Transport maps of cavities

(1) Linearization: $E_r(r,z,t) =_1 -\frac{r}{2}\partial_z E_z(0,z,t) \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{E} = 0$

$$B_\varphi(r,z,t) =_1 \tfrac{1}{c^2}\tfrac{r}{2}\partial_t E_z(0,z,t) \quad \Rightarrow \quad \vec{\nabla}\times\vec{B} = \tfrac{1}{c^2}\partial_t\vec{E}$$

(2) Equation of motion:

$$a = \frac{p_x}{p_0}$$

$$a' = \tfrac{1}{p_0 v}(F_x - aF_z) = -\tfrac{q}{p_0 v}[\tfrac{r}{2}(\partial_z + \tfrac{v}{c^2}\partial_t)E_z + aE_z]$$

$$= -\tfrac{q}{p_0 v}[\tfrac{r}{2}(\tfrac{d}{dz} - \tfrac{1}{v}(1-\tfrac{v^2}{c^2})\partial_t)E_z + aE_z] \approx -\tfrac{1}{p_0}[r\tfrac{1}{2}p_0'' + ap_0']$$

$$u = r\sqrt{p}$$

p denotes p_0 for simplicity

$$u' = a\sqrt{p} + r\sqrt{p}\,\tfrac{p'}{2p}$$

Focusing !

$$u'' \approx -\tfrac{1}{\sqrt{p}}(r\tfrac{1}{2}p'' + ap') + a\sqrt{p}\,\tfrac{p'}{p} + r(\tfrac{p''}{2\sqrt{p}} - \tfrac{p'}{4\sqrt{p}^3}) = -u\left(\tfrac{p'}{2p}\right)^2$$

Transport maps of cavities

(3) Average focusing over one period with relatively little energy change:

$$u'' \approx -u\frac{\Delta^2/4}{p^2}\,,\quad \Delta = \sqrt{\langle p'^2\rangle}$$

(4) Continuous energy change:

$$p' \approx \Omega\,,\quad \Omega = \langle p'\rangle$$

$$\frac{d^2}{dp^2}u \approx \frac{1}{\Omega^2}u'' \approx -u\frac{(\Delta/\Omega)^2}{4p^2}$$

$$\frac{d^2}{dp^2}(r\sqrt{p}) = \frac{d^2}{dp^2}r\sqrt{p} + \frac{d}{dp}r\frac{1}{\sqrt{p}} - r\frac{1}{4\sqrt{p}^3} \approx -r\frac{(\Delta/\Omega)^2}{4\sqrt{p}^3}$$

$$\frac{d^2}{dp^2}r + \frac{d}{dp}r\frac{1}{p} \approx -r\frac{(\Delta/\Omega)^2-1}{4p^2} = -r\frac{\varepsilon^2}{p^2}$$

$$r(p) = \eta(-\ln(p)) \quad\Rightarrow\quad \frac{d^2}{dp^2}r = \frac{1}{p^2}\eta' + \frac{1}{p^2}\eta'' = \frac{1}{p^2}\eta' - \eta\frac{\varepsilon^2}{p^2}$$

Transport maps of traveling wave cavities

$$\eta'' = -\varepsilon^2\eta\,,\quad \eta(\xi) = A\cos(\varepsilon\xi) - B\sin(\varepsilon\xi)$$

$$\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{p'}{p} \end{pmatrix} \begin{pmatrix} \cos(\varepsilon\ln(p)) & \sin(\varepsilon\ln(p)) \\ -\sin(\varepsilon\ln(p)) & \cos(\varepsilon\ln(p)) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{p'}{p} \end{pmatrix} \begin{pmatrix} \cos(\varepsilon\ln(\frac{p}{p_i})) & \sin(\varepsilon\ln(\frac{p}{p_i})) \\ -\sin(\varepsilon\ln(\frac{p}{p_i})) & \cos(\varepsilon\ln(\frac{p}{p_i})) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{p_i}{p'} \end{pmatrix} \begin{pmatrix} r_i \\ a_i \end{pmatrix}$$

$$E_z = \sum_{n=0}^{\infty} g_n \cos(n\tfrac{2\pi}{L}z + \alpha_n)\cos(\omega t - kz + \varphi_0)$$

$$= \sum_{n=0}^{\infty} g_n \cos(n\tfrac{2\pi}{L}z + \alpha_n)\cos(\varphi_0)\,,\quad \alpha_0 = 0$$

$$\langle p'\rangle = g_0\cos(\varphi_0)\,, \langle p'^2\rangle = \left(g_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} g_n^2\right)\cos^2(\varphi_0) \;\Rightarrow\; \varepsilon = \frac{1}{2}\sqrt{\frac{1}{2}\sum_{n=1}^{\infty}\frac{g_n^2}{g_0^2}}$$

Transport maps of standing wave cavities

$$\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{p'}{p} \end{pmatrix} \begin{pmatrix} \cos(\varepsilon \ln(\frac{p}{p_i})) & \sin(\varepsilon \ln(\frac{p}{p_i})) \\ -\sin(\varepsilon \ln(\frac{p}{p_i})) & \cos(\varepsilon \ln(\frac{p}{p_i})) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{p_i}{p'} \end{pmatrix} \begin{pmatrix} r_i \\ a_i \end{pmatrix}$$

$$E_z = \sum_{n=-\infty}^{\infty} g_n e^{in\frac{2\pi}{L}z} \cos(kz)\cos(\omega t + \varphi_0) \ , \quad \omega t = kz \, , k = m\tfrac{\pi}{L} \, , \quad g_{-n} = g_n^*$$

$$= \tfrac{1}{4} \sum_{n=-\infty}^{\infty} g_n [e^{i(n+m)\frac{2\pi}{L}z+\varphi_0} + e^{i(n-m)\frac{2\pi}{L}z-\varphi_0} + e^{in\frac{2\pi}{L}z+\varphi_0} + e^{in\frac{2\pi}{L}z-\varphi_0}]$$

$$= \tfrac{1}{4} \sum_{n=-\infty}^{\infty} [g_{n-m} e^{i\varphi_0} + g_{n+m} e^{-i\varphi_0} + 2 g_n \cos(\varphi_0)] e^{in\frac{2\pi}{L}z} = \sum_{n=-\infty}^{\infty} f_n e^{in\frac{2\pi}{L}z}$$

$$\left.\begin{array}{l} \langle p' \rangle = f_0 \, , \\ \langle p'^2 \rangle = \sum_{n=0}^{\infty} |f_n|^2 \end{array}\right\} \varepsilon = \tfrac{1}{2} \sqrt{\sum_{n=0}^{\infty} \left| \tfrac{f_n}{f_0} \right|^2 - 1}$$

Phase space preservation in cavities

Average focusing over one period with relatively little energy change:

$$\begin{pmatrix} r \\ a \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & \frac{p'}{p} \end{pmatrix} \begin{pmatrix} \cos(\varepsilon \ln(\frac{p}{p_i})) & \sin(\varepsilon \ln(\frac{p}{p_i})) \\ -\sin(\varepsilon \ln(\frac{p}{p_i})) & \cos(\varepsilon \ln(\frac{p}{p_i})) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{p_i}{p'} \end{pmatrix}}_{\underline{M}} \begin{pmatrix} r_i \\ a_i \end{pmatrix}$$

$$\det(\underline{M}) = \frac{p_i}{p}$$

Because the determinant is not 1, the phase space volume is no longer conserved but changes by p_0/p.

A new propagation and definition of Twiss parameters is therefore needed:

$$r = \sqrt{2J \frac{1}{\beta_r \gamma_r} \beta} \sin(\psi + \phi_0)$$

Twiss parameters in accelerating cavities

$$\alpha = -\tfrac{1}{2}\beta' , \quad \gamma = \tfrac{1+\alpha^2}{\beta}$$

$$a = r' = \sqrt{2J\,\tfrac{mc}{p}}[-\tfrac{2\alpha+\beta\frac{p'}{p}}{2\sqrt{\beta}}\sin(\psi+\phi_0) + \tfrac{\beta\psi'}{\sqrt{\beta}}\cos(\psi+\phi_0)]$$

$$a' \approx -\tfrac{1}{p}[r(pK+\tfrac{1}{2}p'') + ap']$$

$$a' = -\sqrt{2J\,\tfrac{mc}{\beta p}}\begin{pmatrix} \frac{(\beta\psi')^2+\alpha^2}{\beta} + \alpha' - \alpha\frac{p'}{p} + \beta\frac{p''}{2p} - \beta\frac{3p'^2}{4p^2} \\ 2\alpha\psi' + \beta\frac{p'}{p}\psi' - \beta\psi'' \end{pmatrix}\begin{pmatrix} \sin(\psi+\phi_0) \\ \cos(\psi+\phi_0) \end{pmatrix}$$

$$= -\sqrt{2J\,\tfrac{mc}{\beta p}}\begin{pmatrix} \beta(K+\frac{1}{2}\frac{p''}{p}) - (\alpha+\beta\frac{p'}{2p})\frac{p'}{p} \\ \beta\frac{p'}{p}\psi' \end{pmatrix}\begin{pmatrix} \sin(\psi+\phi_0) \\ \cos(\psi+\phi_0) \end{pmatrix}$$

$$\Rightarrow \quad \psi' = \frac{A}{\beta}, \text{choice}: A = 1$$

$$\alpha' + \gamma = \beta\left[K + \left(\tfrac{p'}{2p}\right)^2\right]$$

$$\begin{pmatrix} r \\ a \end{pmatrix} = \sqrt{2J\,\tfrac{mc}{p}}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha+\beta\frac{p'}{2p}}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi+\phi_0) \\ \cos(\psi+\phi_0) \end{pmatrix}$$

Beta functions in accelerating cavities

$$\begin{pmatrix} r \\ a \end{pmatrix} = \sqrt{2J_n \tfrac{mc}{p}} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\tilde{\alpha}}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos(\psi + \phi_0) \\ \sin(\psi + \phi_0) \end{pmatrix} \quad , \quad \tilde{\alpha} = \alpha + \beta \tfrac{p'}{2p}$$

For systems with changing energy one uses the normalized Courant-Snyder invariant **J_n = J b_r g_r**

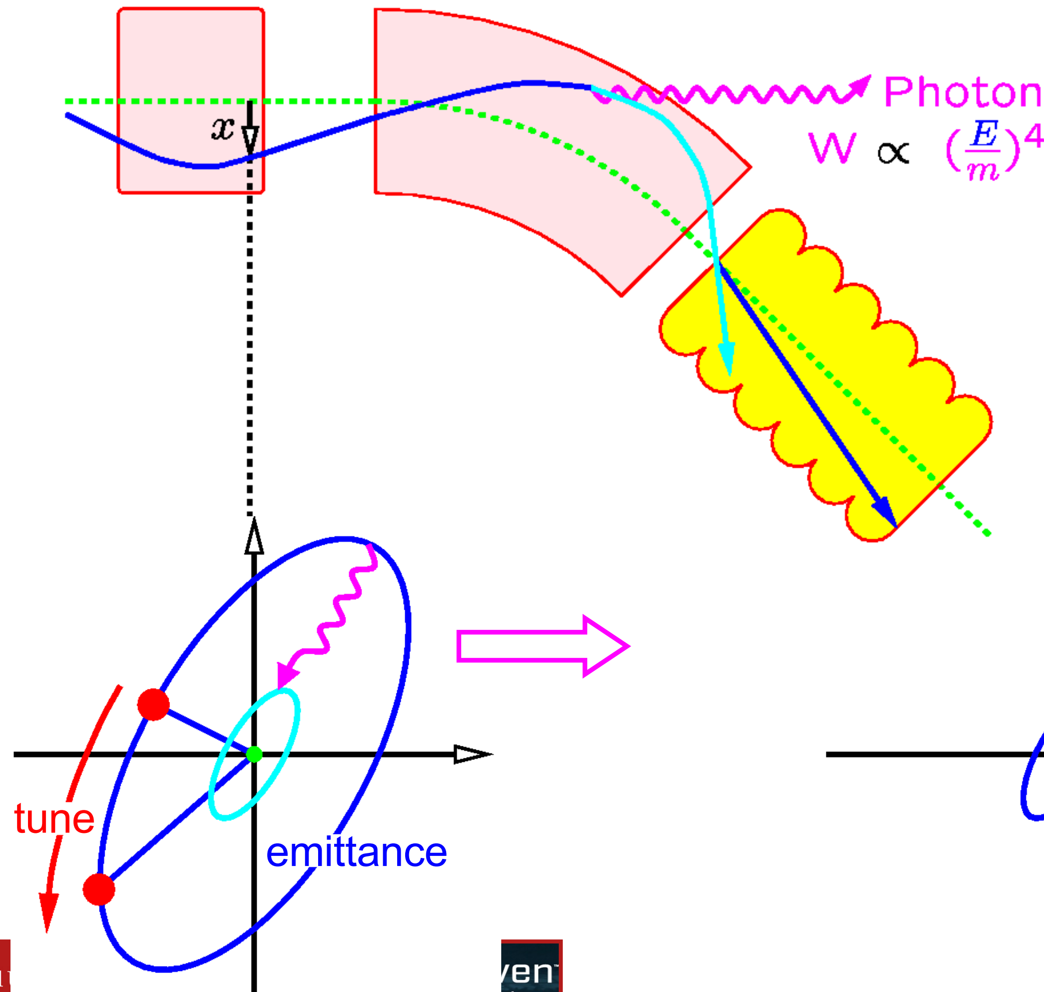
$$(r \quad a) \begin{pmatrix} \frac{1}{\sqrt{\beta}} & \frac{\tilde{\alpha}}{\sqrt{\beta}} \\ 0 & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\tilde{\alpha}}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} r \\ a \end{pmatrix} \frac{p}{2mc} = (r \quad a) \begin{pmatrix} \frac{1+\tilde{\alpha}^2}{\beta} & \tilde{\alpha} \\ \tilde{\alpha} & \beta \end{pmatrix} \begin{pmatrix} r \\ a \end{pmatrix} \frac{p}{2mc} = J_n$$

Reasons:

(1) **J** is the phase space amplitude of a particle in **(x , a)** phase space, which is the area in phase space (over 2p) that its coordinate would circumscribe during many turns in a ring. However, **$a=p_x/p_0$** is not conserved when p0 changes in a cavity. Therefore J is not conserved.

(2) **J_n = J p_0/mc** is therefore proportional to the corresponding area in **(x , p_x)** phase space, and is thus conserved.

Radiative damping of the transverse emittance



1) Energy emission:

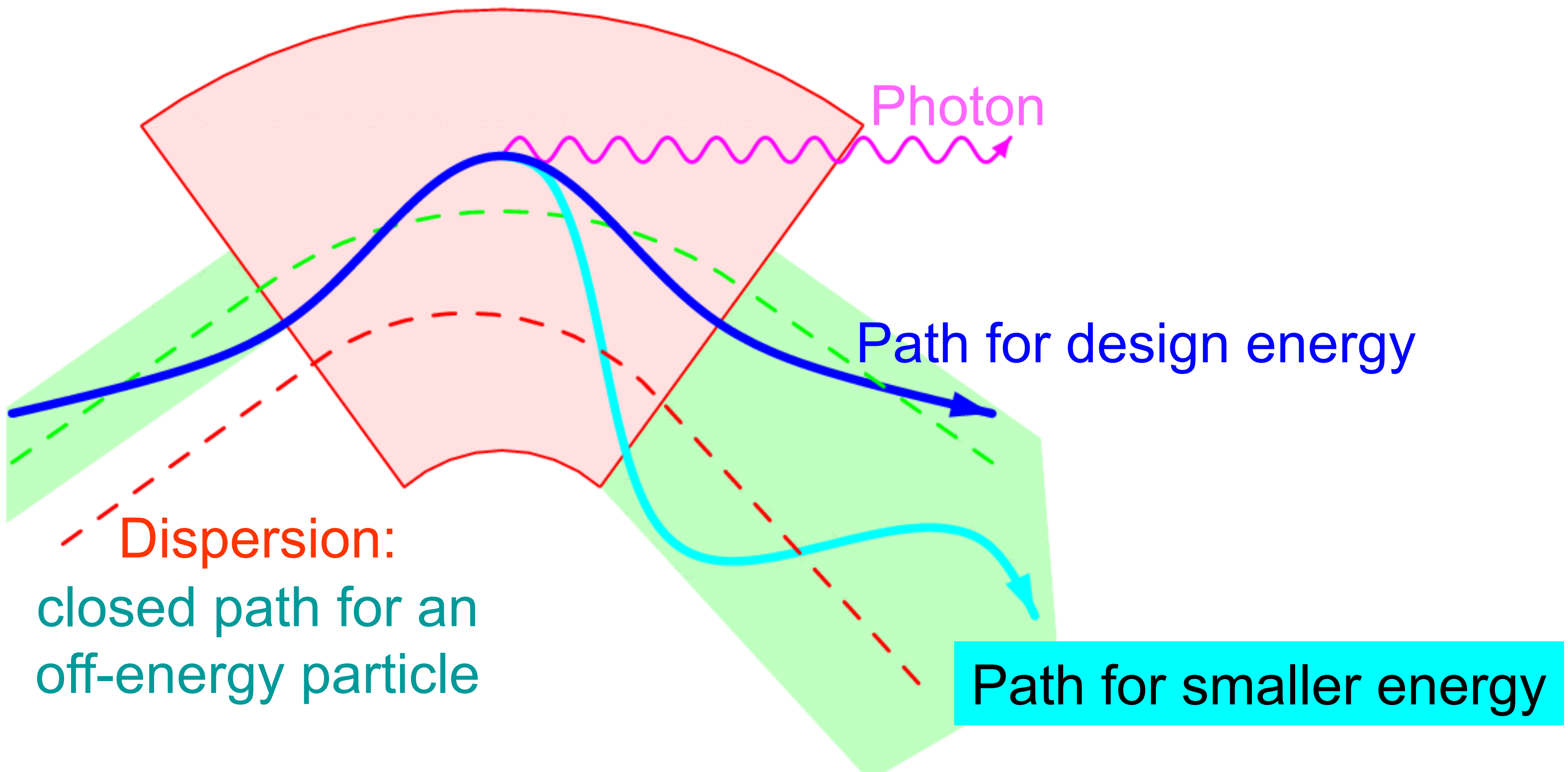
Damping of the beam size

2) Stochastic emission:

Excitation of the beam size

Radiative excitation of the transverse emittance

Smaller dispersion → Smaller emittance



Radiative damping of the longitudinal emittance

1) Energy emission: Particles with larger energy radiate more, leading them closer to the average energy.

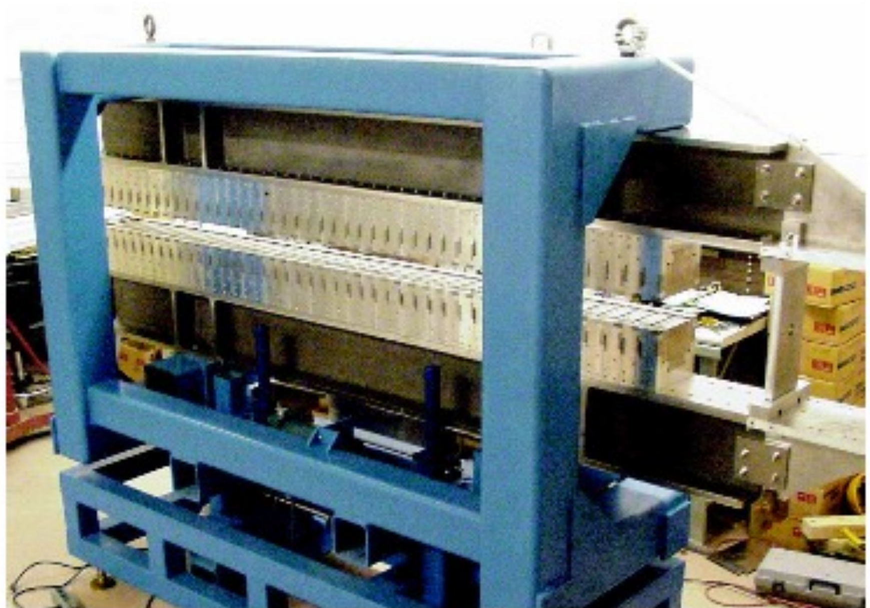
2) Stochastic emission: Random noise in the energy of emitted photons lead to an energy spread.

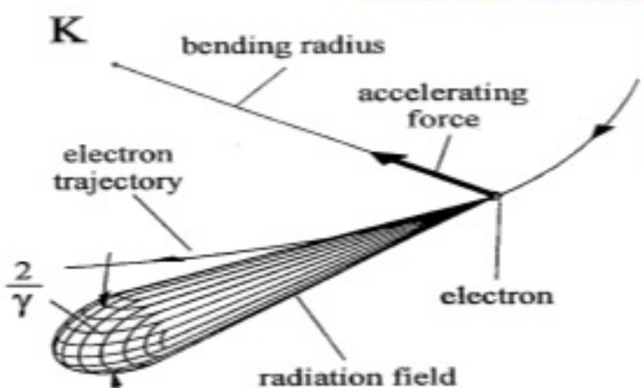
The equilibrium of the two effects leads to an equilibrium longitudinal emittance.

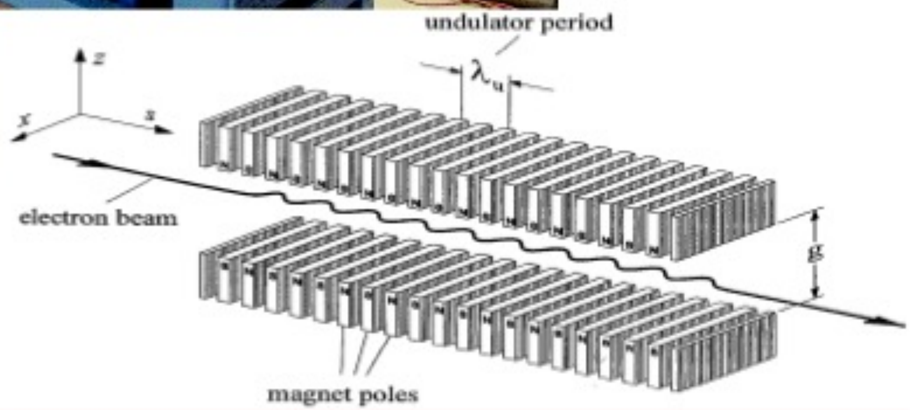
The damping time is the time it takes to radiate off the energy of the beam, while it is kept at constant energy with RF cavities. It is usually a few 100 revolutions.

During this damping time the beam forgets its history, particle coordinates are reshuffled within the beam.

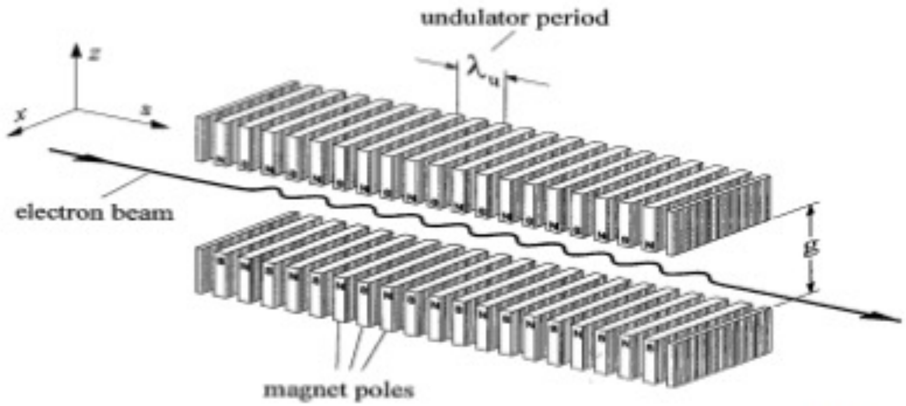
Radiative production by electrons



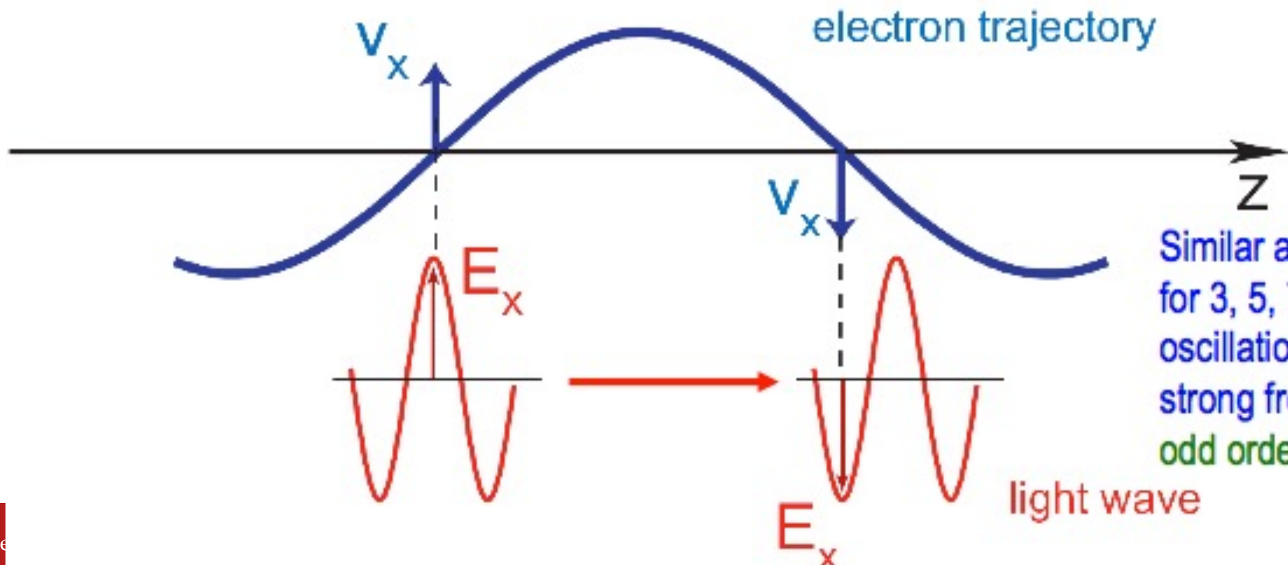




Radiative production in undulators

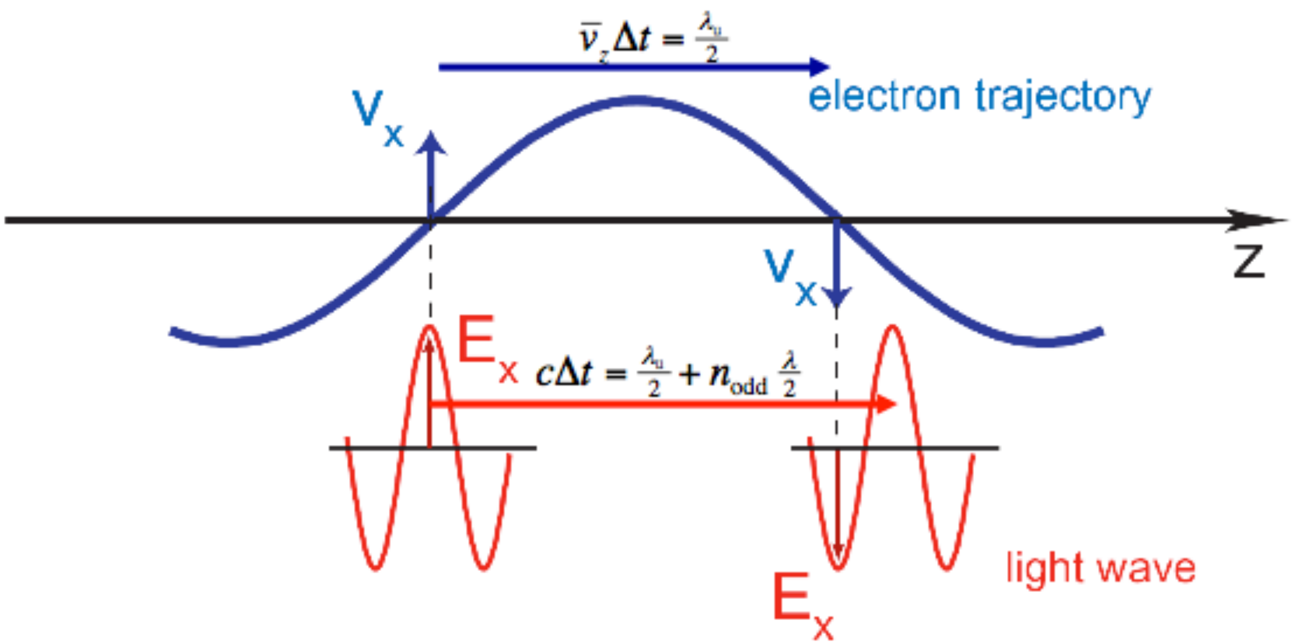


When the light wave passes the electron beam by half a wavelength per half beam oscillation, the radiation from each beam oscillation adds.



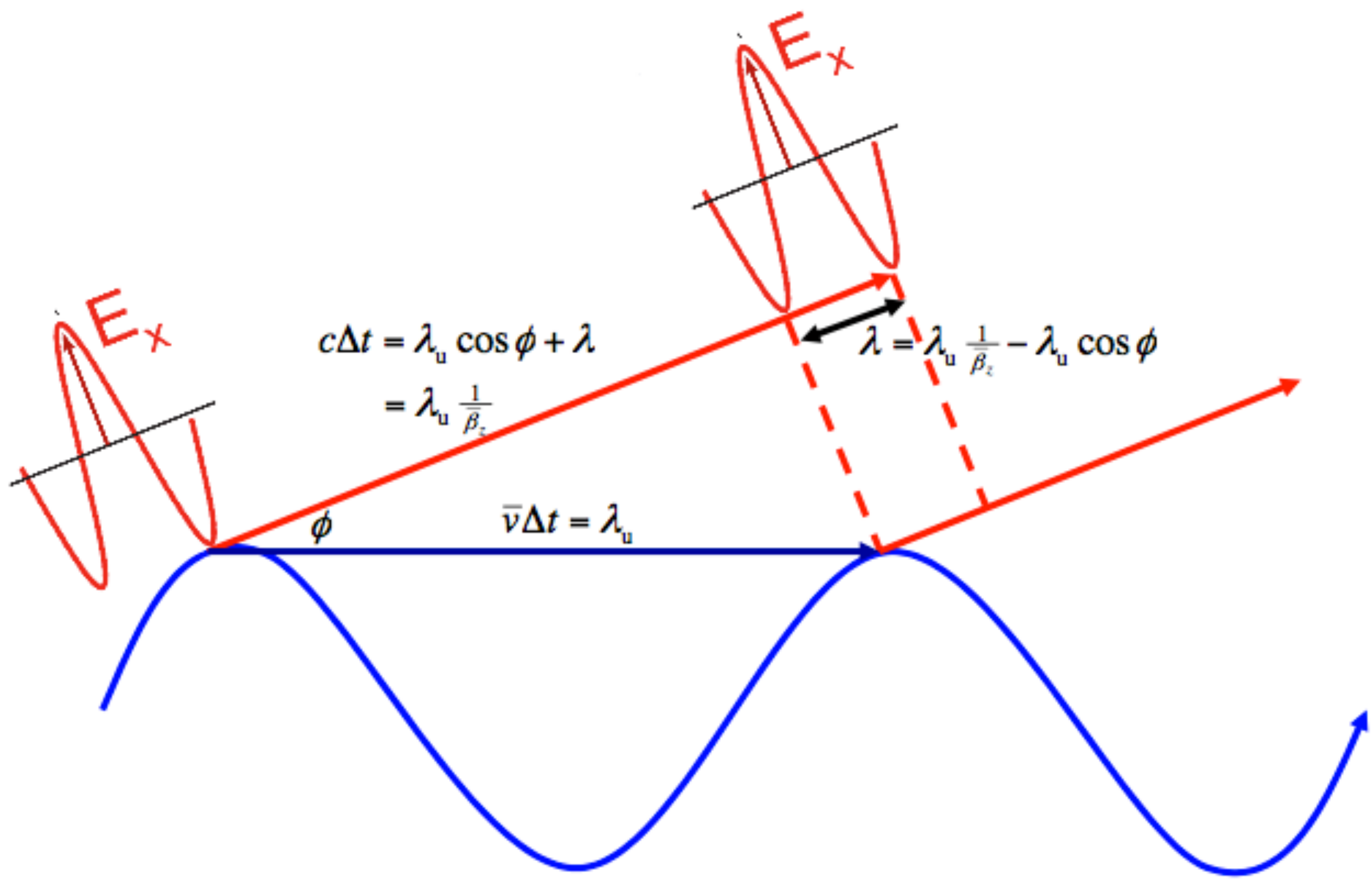
Similar adding can happen for 3, 5, 7, etc. half beam oscillations. For every strong frequency there are odd order harmonics.

Coherent addition of radiation



$$\frac{c}{\bar{v}_z}\frac{\lambda_u}{2} = \frac{\lambda_u}{2} + n_{odd}\frac{\lambda}{2} \quad \Rightarrow \quad \boxed{\lambda = \frac{1}{n_{odd}}\lambda_u\left(\frac{1}{\bar{\beta}_z} - 1\right)}$$

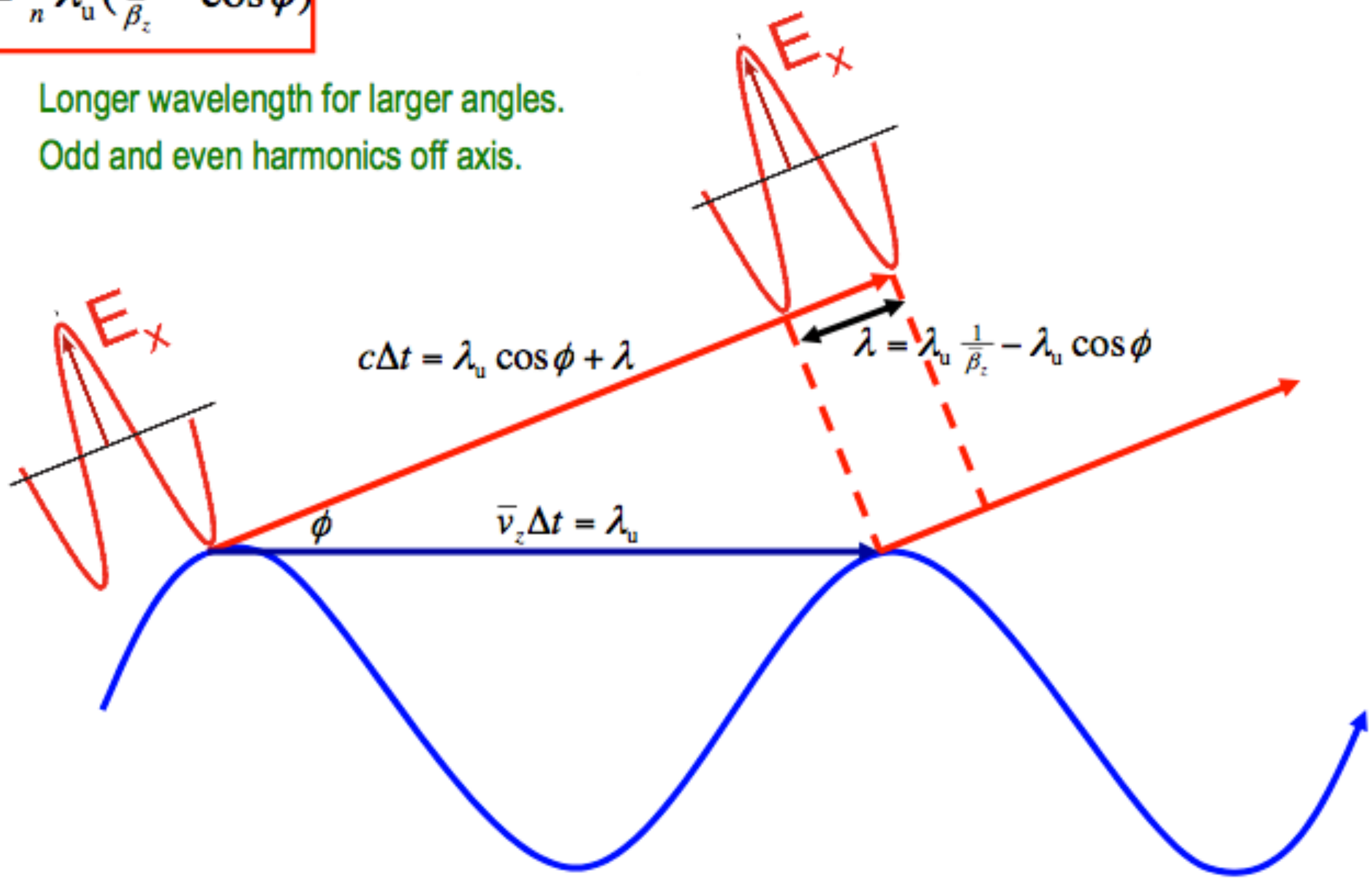
Coherent addition at angles



Radiation production at angles

$$\lambda = \frac{1}{n}\lambda_u\left(\frac{1}{\beta_z} - \cos\phi\right)$$

1) Longer wavelength for larger angles.
2) Odd and even harmonics off axis.



Lasing at the JLAB FEL

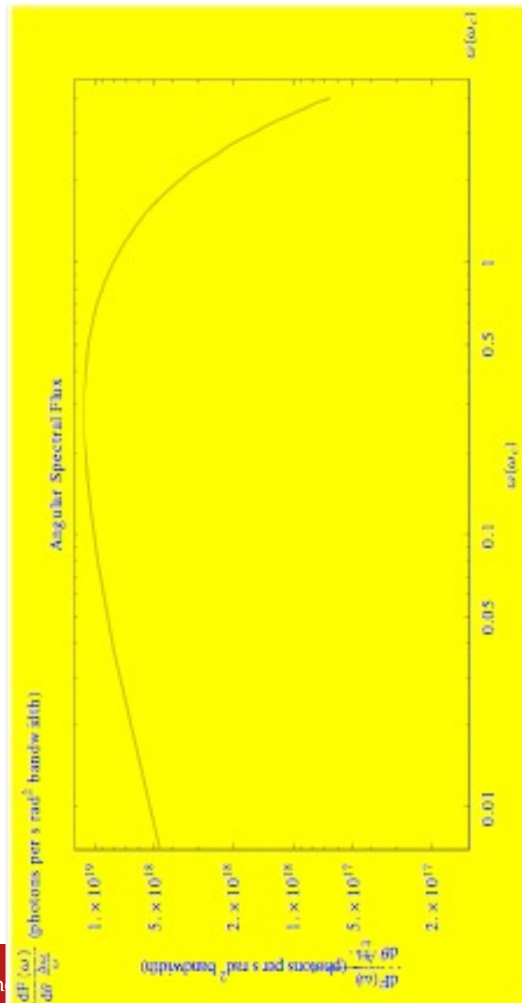
Wiggler gap

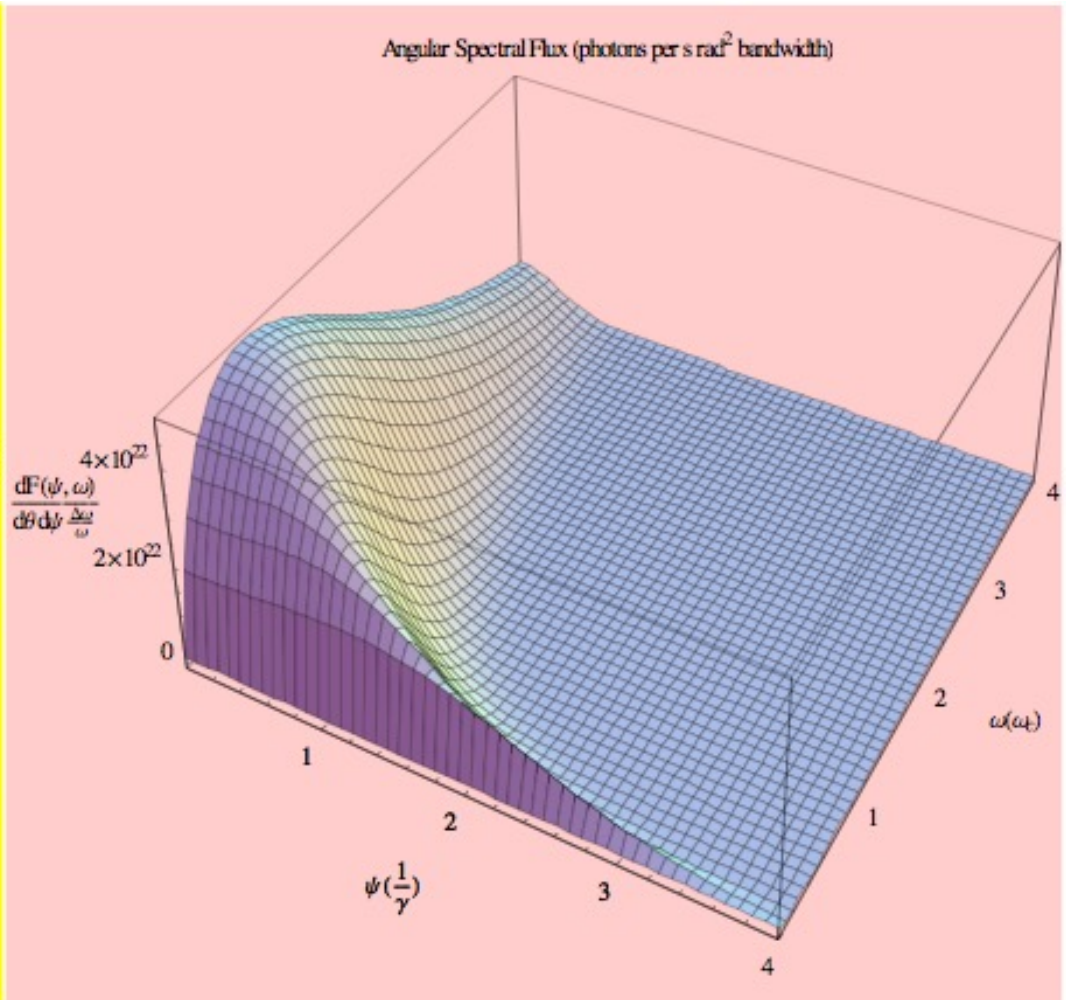
High Reflector

Hole Outcoupler

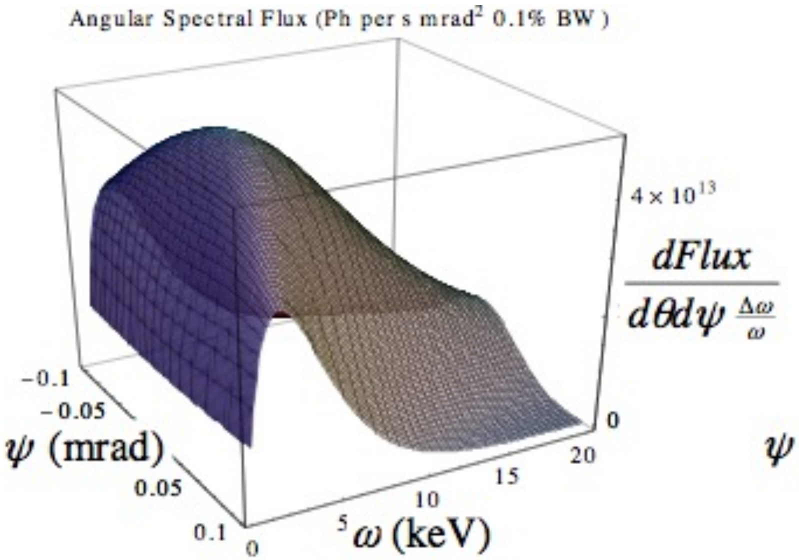
Beam in control room

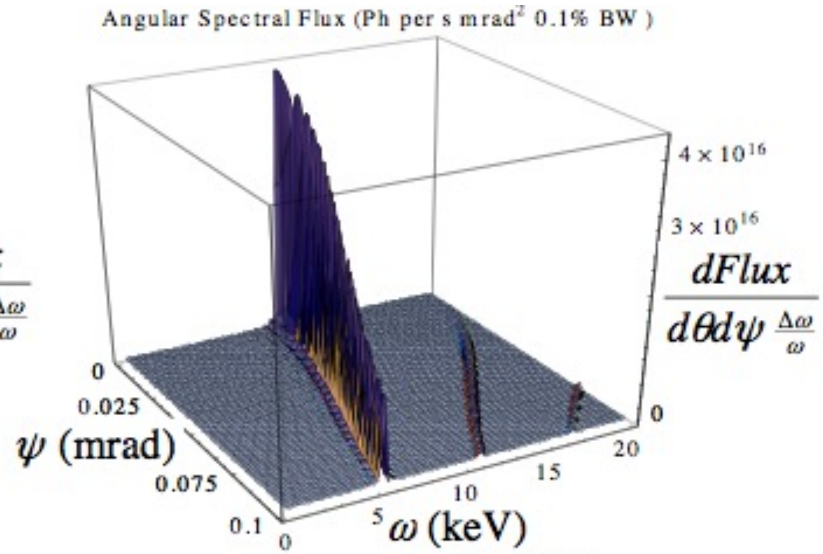
Radiative from bending magnets

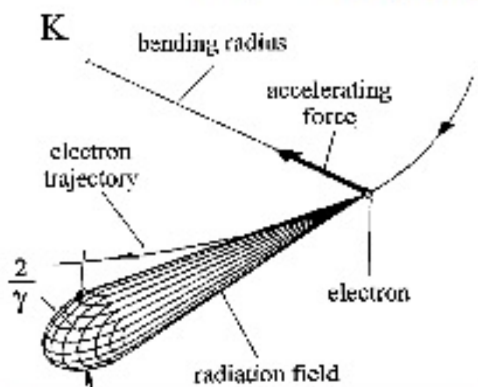


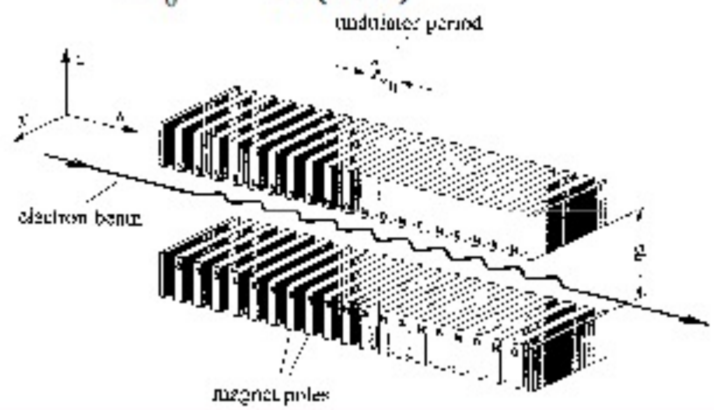


Photon flux from bends and undulators

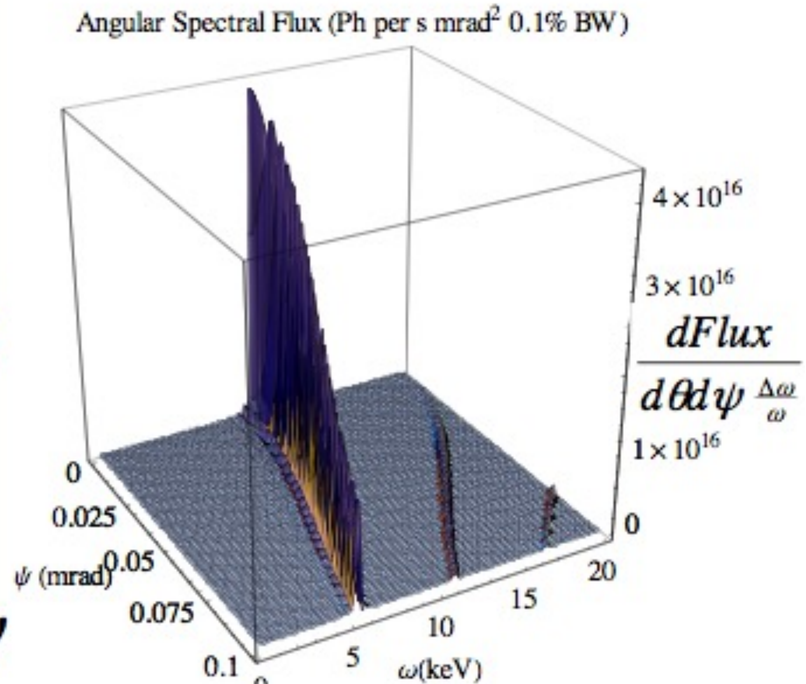








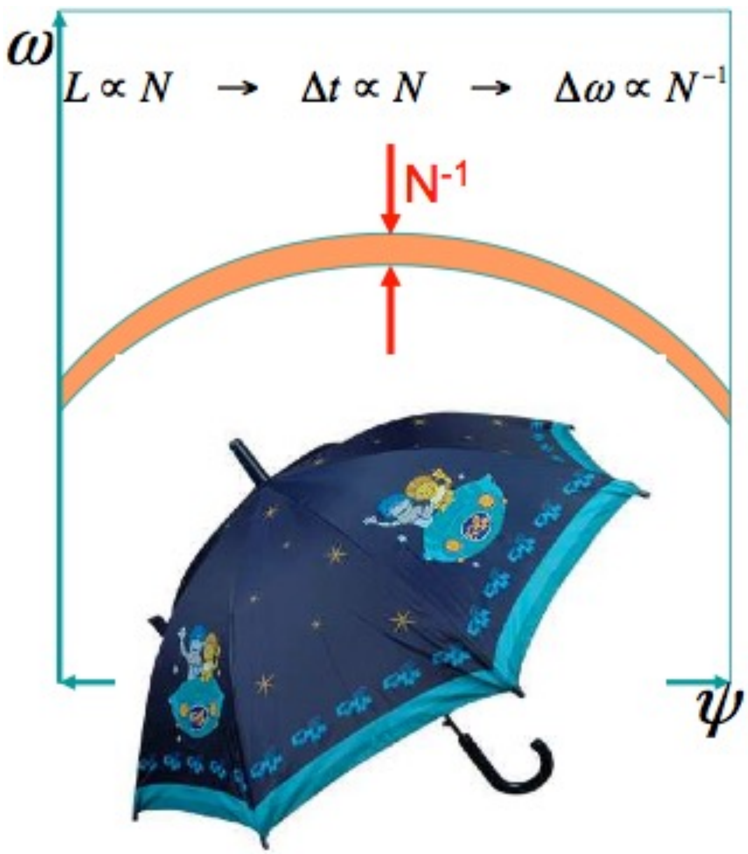
The umbrella of N-pole undulator radiation

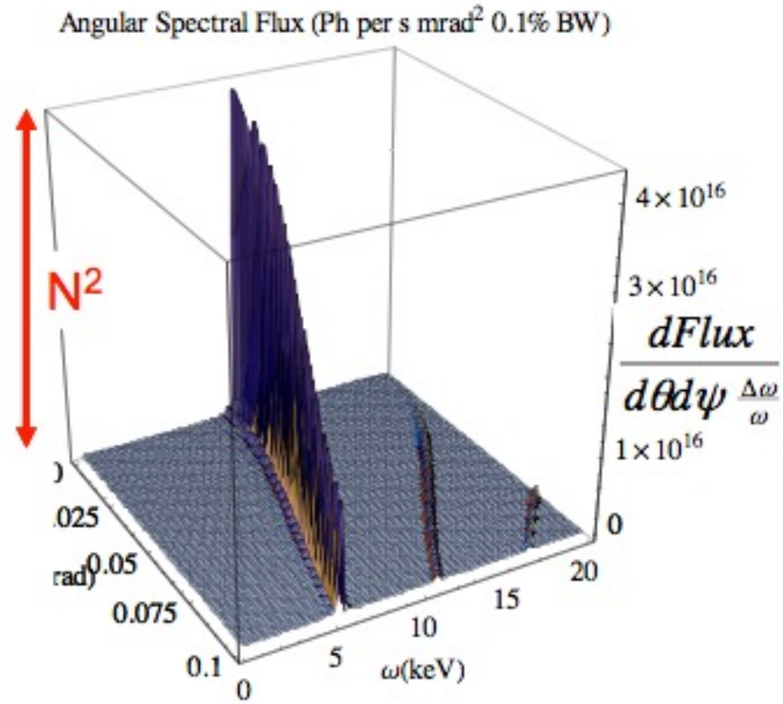


$$F_{total} \propto N$$

Flux from N poles is N times the flux from one pole

The umbrella of N-pole undulator radiation

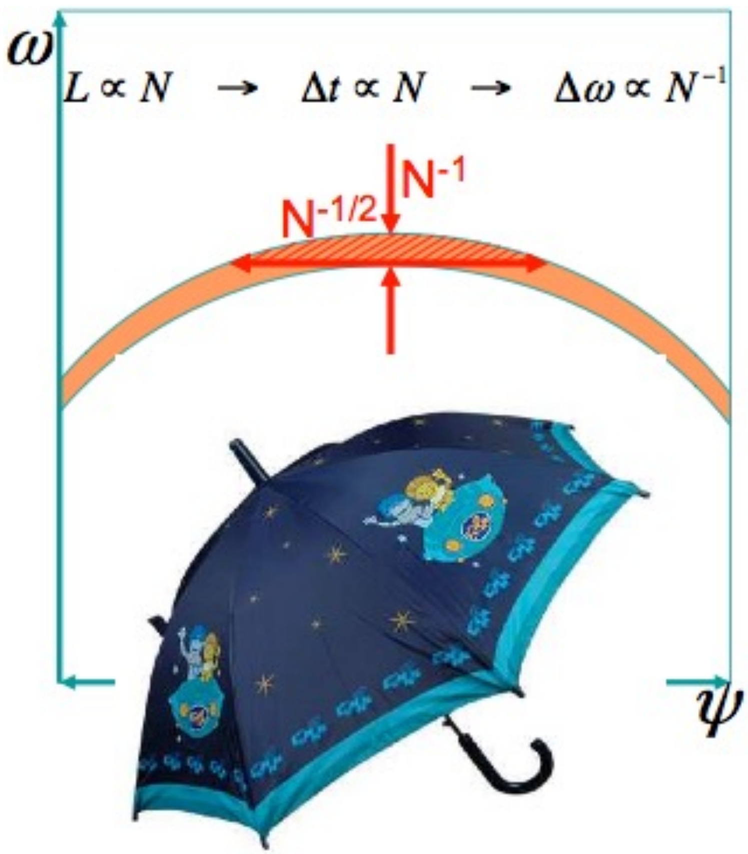


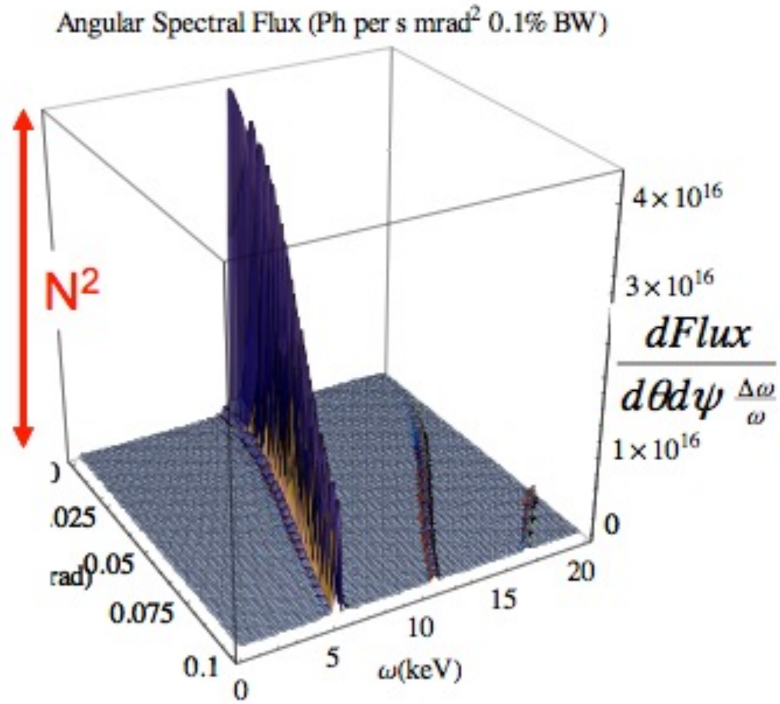


$$F_{total} \propto N$$

Flux from N poles is N times the flux from one pole

The umbrella of N-pole undulator radiation



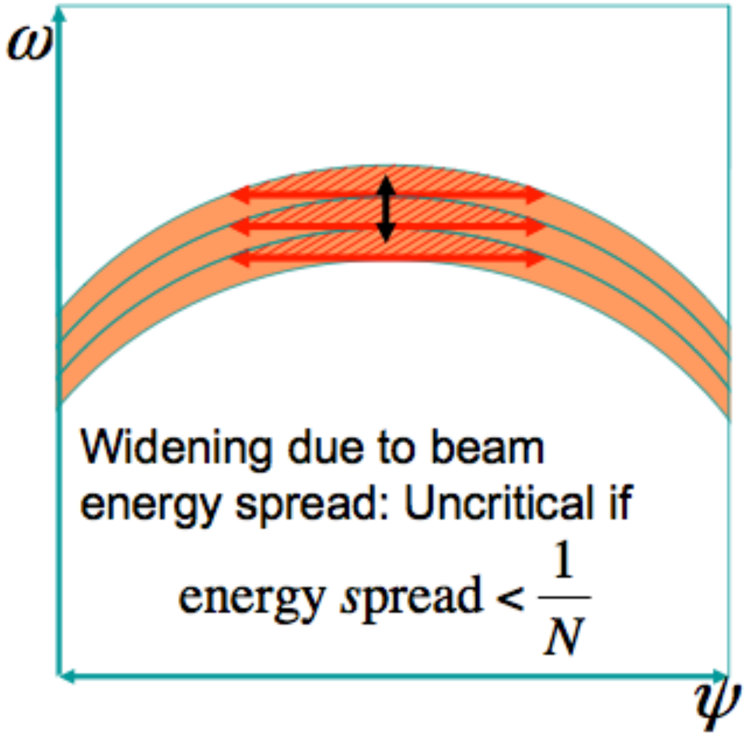


$$F_{total} \propto N$$

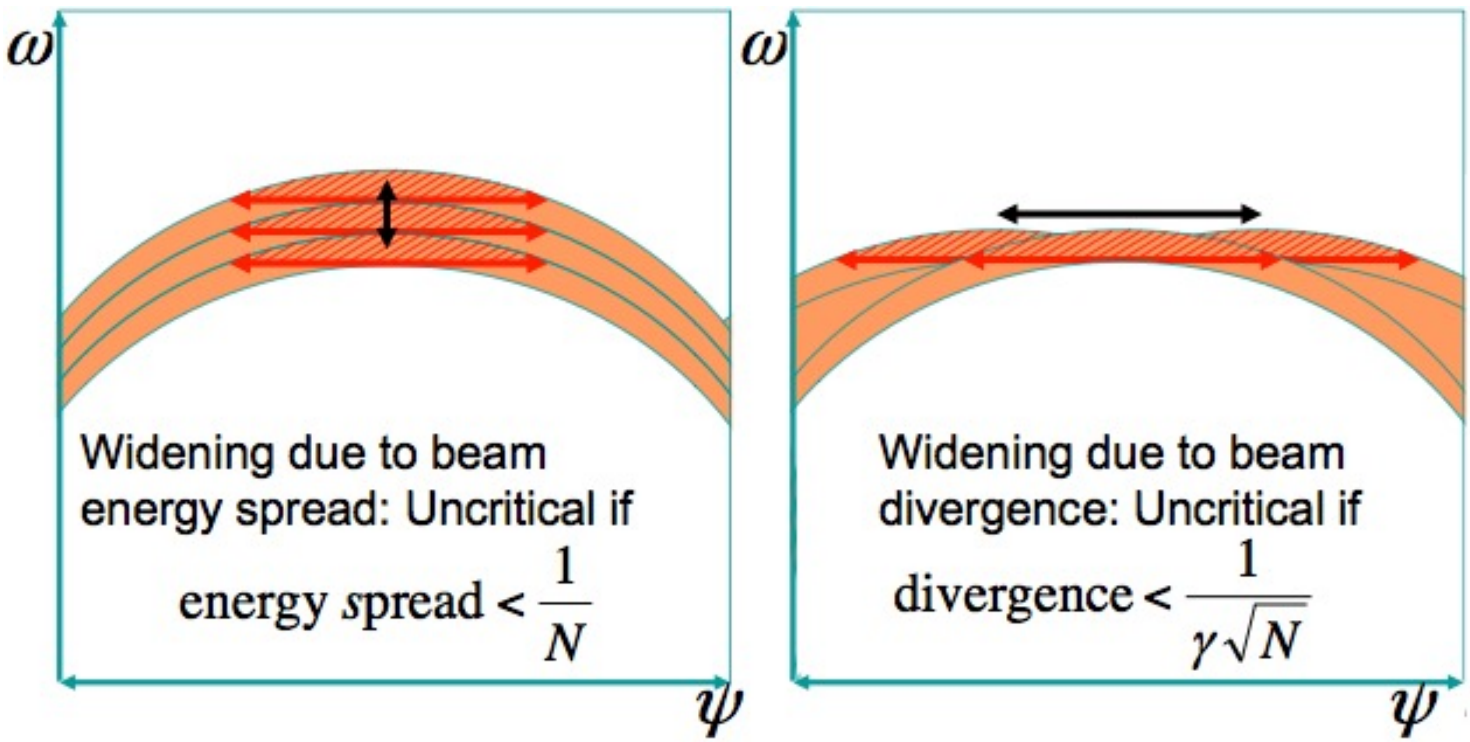
The power in the central cone is in dependent of N

Flux from N poles is N times the flux from one pole

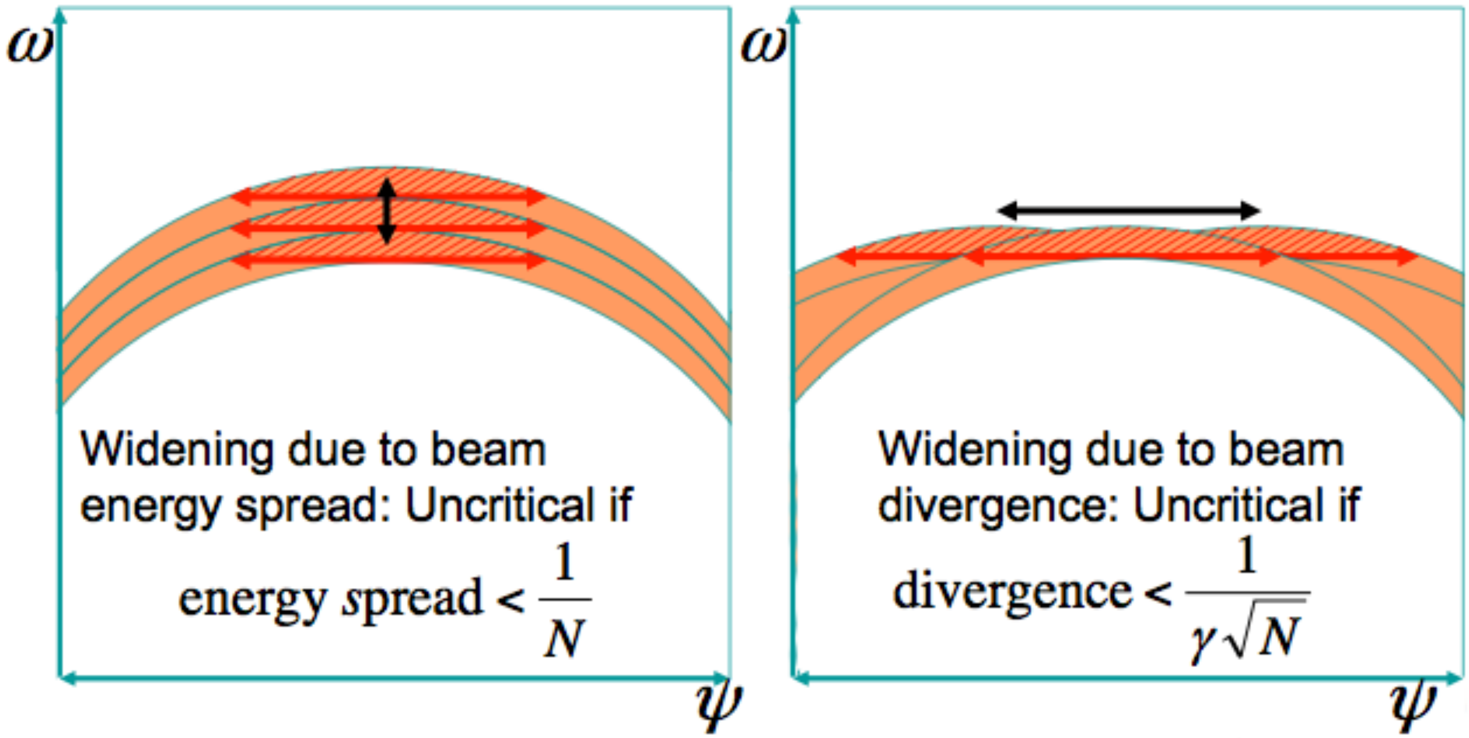
Brightness reduction by beam properties



Brightness reduction by beam properties



Brightness reduction by beam properties



Widening due to beam energy spread: Uncritical if

$$\text{energy spread} < \frac{1}{N}$$

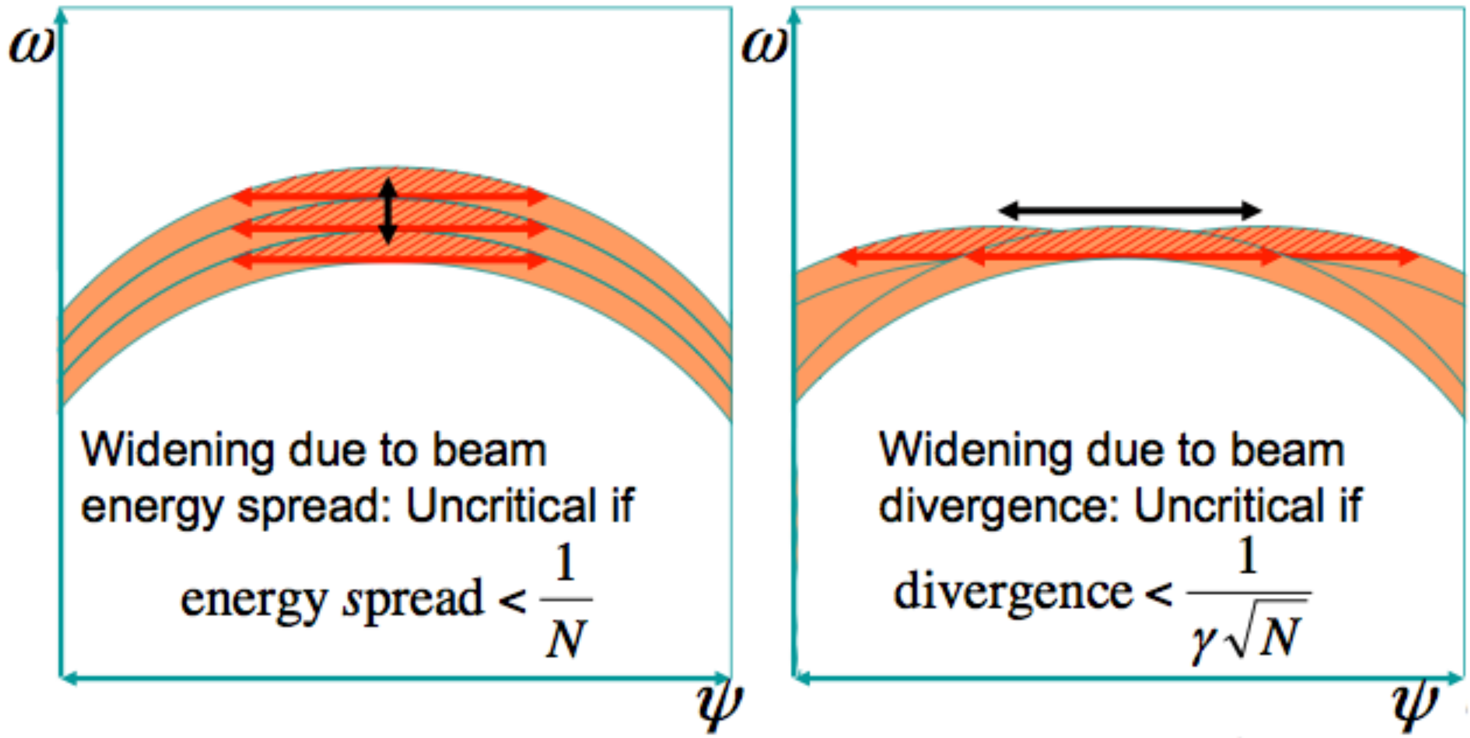
Widening due to beam divergence: Uncritical if

$$\text{divergence} < \frac{1}{\gamma\sqrt{N}}$$

Field from a single electron cannot be distinguished from field from a spot with:

$$\text{spot size} < \frac{\lambda}{\text{divergence}}$$

Brightness reduction by beam properties



Field from a single electron cannot be distinguished from field from a spot with:

$$\text{spot size} < \frac{\lambda}{\text{divergence}}$$

To take advantage of many undulator poles, the electron beam needs to have little energy spread, little divergence, and small beam size.