\sqrt{2J\beta}^{\,3}\sin^3(\tilde{\psi}+\varphi)$$

$$\langle\Delta H\rangle_{\varphi,\vartheta}=0\quad\Rightarrow\quad\Delta\upsilon=0$$

Octupole: $\Delta f=-k_3\tfrac{1}{3!}x^3$

$$\Delta H=\tfrac{L}{2\pi}k_3\tfrac{1}{4!}x^4=\tfrac{L}{2\pi}k_3\tfrac{1}{3!}(J\beta)^2\sin^4(\tilde{\psi}+\varphi)$$

$$\langle\Delta H\rangle_{\varphi,\vartheta}=\tfrac{J^2}{3!2\pi}\oint k_3\beta^2ds\left\langle\tfrac{1}{2^4}(e^{i\varphi}-e^{-i\varphi})^4\right\rangle_{\varphi}\Rightarrow\Delta\upsilon=J\tfrac{1}{16\pi}\oint k_3\beta^2ds$$

Nonlinear resonances

$$\frac{d}{d\vartheta}J=\cos(\tilde{\psi}+\varphi)\sqrt{2J\beta}\Delta f\,\frac{L}{2\pi}\quad,\quad \frac{d}{d\vartheta}\varphi=\upsilon-\sin(\tilde{\psi}+\varphi)\sqrt{\frac{\beta}{2J}}\Delta f\,\frac{L}{2\pi}$$

$$\frac{d}{d\vartheta}\varphi=\partial_J H\quad,\quad \frac{d}{d\vartheta}J=-\partial_\phi H\quad,\quad H(\varphi,J,\vartheta)=\upsilon\cdot J-\frac{L}{2\pi}\int_0^x\Delta f(\hat{x},s)\,d\hat{x}$$

The effect of the perturbation is especially strong when

$\cos(\tilde{\psi}+\varphi)\sqrt{\beta}\Delta f$ or $\sin(\tilde{\psi}+\varphi)\sqrt{\beta}\Delta f$

has contributions that hardly change, i.e. the change of

$\sqrt{\beta(\vartheta)}\Delta f(x(\vartheta),\vartheta)$ is in resonance with the rotation angle $\varphi(\vartheta)$.

Periodicity allows Fourier expansion:

$$H(\varphi,J,\vartheta)=\sum_{n,m=-\infty}^{\infty}\widehat{H}_{nm}(J)\,e^{i[n\vartheta+m\varphi]}=\sum_{n,m=-\infty}^{\infty}H_{nm}(J)\cos(n\vartheta+m\varphi+\Psi_{nm}(J))$$

$H_{00}(J)=\langle H(\varphi,J,s)\rangle_{\varphi,s}\Rightarrow$ Tune shift

Choosing: $\Psi_{00}(J)=0$

Nonlinear motion

Sextupoles cause nonlinear dynamics, which can be chaotic and unstable.

$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \underline{M}_0 \left[\begin{pmatrix} x_n \\ x'_n \end{pmatrix} - \frac{k_2 l_s}{2} \begin{pmatrix} 0 \\ x_n^{\,2} \end{pmatrix} \right] \qquad \begin{pmatrix} x_n \\ x'_n \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \hat{x}_n \\ \hat{x}'_n \end{pmatrix}$$

$$\begin{pmatrix} \hat{x}_{n+1} \\ \hat{x}'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \left[\begin{pmatrix} \hat{x}_n \\ \hat{x}'_n \end{pmatrix} - \frac{k_2 l_s}{2} \sqrt{\beta} \begin{pmatrix} 0 \\ \beta\, \hat{x}_n^2 \end{pmatrix} \right]$$

$$\begin{pmatrix} \hat{x}_f \\ \hat{x}'_f \end{pmatrix} = \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \begin{pmatrix} 1-\cos\mu & \sin\mu \\ -\sin\mu & 1-\cos\mu \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \hat{x}_f^2 \end{pmatrix} = \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \frac{1}{2\sin\frac{\mu}{2}} \begin{pmatrix} -\cos\frac{\mu}{2} \\ \sin\frac{\mu}{2} \end{pmatrix} \hat{x}_f^2$$

$$\left.\begin{aligned} \hat{x}_f &= -\tfrac{4}{k_2 l_s} \beta^{-\frac{3}{2}} \tan\tfrac{\mu}{2} \\ \hat{x}'_f &= \tfrac{4}{k_2 l_s} \beta^{-\frac{3}{2}} \tan^2\tfrac{\mu}{2} \end{aligned}\right\} \hat{x} = \hat{x}_f + \Delta\hat{x} \qquad J_f = \tfrac{1}{2}(\hat{x}_f^2 + \hat{x}'^2_f) = \tfrac{1}{2\beta^3}\left(\tfrac{4}{k_2 l_s} \tfrac{\tan\frac{\mu}{2}}{\cos\frac{\mu}{2}}\right)^2$$

$$\begin{pmatrix} \Delta\hat{x}_{n+1} \\ \Delta\hat{x}'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \left[\begin{pmatrix} \Delta\hat{x}_n \\ \Delta\hat{x}'_n \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \Delta\hat{x}_n^2 - 4\tan\frac{\mu}{2}\, \Delta\hat{x}_n \end{pmatrix} \right]$$

Dynamic aperture (e.g., close to integer tunes)

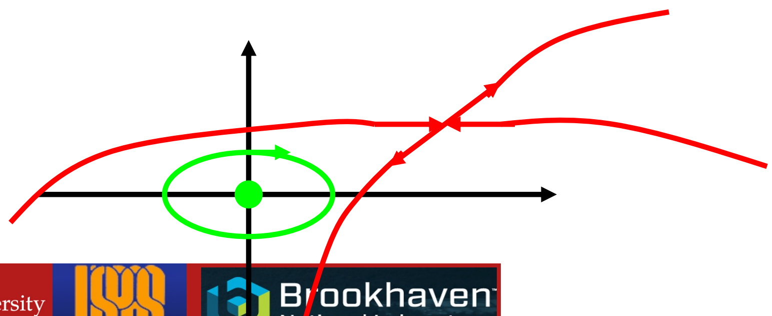
$$\begin{pmatrix}\Delta\hat{x}_{n+1}\\ \Delta\hat{x}'_{n+1}\end{pmatrix}=\begin{pmatrix}\cos\mu & \sin\mu\\ -\sin\mu & \cos\mu\end{pmatrix}\left[\begin{pmatrix}\Delta\hat{x}_n\\ \Delta\hat{x}'_n\end{pmatrix}-\begin{pmatrix}0\\ \frac{k_2 l_s}{2}\beta^{\frac{3}{2}}\Delta\hat{x}_n^2-4\tan\frac{\mu}{2}\Delta\hat{x}_n\end{pmatrix}\right]$$

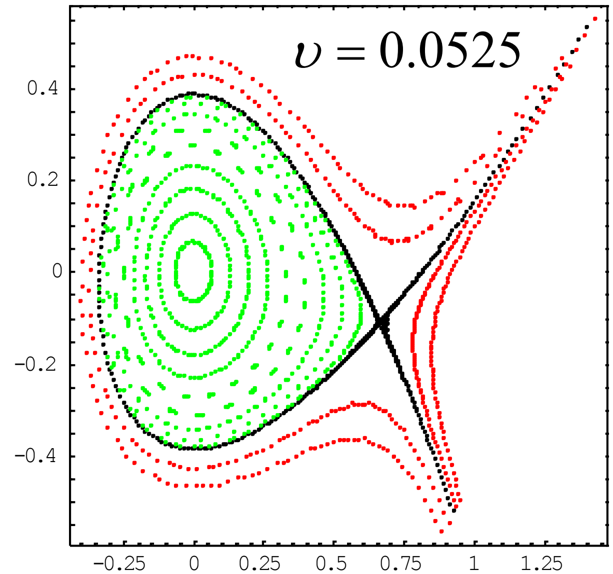
$$\begin{pmatrix}\Delta\hat{x}_{n+1}\\ \Delta\hat{x}'_{n+1}\end{pmatrix}=\begin{pmatrix}\cos\mu+4\sin\mu\tan\frac{\mu}{2} & \sin\mu\\ -\sin\mu+4\cos\mu\tan\frac{\mu}{2} & \cos\mu\end{pmatrix}\left[\begin{pmatrix}\Delta\hat{x}_n\\ \Delta\hat{x}'_n\end{pmatrix}-\begin{pmatrix}0\\ \frac{k_2 l_s}{2}\beta^{\frac{3}{2}}\Delta\hat{x}_n^2\end{pmatrix}\right]$$

Example of one sextupole:

$$\frac{1}{2}Tr[\underline{M}]=1-2\sin^2\frac{\mu}{2}+4\sin^2\frac{\mu}{2}=1+2\sin^2\frac{\mu}{2}\geq 1$$

The additional fixed point is unstable !



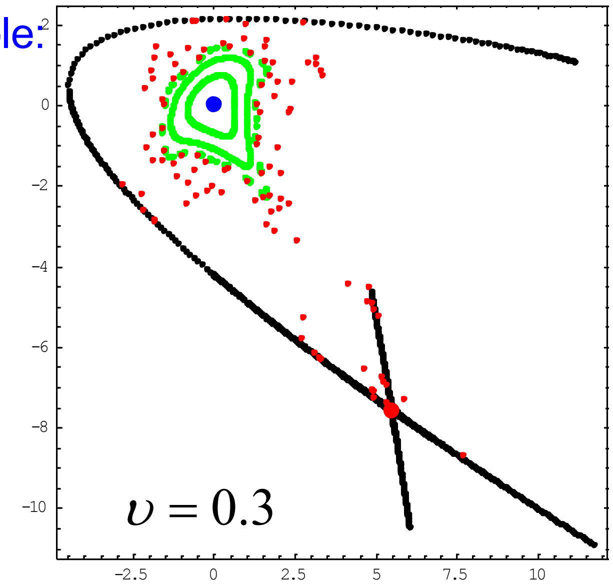


Sextupole Aperture

If the chormaticity is corrected by a single sextupole:

$$\xi_x = \xi_{0x} + \frac{1}{4\pi}\beta_x\eta_x k_2 l \approx 0$$

$$J_f = \frac{1}{2\beta^3}\left(\frac{4}{k_2 l_s}\frac{\tan\frac{\mu}{2}}{\cos\frac{\mu}{2}}\right)^2 \approx \frac{1}{2\beta}\left(\frac{\eta}{\xi_0\pi}\frac{\sin\frac{\mu}{2}}{\cos^2\frac{\mu}{2}}\right)^2$$



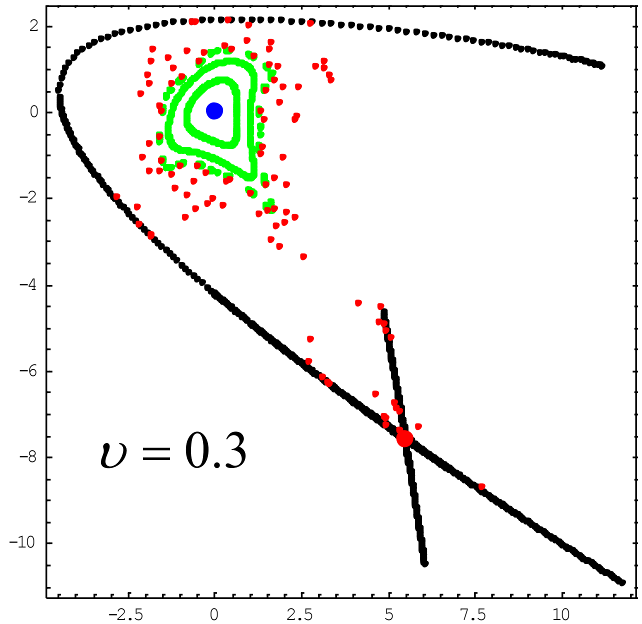
Often the dynamic aperture is much smaller than the fixed point indicates !

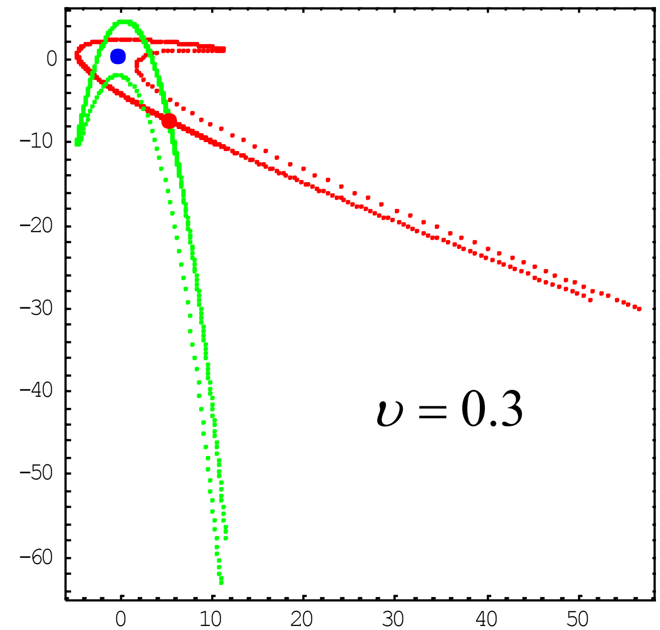
When many sextupoles are used: $\xi_{0x} + \frac{N}{4\pi}\beta_x\eta_x k_2 l \approx 0$

The sum of all k_2^2 is then reduced to about $\sum (k_2 l\beta)^2 \approx N(k_2 l\beta)^2 \approx \frac{1}{N}\left(\frac{4\pi}{\eta}\xi_0\right)^2$

The dynamic aperture is therefore greatly increased when distributed sextupoles are used.

Sextupole extraction





Due to the narrow region of unstable trajectories, sextupoles are used for slow particle extraction at a tune of 1/3.

The intersection of stable and unstable manifolds is a certain indication of chaos.