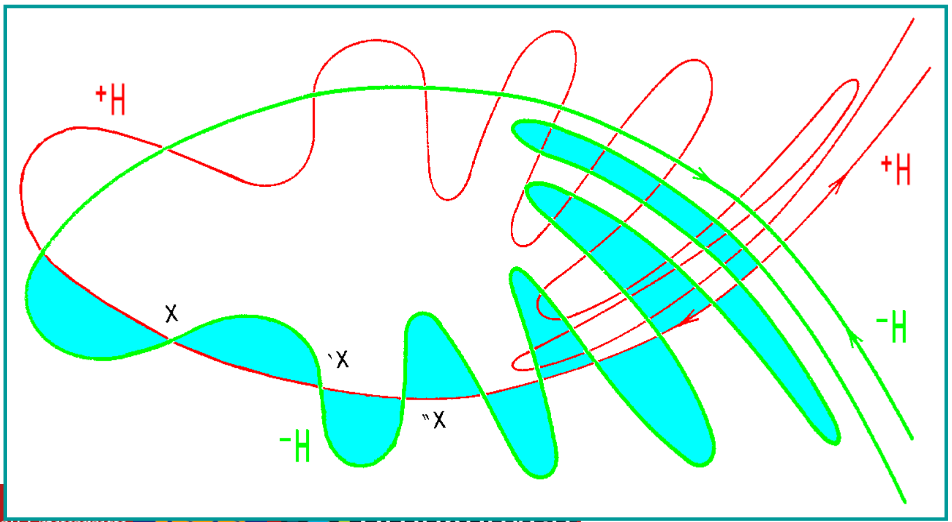
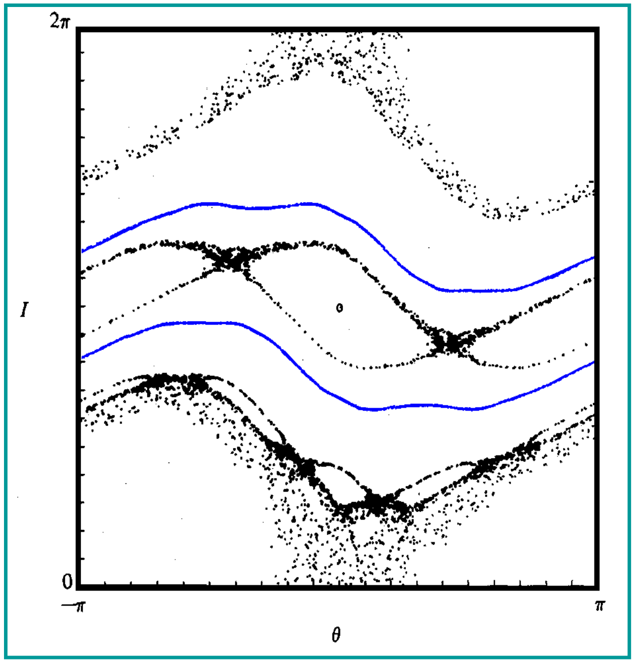
Homoclinic points

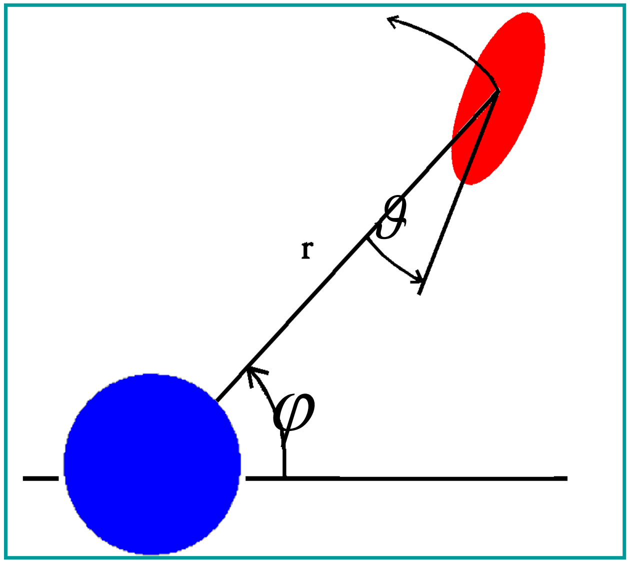
- At instable fixed points, there is a stable and an instabile invariant curve.
- Intersections of these curves (homoclinic points) lead to chaos.

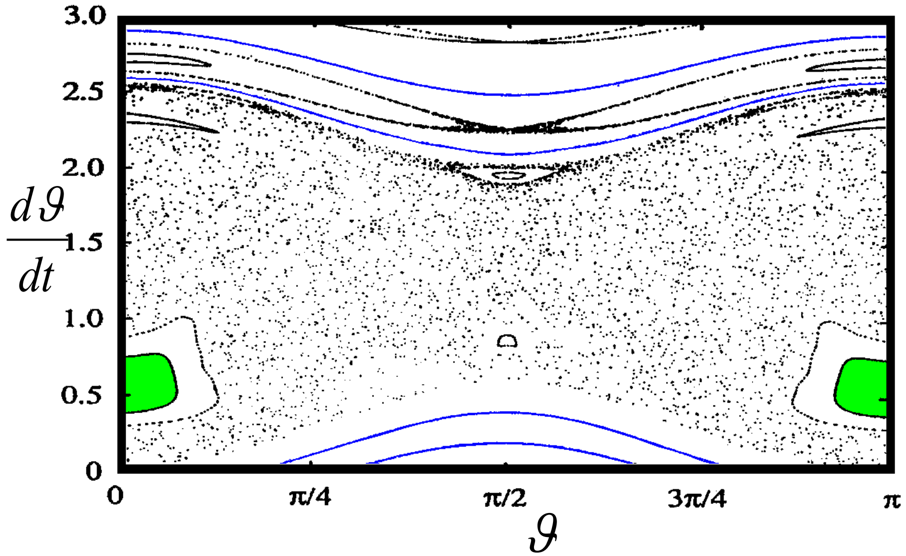




Hyperion: rotation around the vertical

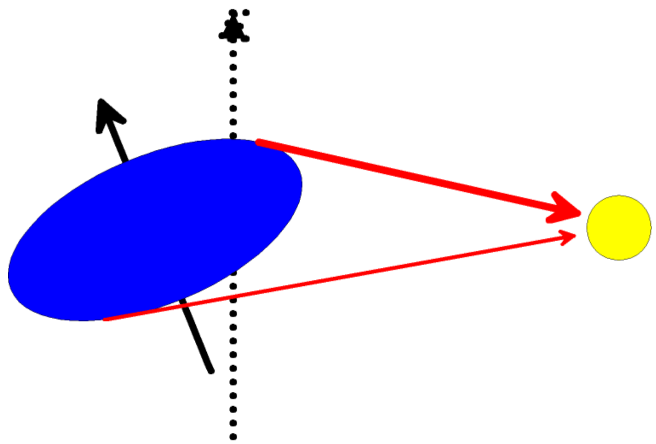
$$\frac{d^2(\vartheta+\varphi(t))}{dt^2} = -\alpha\left(\frac{a}{r(t)}\right)^3 \sin 2\vartheta$$

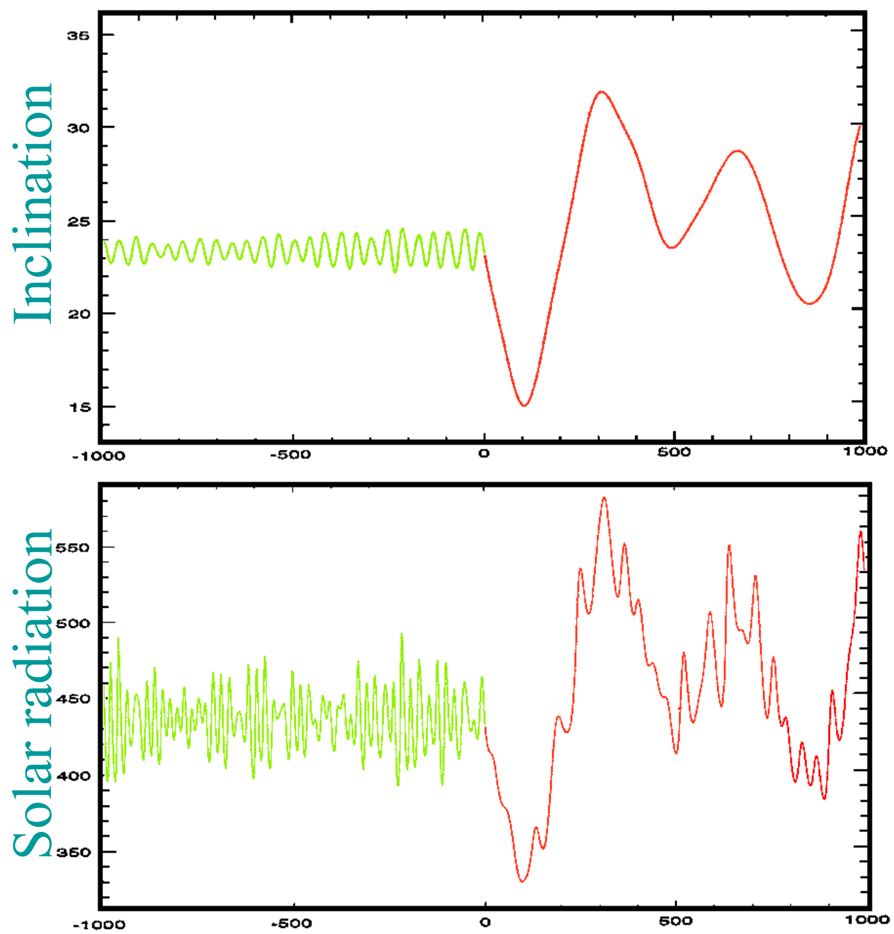




- On the path from Rotation to Libration around the Spin-Orbit-Coupling is a strong chaotic region.

Tilt of the earth





-1Ma with and without moon 1M
a

- Tidal forces from moon and sun cause a stabilization of the rotation axis.

Single resonance model

$$\frac{d}{d\vartheta} J = \sum_{n,m=-\infty}^{\infty} m H_{nm}(J) \sin(n\vartheta + m\varphi + \Psi_{nm}(J))$$

$$\frac{d}{d\vartheta} \varphi = \upsilon + \partial_J \sum_{n,m=-\infty}^{\infty} H_{nm}(J) \cos(n\vartheta + m\varphi + \Psi_{nm}(J))$$

Strong deviation from: $J = J_0 \ , \quad \varphi = \upsilon\vartheta + \varphi_0$

Occur when there is coherence between the perturbation and the phase space rotation: $n + m \frac{d}{ds}\varphi \approx 0$

Resonance condition: tune is rational $\boxed{n + m\,\upsilon = 0}$

On resonance the integral would increases indefinitely !

Neglecting all but the most important term

$$H(\varphi, J, \vartheta) \approx \upsilon J + H_{00}(J) + H_{nm}(J) \cos(n\vartheta + m\varphi + \Psi_{nm}(J))$$

Fixed points

$$\frac{d}{d\vartheta}J = mH_{nm}(J)\sin(n\vartheta + m\varphi + \Psi_{nm}(J))$$

$$\frac{d}{d\vartheta}\varphi = \upsilon + \Delta\upsilon(J) + \partial_J[H_{nm}(J)\cos(n\vartheta + m\varphi + \Psi_{nm}(J))]$$

$$\Phi = \frac{1}{m}[n\vartheta + m\varphi + \Psi_{nm}(J)] \ , \quad \delta = \upsilon + \frac{n}{m}$$

$$\frac{d}{d\vartheta}J = mH_{nm}(J)\sin(m\Phi) \ , \quad \frac{d}{d\vartheta}\Phi = \delta + \Delta\upsilon(J) + H'_{nm}(J)\cos(m\Phi)$$

$$H(\Phi, J, \vartheta) \approx \delta J + H_{00}(J) + H_{nm}(J)\cos(m\Phi)$$

Fixed points: $\frac{d}{d\vartheta}J = mH_{nm}(J_f)\sin(m\Phi_f) = 0 \quad \Rightarrow \quad \Phi_f = \frac{k}{m}\pi$

If $\delta + \Delta\upsilon(J_f) \pm H'_{nm}(J_f) = 0$ has a solution.

$$\frac{d}{d\vartheta}\Delta J = \pm m^2 H_{nm}(J_f)\Delta\Phi \ , \quad \frac{d}{d\vartheta}\Delta\Phi = [\Delta\upsilon'(J_f) \pm H''_{nm}(J_f)]\Delta J$$

Stable fixed point for: $H_{nm}(J_f)[H''_{nm}(J_f) \pm \Delta\upsilon'(J_f)] < 0$

Third integer resonances

Sextupole: $\boxed{\Delta f = -k_2 \frac{1}{2} x^2}$

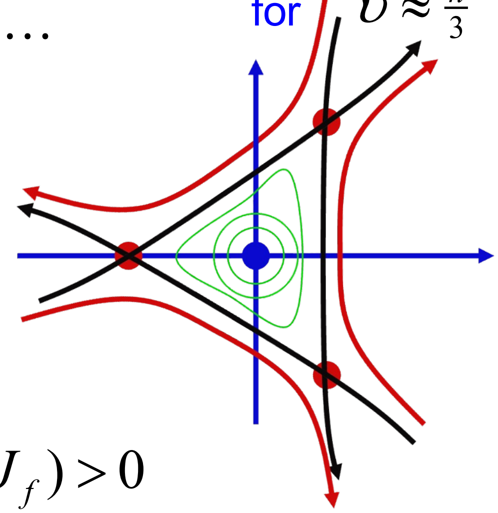
$$\Delta H = \frac{L}{2\pi} k_2 \frac{1}{3!} x^3 = \frac{L}{2\pi} k_2 \frac{1}{3!} \sqrt{2J\beta}^3 \sin^3(\tilde{\psi} + \varphi)$$

$$= \frac{L}{2\pi} k_2 \frac{1}{3!4} \sqrt{2J\beta}^3 [\sin(3[\tilde{\psi} + \varphi]) + 3\sin(\tilde{\psi} + \varphi)]$$

Simplification: one sextupole $\quad k_2(\vartheta) = k_2 \delta(\vartheta) = k_2 \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \cos(n\vartheta)$

$$\Delta H = \frac{L}{2\pi} k_2 \frac{1}{3!4} \sqrt{2J\beta}^3 \frac{1}{2\pi} \cos(-n\vartheta + 3\varphi + 3\tilde{\psi} - \tfrac{\pi}{2}) + \dots$$

for $\upsilon \approx \frac{n}{3}$



$$\Delta H \approx A_2 \sqrt{J}^3 \cos(3\Phi)$$

$$\left.\begin{array}{l} \Phi_f = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \dots \\ \delta \pm A_2 \frac{3}{2} \sqrt{J} = 0 \end{array}\right\} \begin{array}{l} \Phi_f = \frac{1}{3}\pi, \pi, \frac{5}{3}\pi \\ \text{for } \delta > 0 \end{array}$$

All these fixed points are instable since $H_{nm}(J_f) H''_{nm}(J_f) > 0$

Fourth integer resonances

Octupole: $\Delta f = -k_3 \frac{1}{3!} x^3$, $\Delta H = \frac{L}{2\pi} k_3 \frac{1}{4!} x^4 = \frac{L}{2\pi} k_3 \frac{1}{3!} J^2 \beta^2 \sin^4(\tilde{\psi} + \varphi)$

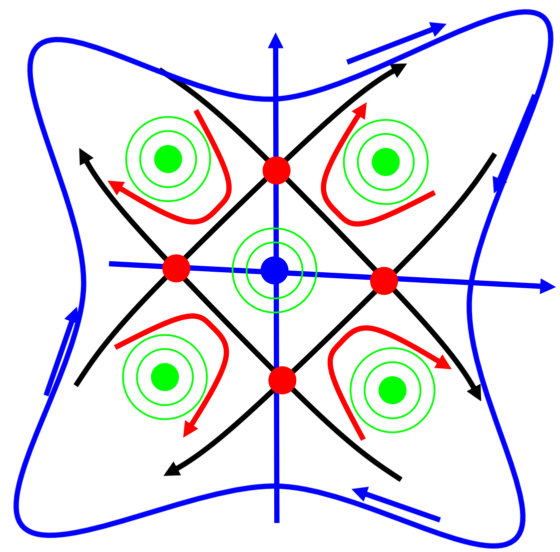
$$= \frac{L}{2\pi} k_3 \frac{1}{3!8} J^2 \beta^2 [\cos(4[\tilde{\psi} + \varphi]) - 4\cos(2[\tilde{\psi} + \varphi]) + 3]$$

Simplification: one octupole $k_3(\vartheta) = k_3\delta(\vartheta) = k_3 \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \cos(n\vartheta)$

$\Delta H \approx A_3 J^2 [3 + \cos(4\Phi)]$ for $\upsilon \approx \frac{n}{4}$

$\Phi_f = 0, \frac{1}{4}\pi, \frac{2}{4}\pi, \ldots$ Either 8 fixed points: $\delta < 0$

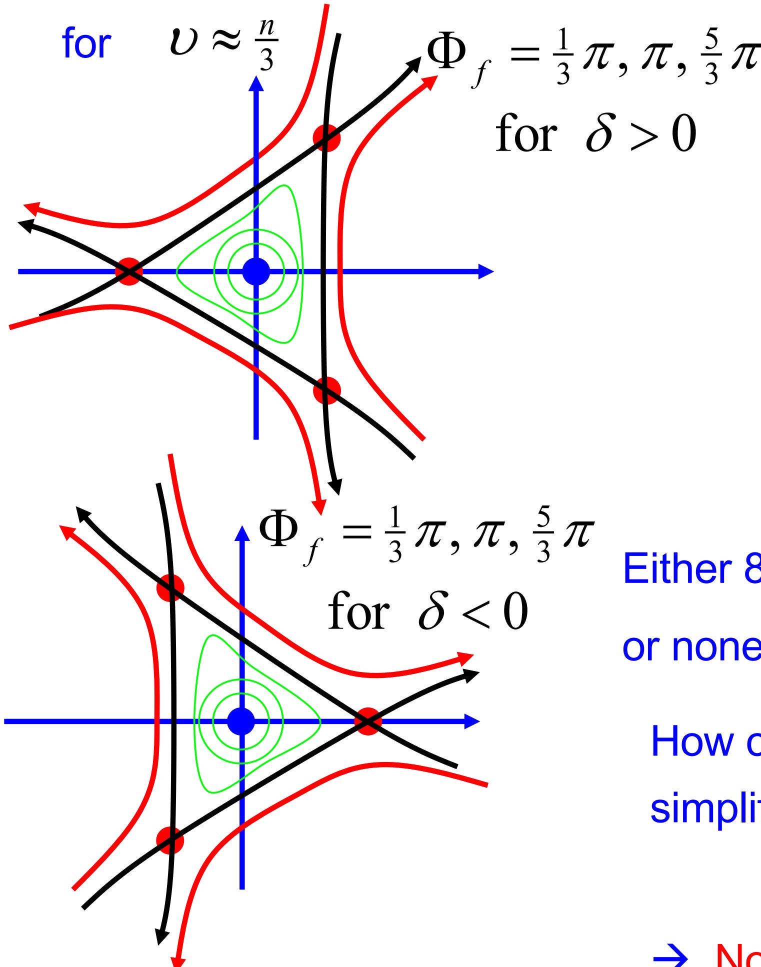
$\delta + A_3 2J(3 \pm 1) = 0$ or none for: $\delta > 0$

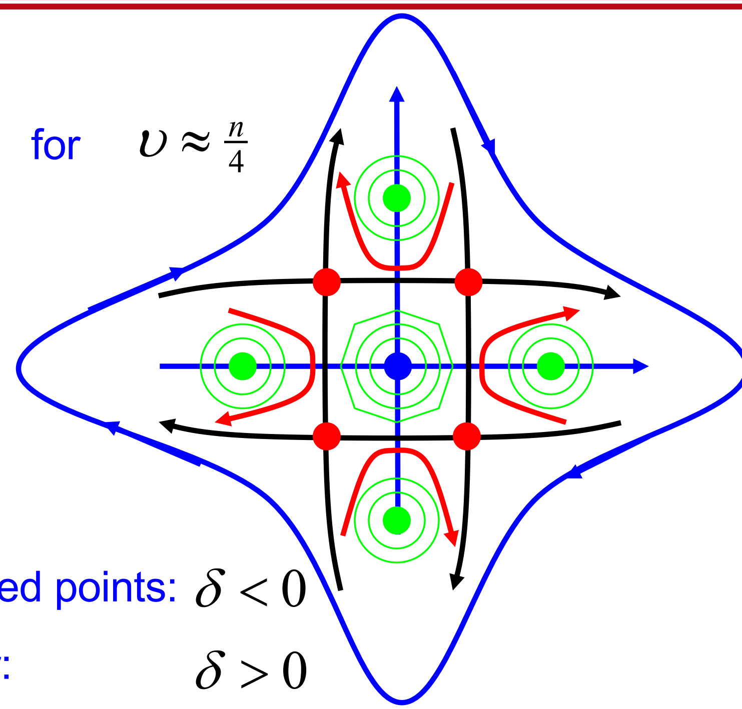


$$H_{nm}(J_f)[H''_{nm}(J_f) \pm \Delta\upsilon'(J_f)] < 0$$

Stability for $(2A_3J)^2[1 \pm 3] < 0$,
i.e. for the 4 outer fixed points.

Particle motion in the single resonance model





Either 8 fixed points: $\delta < 0$

or none for: $\delta > 0$

How can the motion inside the fixed points be simplified for a real accelerator ?

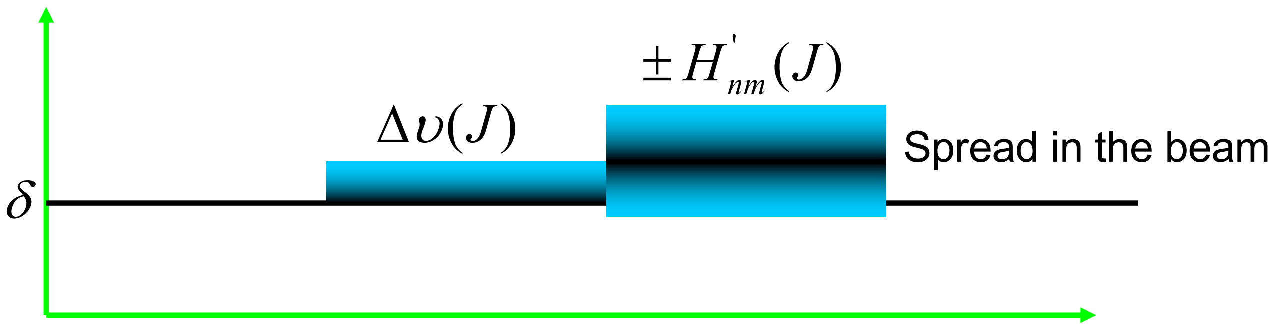
→ Normal From Theory

Resonance width (strength)

Fixed points: $\frac{d}{d\vartheta} J = mH_{nm}(J_f)\sin(m\Phi_f) = 0 \quad \Rightarrow \quad \Phi_f = \frac{k}{m}\pi$

If $\delta + \Delta\upsilon(J_f) \pm H'_{nm}(J_f) = 0$ has a solution.

δ has to avoid the region $\delta + \Delta\upsilon(J) \pm H'_{nm}(J) = 0$ for all particles.



Assuming that the tune shift and perturbation are monotonous in J:

This tune region has the width $\Delta_{nm} = 2\,|\,H'_{nm}(J_{\max})|$ for strong resonances.

Δ_{nm} Is called Resonance Width, Resonance Strength, or Stop-Band Width

Coupling resonances

$$\tfrac{d}{d\vartheta}J_x = \cos(\tilde{\psi}_x + \varphi_x)\sqrt{2J_x\beta_x}\Delta f_x \tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi_x = \upsilon_x - \sin(\tilde{\psi}_x + \varphi_x)\sqrt{\tfrac{\beta_x}{2J_x}}\Delta f_x \tfrac{L}{2\pi}$$

$$\tfrac{d}{d\vartheta}J_y = \cos(\tilde{\psi}_y + \varphi_y)\sqrt{2J_y\beta_y}\Delta f_y \tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi_y = \upsilon_y - \sin(\tilde{\psi}_y + \varphi_y)\sqrt{\tfrac{\beta_y}{2J_y}}\Delta f_y \tfrac{L}{2\pi}$$

$$\tfrac{d}{d\vartheta}\vec{\varphi} = \vec{\partial}_J H \quad , \quad \tfrac{d}{d\vartheta}\vec{J} = -\vec{\partial}_\varphi H \quad , \quad H(\vec{\varphi},\vec{J},\vartheta) = \vec{\upsilon}\cdot\vec{J} - \tfrac{L}{2\pi}\int_0^{\vec{x}} \Delta\vec{f}(\hat{\vec{x}},s)\,d\hat{\vec{x}}$$

The integral form can be chosen since it is path independent. This is due to the Hamiltonian nature of the force: $\Delta f_{x,y}(x,y,s) = -\partial_{x,y}\Delta H(x,y,s)$

Single Resonance model for two dimensions means retaining only the amplitude dependent tune shift and one term in the two dimensional Fourier expansion:

$$H(\vec{\varphi},\vec{J},\vartheta) = \vec{\upsilon}\cdot\vec{J} + H_{00}(\vec{J}) + H_{n\vec{m}}(\vec{J})\cos(n\vartheta + m_x\varphi_x + m_y\varphi_y + \Psi_{n\vec{m}}(\vec{J}))$$

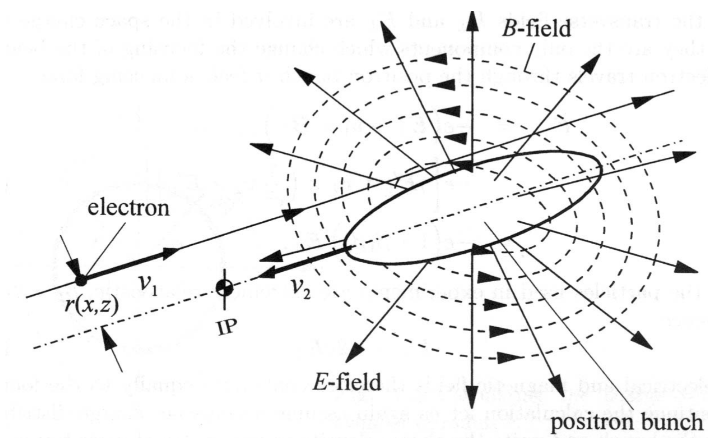
For $n + m_x\upsilon_x + m_y\upsilon_y \approx 0$

$$m_x\varphi_x + m_y\varphi_y = \vec{m}\cdot\vec{\varphi}$$

Beam-beam tune shift

The force that acts from one beam to the other during collisions is focusing or defocusing in both planes for small distances.

For large distances it is very nonlinear.



The effects of E and B forces add.
Whereas they subtract for
co-moving particles.

$$\Delta \nu_x^{(1)} = \frac{r_{\rm cl}^{(1)} N_{\rm cpb}^{(2)}}{2\pi} \frac{\beta_x^{(1)}}{\sigma_x^{(2)}(\sigma_x^{(2)} + \sigma_y^{(2)})}$$

Beam-beam force

$$\left.\begin{aligned}\rho_{\text{lab}} &= \frac{1}{2\pi\sigma_x\sigma_y} e^{-(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2})} \rho_{\text{lab}\,z}(z+v^{(2)}t)\\ \vec{j}_{\text{lab}} &= \vec{\beta} c \rho_{\text{lab}\,z}\end{aligned}\right\}\left\{\begin{aligned}\rho_{\text{rest}} &= \frac{1}{2\pi\sigma_x\sigma_y} e^{-(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2})} \rho_z(z)\\ \vec{j}_{\text{rest}} &= 0\end{aligned}\right.$$

$$E_x(x,y,z) = \frac{Q\rho_z}{2\pi\varepsilon_0}\frac{1}{2\pi\sigma_x\sigma_y}\int \frac{x-x_0}{(x-x_0)^2+(y-y_0)^2} e^{-(\frac{x_0^2}{2\sigma_x^2}+\frac{y_0^2}{2\sigma_y^2})} dx_0 dy_0$$

$$\approx \frac{Q\rho_z}{2\pi\varepsilon_0}\frac{1}{2\pi\sigma_x\sigma_y}\int \frac{u}{u^2+v^2} e^{-(\frac{(x-u)^2}{2\sigma_x^2}+\frac{(y-v)^2}{2\sigma_y^2})} dudv$$

$$= \frac{Q\rho_z}{2\pi\varepsilon_0}\frac{1}{2\pi\sigma_x\sigma_y}(\sigma_x^2\partial_x + x)\int \frac{1}{u^2+v^2} e^{-(\frac{(x-u)^2}{2\sigma_x^2}+\frac{(y-v)^2}{2\sigma_y^2})} dudv$$

$$= \frac{Q\rho_z}{2\pi\varepsilon_0}\frac{1}{2\pi\sigma_x\sigma_y}(\sigma_x^2\partial_x + x)\int [\int_0^\infty e^{-t(u^2+v^2)} dt] e^{-(\frac{(x-u)^2}{2\sigma_x^2}+\frac{(y-v)^2}{2\sigma_y^2})} dudv$$

$$= \frac{Q\rho_z}{2\pi\varepsilon_0}\frac{1}{2\pi\sigma_x\sigma_y}(\sigma_x^2\partial_x + x)\int e^{-(u^2(\frac{1}{2\sigma_x^2}+t)-\frac{2xu}{2\sigma_x^2}+v^2(\frac{1}{2\sigma_y^2}+t)-\frac{2yv}{2\sigma_y^2})} e^{-(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2})} dudvdt$$

Beam-beam force

$$E_x(x,y,z) \approx \frac{Q\rho_z}{4\pi\varepsilon_0}\frac{1}{\pi\sigma_x\sigma_y}(\sigma_x^2\partial_x + x)\int e^{-(u^2(\frac{1}{2\sigma_x^2}+t)-\frac{2xu}{2\sigma_x^2}+v^2(\frac{1}{2\sigma_y^2}+t)-\frac{2yv}{2\sigma_y^2})}e^{-(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2})}dudvdt$$

$$= \frac{Q\rho_z}{4\pi\varepsilon_0}\frac{1}{\pi\sigma_x\sigma_y}(\sigma_x^2\partial_x + x)\int e^{-(\tilde{u}^2(\frac{1}{2\sigma_x^2}+t)+\tilde{v}^2(\frac{1}{2\sigma_y^2}+t))}e^{-(\frac{x^2}{2\sigma_x^2}-\frac{x^2}{4\sigma_x^4(\frac{1}{2\sigma_x^2}+t)}+\frac{y^2}{2\sigma_y^2}-\frac{y^2}{4\sigma_y^4(\frac{1}{2\sigma_y^2}+t)})}dudvdt$$

$$= \frac{Q\rho_z}{4\pi\varepsilon_0}2(\sigma_x^2\partial_x + x)\int_0^\infty \frac{e^{-(\frac{x^2}{2\sigma_x^2+\frac{1}{t}}+\frac{y^2}{2\sigma_y^2+\frac{1}{t}})}}{\sqrt{2\sigma_x^2+\frac{1}{t}}\sqrt{2\sigma_y^2+\frac{1}{t}}}\frac{dt}{t}$$

$$= -\frac{Q\rho_z}{4\pi\varepsilon_0}\partial_x\int_0^\infty \frac{e^{-(\frac{x^2}{2\sigma_x^2+q}+\frac{y^2}{2\sigma_y^2+q})}}{\sqrt{2\sigma_x^2+q}\sqrt{2\sigma_y^2+q}}dq = -\partial_x U$$

$$\approx \frac{Q\rho_z}{2\pi\varepsilon_0}x\int_0^\infty \frac{1}{\sqrt{2\sigma_x^2+q}^3\sqrt{2\sigma_y^2+q}}dq = \frac{Q\rho_z}{2\pi\varepsilon_0}\frac{1}{\sigma_x(\sigma_x+\sigma_y)}x$$

Beam-beam force

$$\vec{E}(x,y,z)\approx\frac{Q\rho_z}{2\pi\varepsilon_0}\frac{1}{\sigma_x+\sigma_y}\begin{pmatrix}\frac{x}{\sigma_x}\\ \frac{y}{\sigma_y}\end{pmatrix}$$

$$\vec{B}(x,y,z)=0$$

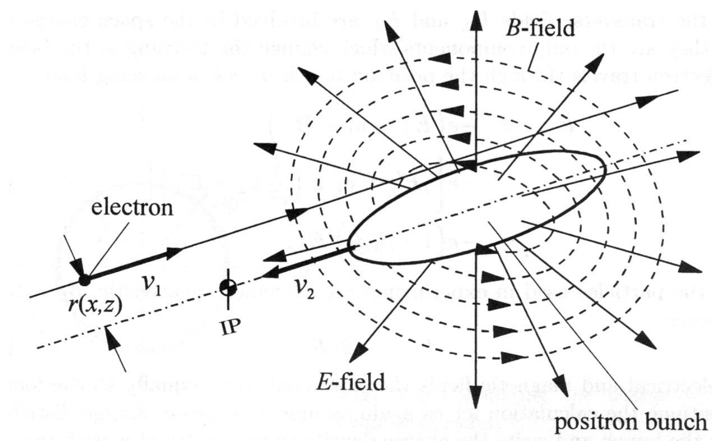
$$\left\{\begin{array}{l}\vec{E}_{\text{lab}}(x,y,z)\approx\frac{Q\rho_{\text{lab}\,z}}{2\pi\varepsilon_0}\frac{1}{\sigma_x+\sigma_y}\begin{pmatrix}\frac{x}{\sigma_x}\\ \frac{y}{\sigma_y}\end{pmatrix}\\ \vec{B}_{\text{lab}}(x,y,z)=\frac{1}{c}\vec{\beta}\times\vec{E}_{\text{lab}}(x,y,z)\end{array}\right.$$

$$\Delta\vec{p}(x,y)=\int\vec{F}(x,y,v^{(1)}t)dt$$

$$\approx\frac{Q}{2\pi\varepsilon_0}\frac{1}{\sigma_x+\sigma_y}\begin{pmatrix}\frac{x}{\sigma_x}\\ \frac{y}{\sigma_y}\end{pmatrix}\int\rho_{\text{lab}\,z}(v^{(1)}t+v^{(2)}t)dt(1+\beta^{(1)}\beta^{(2)})$$

$$=\frac{Q}{2\pi\varepsilon_0}\frac{1}{\sigma_x+\sigma_y}\begin{pmatrix}\frac{x}{\sigma_x}\\ \frac{y}{\sigma_y}\end{pmatrix}\frac{1+\beta^{(1)}\beta^{(2)}}{v^{(1)}+v^{(2)}}$$

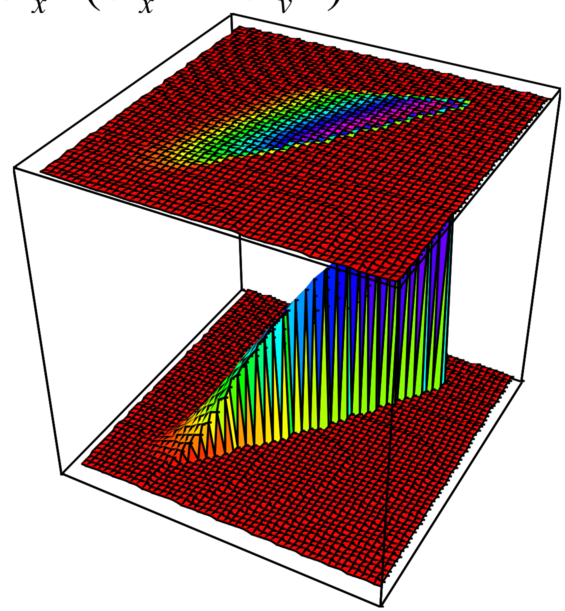
$$\approx\frac{Q}{2\pi\varepsilon_0 c}\frac{1}{\sigma_x+\sigma_y}\begin{pmatrix}\frac{x}{\sigma_x}\\ \frac{y}{\sigma_y}\end{pmatrix}$$



Beam-beam tune shift

$$\begin{pmatrix} \Delta x' \\ \Delta y' \end{pmatrix} = \frac{\Delta \vec{p}}{p} \approx \frac{q^{(1)}Q}{2\pi\varepsilon_0 p^{(1)}c} \frac{1}{\sigma_x + \sigma_y} \begin{pmatrix} \frac{x}{\sigma_x} \\ \frac{y}{\sigma_y} \end{pmatrix} = \begin{pmatrix} k_x x \\ k_y y \end{pmatrix}$$

$$\Delta \nu_x^{(1)} \approx \frac{q^{(1)}Q}{8\pi^2\varepsilon_0 p^{(1)}c} \frac{\beta_x^{(1)}}{\sigma_x(\sigma_x + \sigma_y)} \approx \frac{r_{cl}^{(1)} N_{cpb}^{(2)}}{2\pi} \frac{\beta_x^{(1)}}{\sigma_x^{(2)}(\sigma_x^{(2)} + \sigma_y^{(2)})}$$



Tune spread over the beam,
amplitude dependent tune shift
and the tune shift cravat