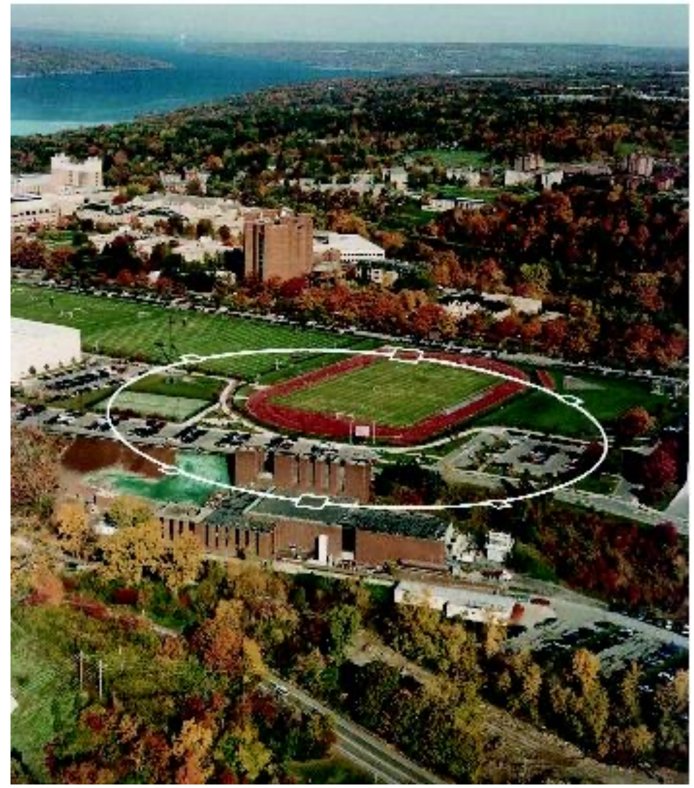
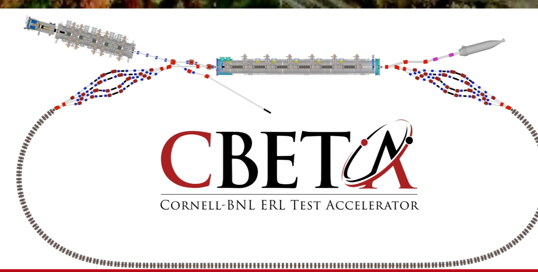
USPAS graduate Accelerator Physics

Content

1. Typical Particle Accelerators
2. Typical Accelerator Components
3. Linear Beam Optics (Circular & Straight)
4. Beam distributions
5. Nonlinear Beam Optics
6. Differential Algebra for Nonlinear Maps
7. Longitudinal Phase Space
8. Synchrotron Radiation
9. Polarization
10. Waveguides and Cavities

Accompanied by homework question and applied accelerator-optics design.





Cornell University Brookhaven National Laboratory

Class data

Dates:

Monday – Friday June 5 – 9, 2023, 9 - 12 and 2 - 5pm, HW after 7:30pm

Monday – Thursday June 12 – 15, 2023, 9 - 12 and 2 - 5pm, HW after 7:30pm

Friday: summary and final exam, 9am - noon

Expectation:

Collaborate on homework.

Consult TAs for homework and optics design.

Co-instructor for Bmad, design, and simulation: David Sagan dcs16@cornell.edu

TAs: Ningdong Wang nw285@cornell.edu and Matthew Signorelli mgs255@cornell.edu

Lecture notes, homework, and homework solution will be on the class web page

https://www.classe.cornell.edu/~hoff/LECTURES/23USPAS/

Accelerator optics and simulations

To practice what will be learned, accelerators will be simulated with the programs Tao and Bmad.

You all have accounts to run Bmad, which will be introduced Monday afternoon.

There will be Homework of analytical nature and design / optimization / simulation assignments.

Images are taken from many sources, including:

The Physics of Particle Accelerators, Klaus Wille, Oxford University Press, 2000, ISBN: 19 850549 3

Particle Accelerator Physics I, Helmut Wiedemann, Springer, 2nd edition, 1999, ISBN 3 540 64671 x

Teilchenbeschleuniger und Ionenoptic, Frank Hinterberger, 1997, Springer, ISBN 3 540 61238 6

Introduction to Ultraviolet and X-ray Free-Electron Lasers, Martin Dohlus, Peter Schmüser, Jörg Rossbach, Springer, 2008

Various web pages, 2003 – 2023

Literature

Recommended as

Introduction

The Physics of Particle Accelerators: An Introduction, Klaus Wille, Oxford University Press

Wide selection of well explained topics

Particle Accelerator Physics, Helmut Wiedemann, Springer, (preferably 3rd edition)

Tremendous overview, with references for derivations and explanations

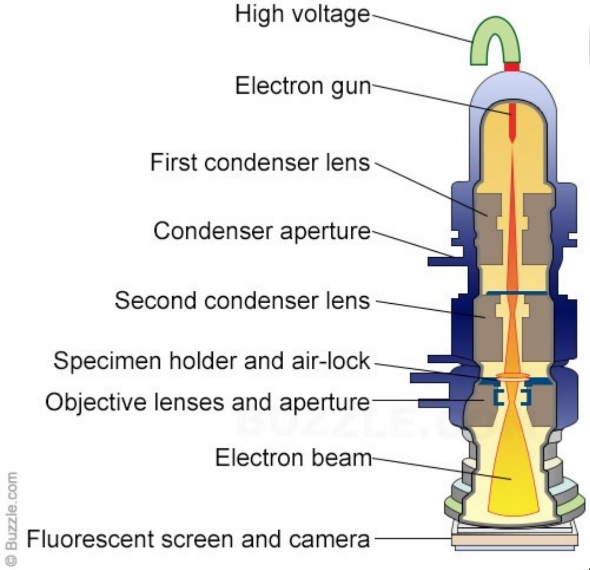
Handbook of Accelerator Physics and Engineering, Alexander Wu Cao, Maury Tigner, Hans Weise, Frank Zimmermann (3rd edition)

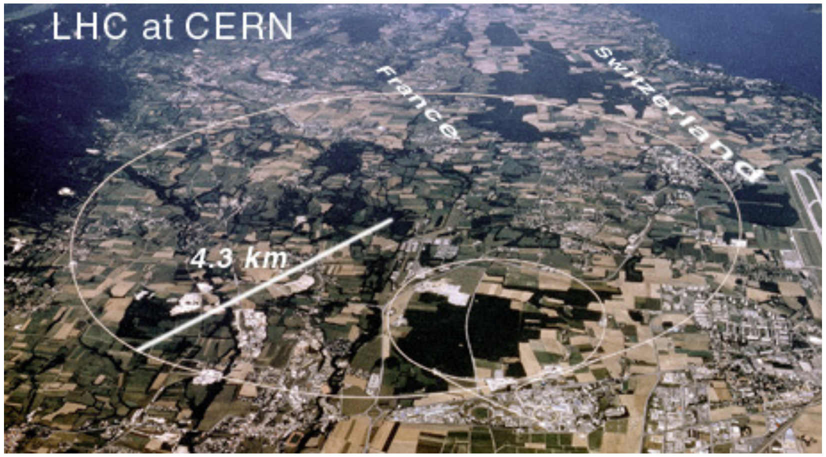
What is accelerator physics ?

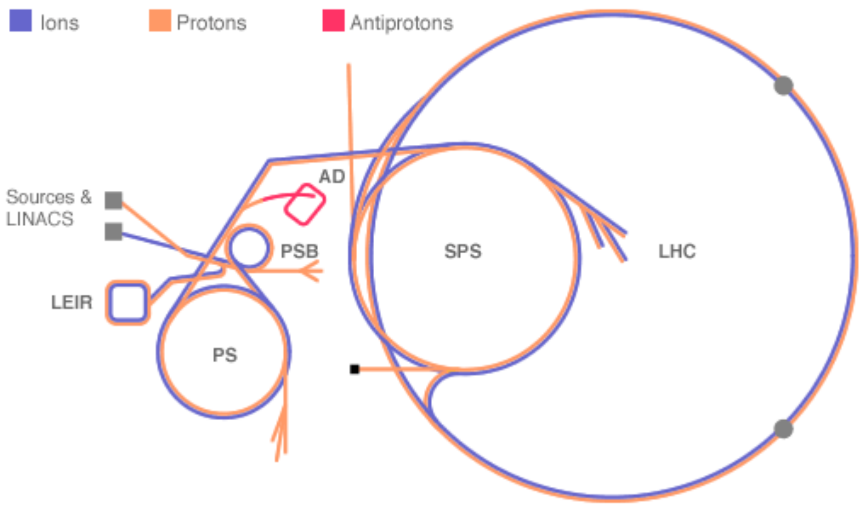
Accelerator Physics has applications in particle accelerators for high energy physics or for x-ray science, in spectrometers, in electron microscopes, and in lithographic devices. These instruments have become so complex that an empirical approach to properties of the particle beams is by no means sufficient and a detailed theoretical understanding is necessary. This course will introduce into theoretical aspects of charged particle beams and into the technology used for their acceleration.

- Physics of beams
- Physics of non-neutral plasmas
- Physics of involved in the technology:
  - Superconductivity in magnets and radiofrequency (RF) devices
  - Surface physics in particle sources, vacuum technology, RF devices
  - Material science in collimators, beam dumps, superconducting materials

Particle accelerators, large and small







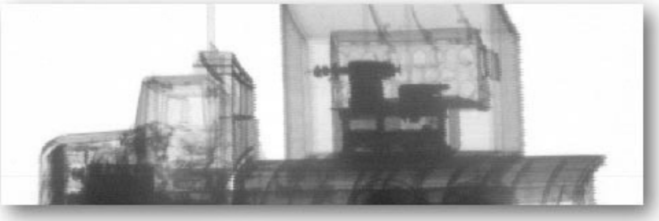
Why accelerator physics ?

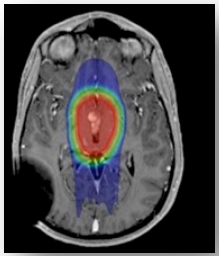
- Industry
  - Food & product safety
  - Contraband detection
  - Semiconductor fabrication
  - Bridge safety
- Medicine
  - Tumor detection and treatment.

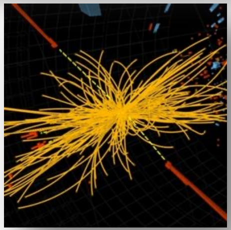
~30,000 industrial and medical accelerators are in use, with annual sales of $3.5 B and 10% growth per year.

- Research
  - X ray sources and colliders for nuclear & particle physics
  - Electron microscopes

Since 1943, a Nobel Prize in **Physics** has been awarded to research benefiting from accelerators every 3 years.
Since 1997, the same has been true of **Chemistry**.



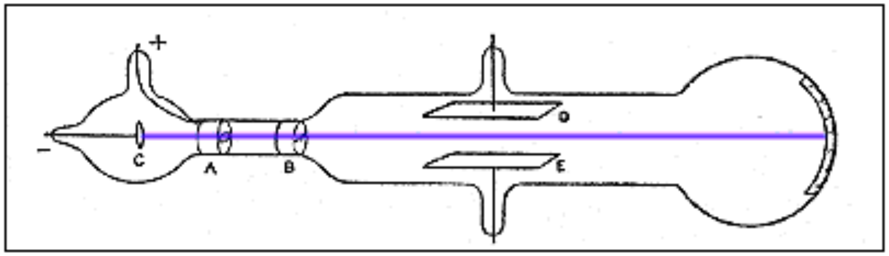




A short history of accelerators

- 1862: Maxwell theory of electromagnetism
- 1887: Hertz discovery of the electromagnetic wave
- 1886: Goldstein discovers positively charged rays (ion beams)
- 1894: Lenard extracts cathode rays (with a 2.65μm Al Lenard window)
- 1897: JJ Thomson shows that cathode rays are particles since they followed the classical Lorentz force $m\vec{a} = e(\vec{E} + \vec{v} \times \vec{B})$ in an electromagnetic field
- 1926: GP Thomson shows that the electron is a wave (1929-1930 in Cornell, NP in 1937)

NP 1905, Philipp E.A. von Lenard

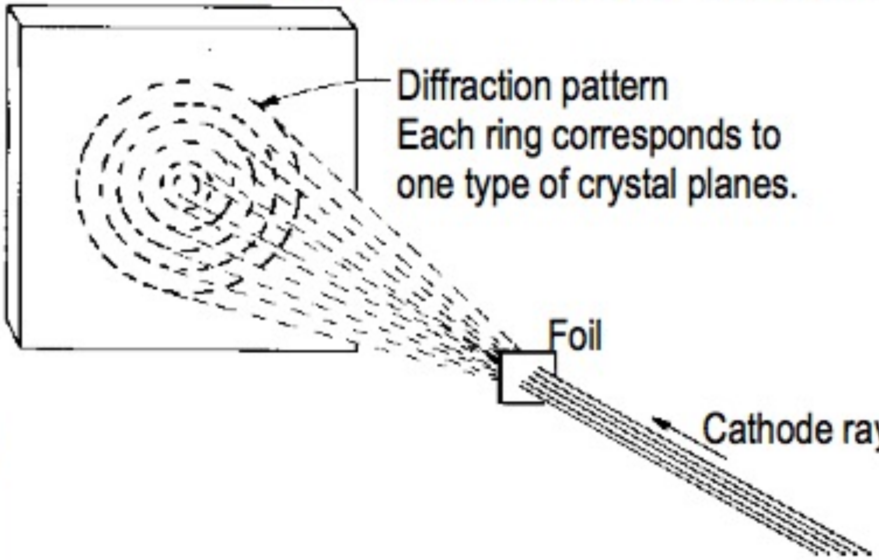


NP 1906
Joseph J. Thomson

Discoveries with accelerated beams …

George P.Thomson
(1892-1975)
1937 Nobel prize
Son of Joseph J. T.

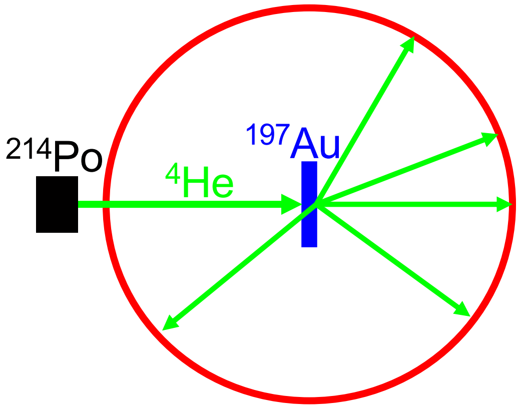
In a powdered, microcrystalline substance there is always some crystal which has the correct angle for constructive interference $2d\cos\alpha = n\lambda$



A magnetic field can change the rings, showing the the waves are associated with the electron charge.

The need for higher energies

- 1911: Rutherford discovers the nucleus with 7.7MeV ^{4}He from ^{214}Po alpha decay measuring the elastic crossection of ^{197}Au + ^{4}He $\mapsto$ ^{197}Au + ^{4}He.



$$E = \frac{Z_1 e Z_2 e}{4\pi\varepsilon_0 d} = Z_1 Z_2 m_e c^2 \frac{r_e}{d},$$

$$r_e = 2.8\text{fm}, \quad m_e c^2 = 0.511\text{MeV}$$

d = smalles approach for back scattering

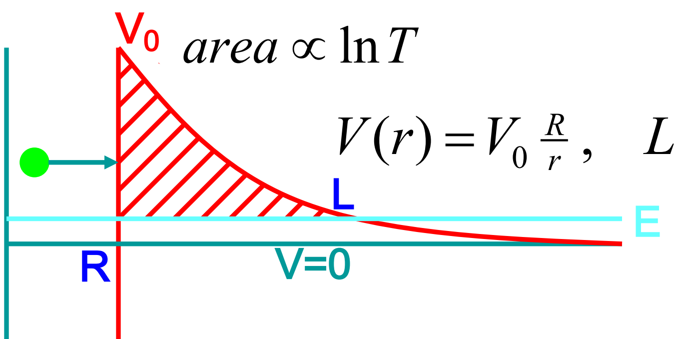
- 1919: Rutherford produces first nuclear reactions with natural ^{4}He
^{14}N + ^{4}He $\mapsto$ ^{17}O + p
- 1921: Greinacher invents the cascade generator for several 100 keV
- Rutherford is convinced that several 10 MeV are in general needed for nuclear reactions. He therefore gave up the thought of accelerating particles.

Tunneling allows low energies

- 1928: Explanation of alpha decay by Gamov as tunneling showed that several 100keV protons might suffice for nuclear reactions

Schroedinger equation: $\frac{\partial^2}{\partial r^2}u(r) = \frac{2m}{\hbar^2}[V(r)-E]u(r)\,,\quad T = \left|\frac{u(L)}{u(0)}\right|^2$

The transmission probability T for an alpha particle traveling from the inside towards the potential well that keeps the nucleus together determines the lifetime for alpha decay.

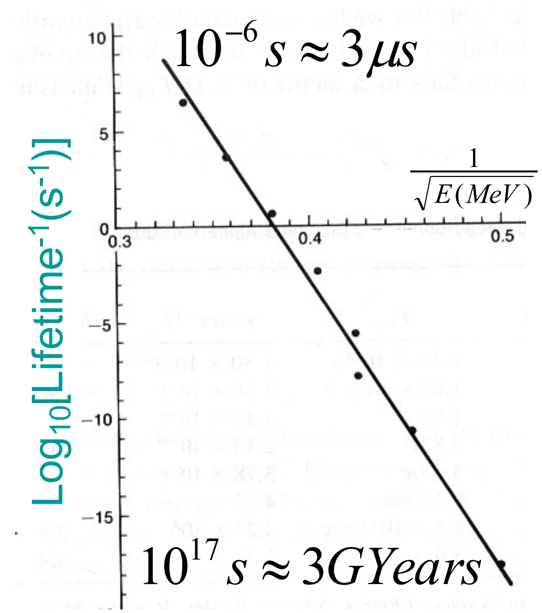


$area \propto \ln T$

$V(r) = V_0\frac{R}{r}\,,\quad L = \frac{V_0}{E}R$

$T \approx \exp[-2\int_R^L \frac{\sqrt{2m[V(r)-E]}}{\hbar}dr]$

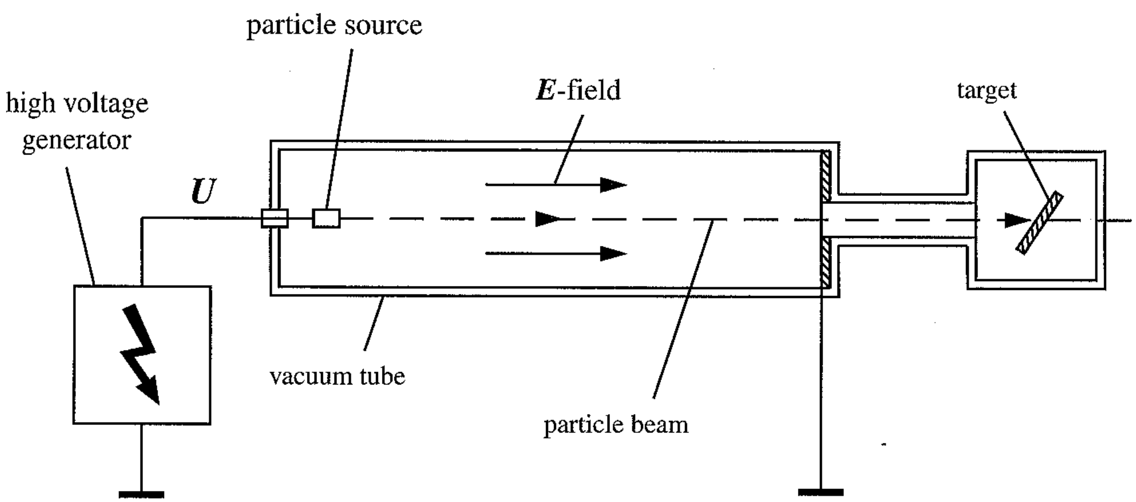
$\ln T \approx A - \frac{C}{\sqrt{E}}$



Three historic lines of accelerators

Direct Voltage Accelerators

Resonant Accelerators   Transformer Accelerator

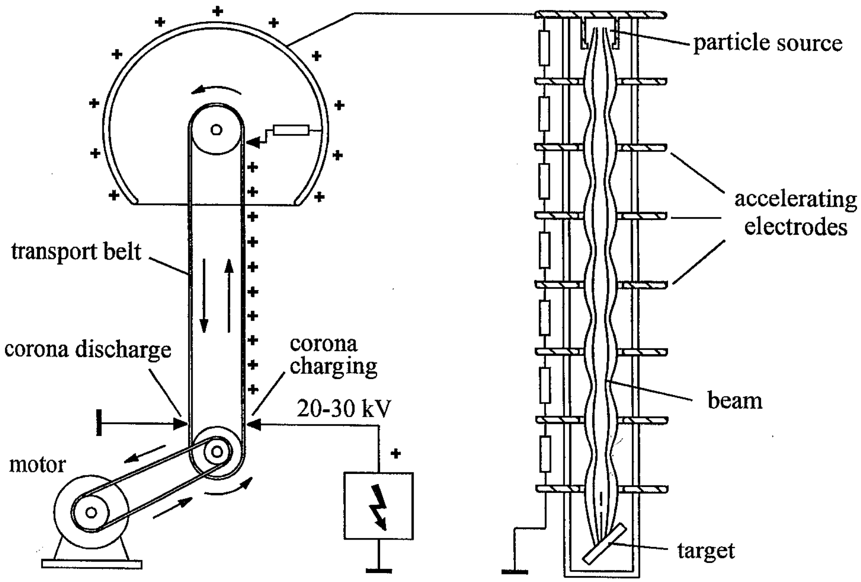


Voltage 1MV
Charge Ze
Energy Z MeV

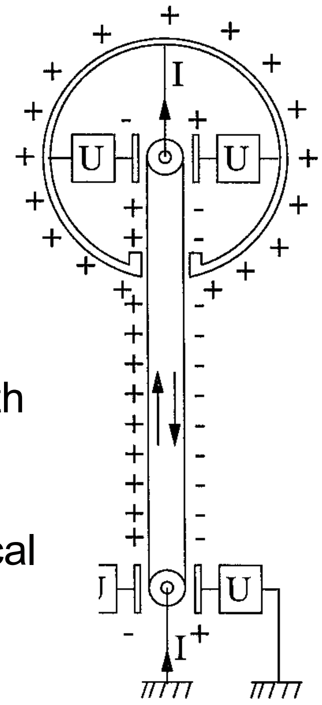
The energy limit is given by the maximum possible voltage. At the limiting voltage, electrons and ions are accelerated to such large energies that they hit the surface and produce new ions. An avalanche of charge carries causes a large current and therefore a breakdown of the voltage.

The Van de Graaff Accelerator

- 1930: van de Graaff builds the first 1.5MV high voltage generator



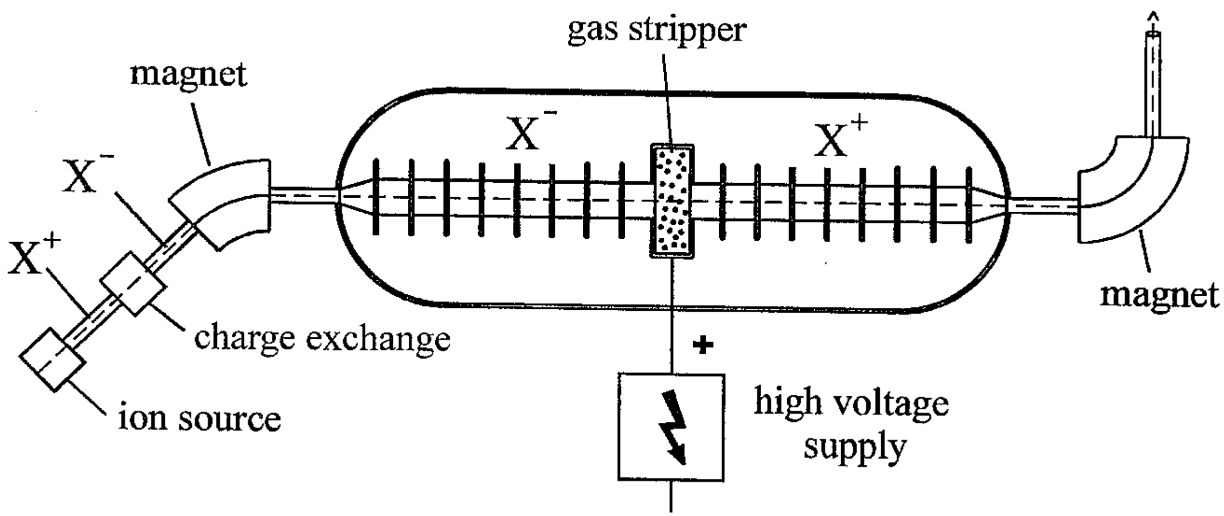
Van de Graaff



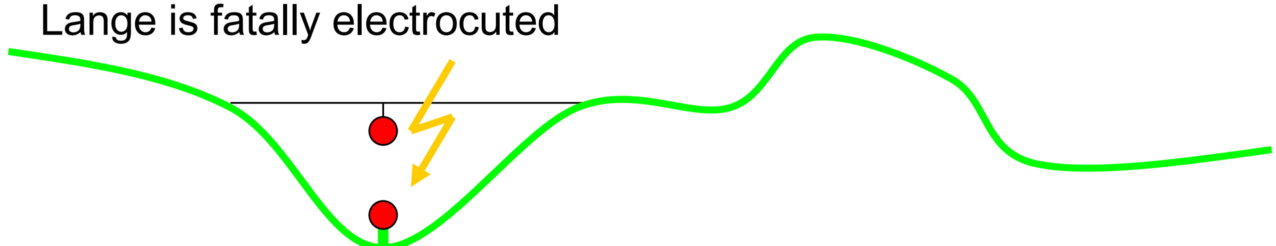
- Today Peletrons (with chains) or Laddertron (with stripes) that are charged by influence are commercially available.
- Used as injectors, for electron cooling, for medical and technical n-source via d + t $\mapsto$ n + α
- Up to 17.5 MV with insulating gas (1MPa SF_6)

The Tandem (Van de Graaff) Accelerator

- Two Van de Graaffs, one + one –
- The Tandem Van de Graaff, highest energy 35MeV

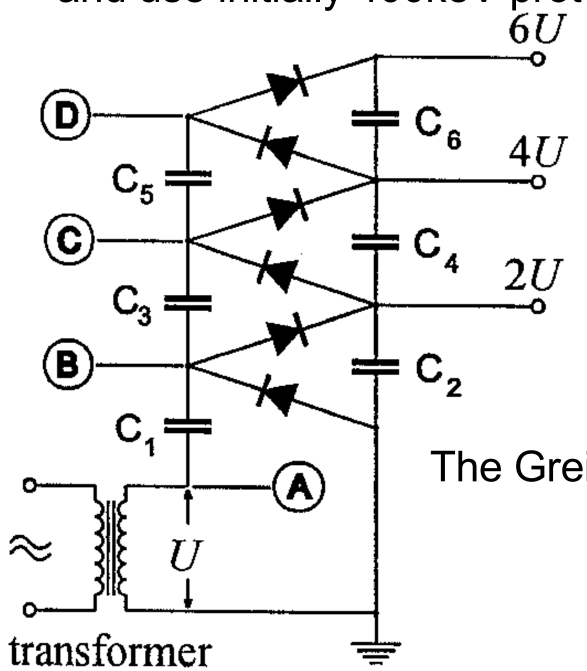


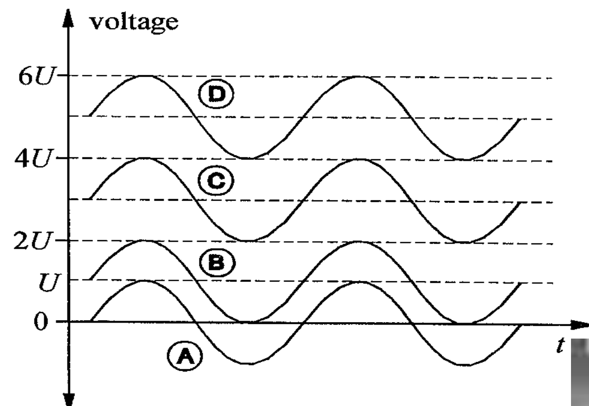
- 1932: Brasch and Lange use potential from lightening, in the Swiss Alps, Lange is fatally electrocuted



The Cockcroft-Walton Accelerator

1932: Cockcroft and Walton 1932: 700keV cascate generator (planed for 800keV) and use initially 400keV protons for ^{7}Li + p $\mapsto$ ^{4}He + ^{4}He and ^{7}Li + p $\mapsto$ ^{7}Be + n





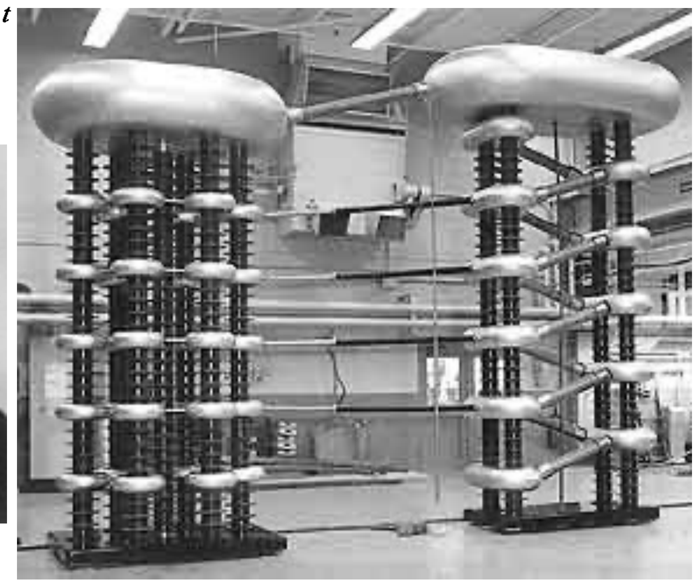
The Greinacker circuit

Up to 4MeV, 1A

NP 1951

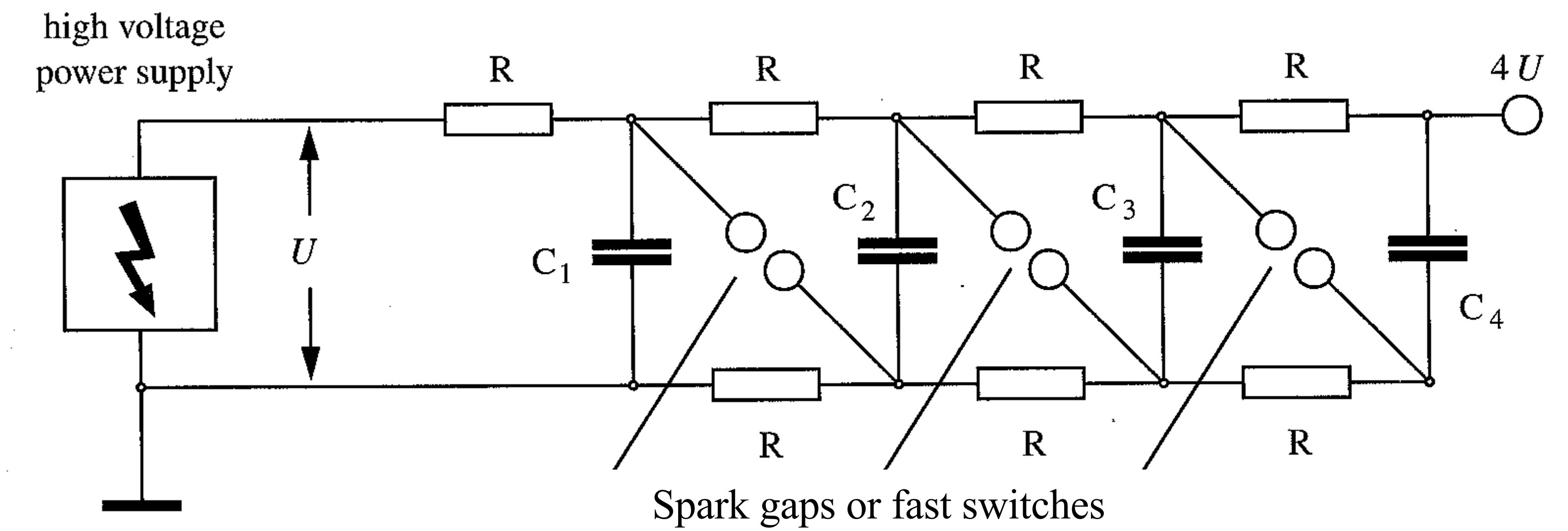
Sir John D Cockcrof

Ernest T S Walton



The Marx Generator

- 1932: Marx Generator achieves 6MV at General Electrics



After capacitors of around 2uF are filled to about 20kV, the spark gaps or switches close as fast as 40ns, allowing up to 500kA.

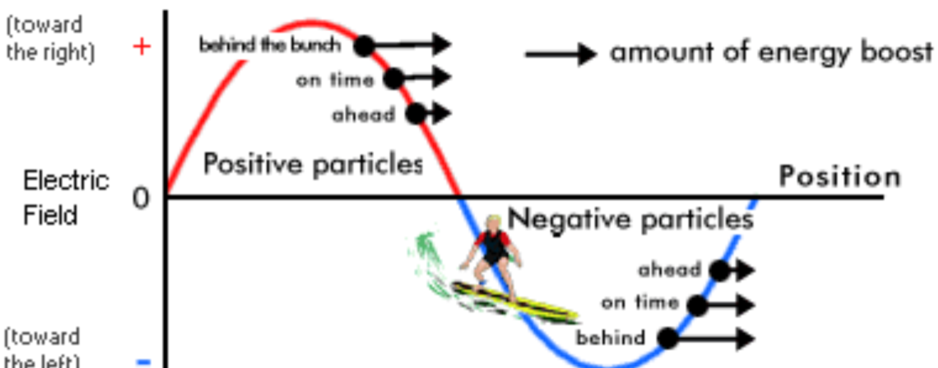
Today: The Z-machine (Physics Today July 2003) for z-pinch initial confinement fusion has 40TW for 100ns from 36 Marx generators

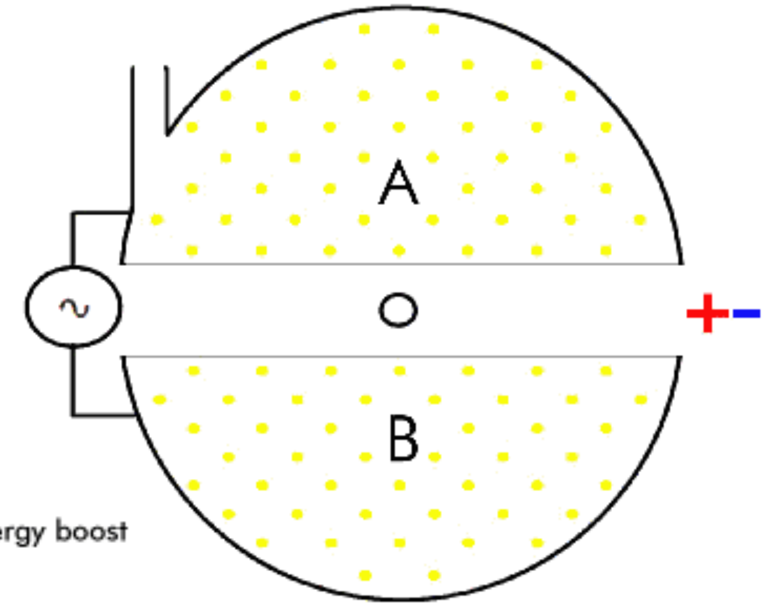
Three historic lines of accelerators

Resonant Accelerators

Direct Voltage Accelerators

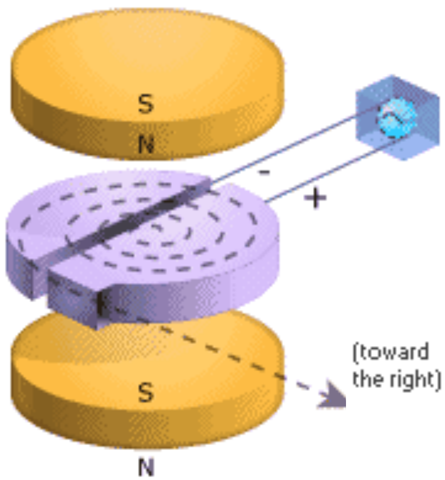
Transformer Accelerator



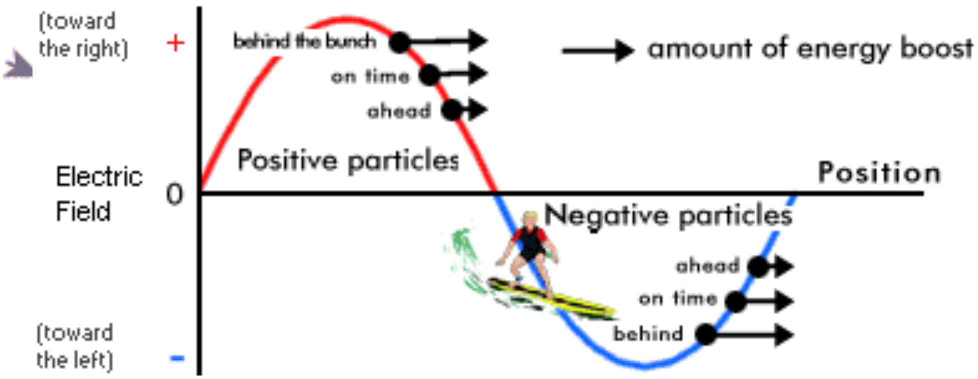


Particles must have the correct phase relation to the accelerating voltage.

The Cyclotron



- 1930: Lawrence proposes the Cyclotron
  (before he develops a workable color TV screen)
- 1932: Lawrence and Livingston use a cyclotron for
  1.25MeV protons and mention longitudinal
  (phase) focusing



NP 1939

Ernest O Lawrence

- 1934: Livingston builds the first Cyclotron
  away from Berkely (2MeV protons)
  at Cornell (in room B54)

M Stanley Livingston

The Cyclotron frequency

$$F_r = m_0\gamma\omega_z v = qvB_z$$

$$\omega_z = \tfrac{q}{m_0\gamma} B_z = \text{const}$$

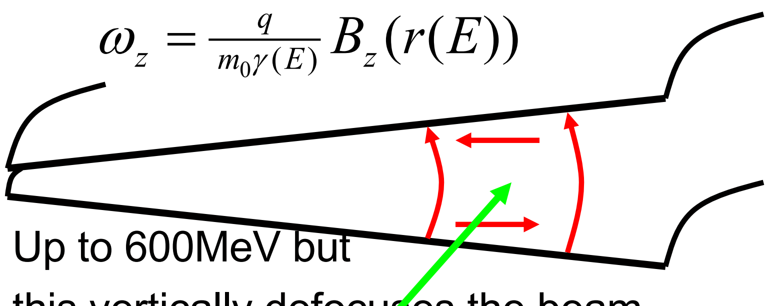
Condition: Non-relativistic particles.
Therefore, not for electrons.

- The synchrocyclotron:
  Acceleration of bunches with decreasing

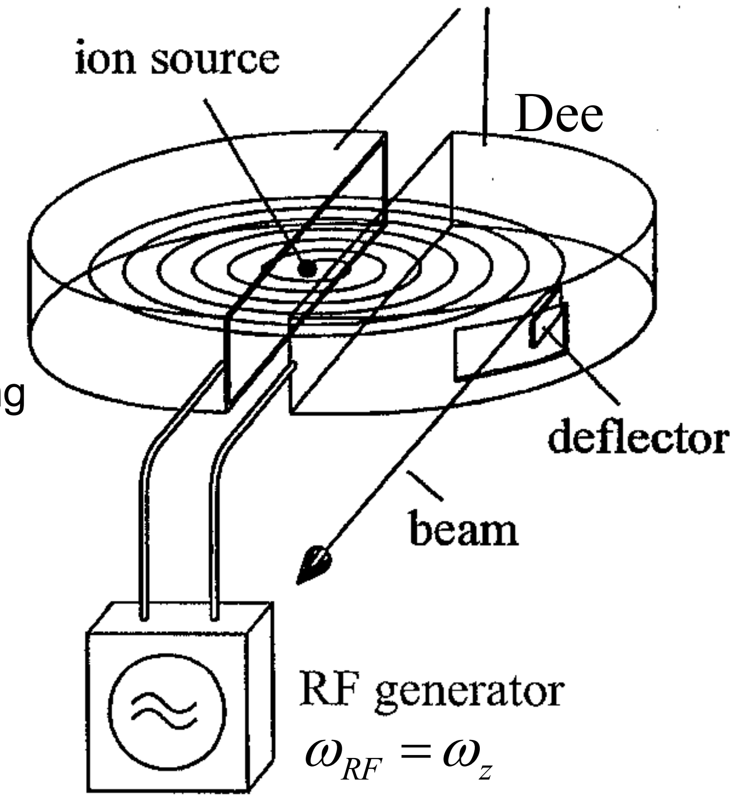
$$\omega_z(E) = \tfrac{q}{m_0\gamma(E)} B_z$$

- The isocyclotron with constant

$$\omega_z = \tfrac{q}{m_0\gamma(E)} B_z(r(E))$$

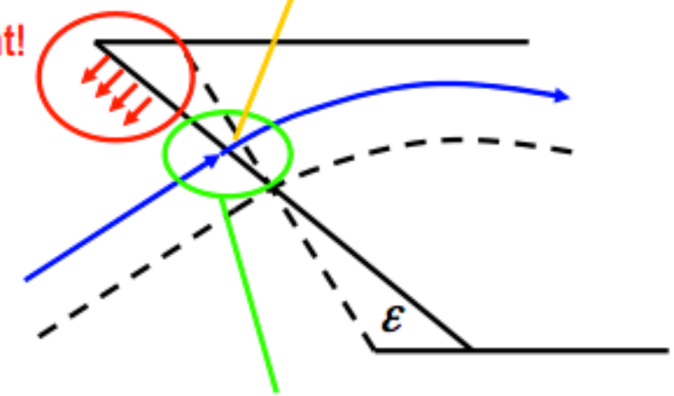


Up to 600MeV but
this vertically defocuses the beam



- 1938: Thomas proposes strong (transverse) focusing for a cyclotron

The Edge Focusing



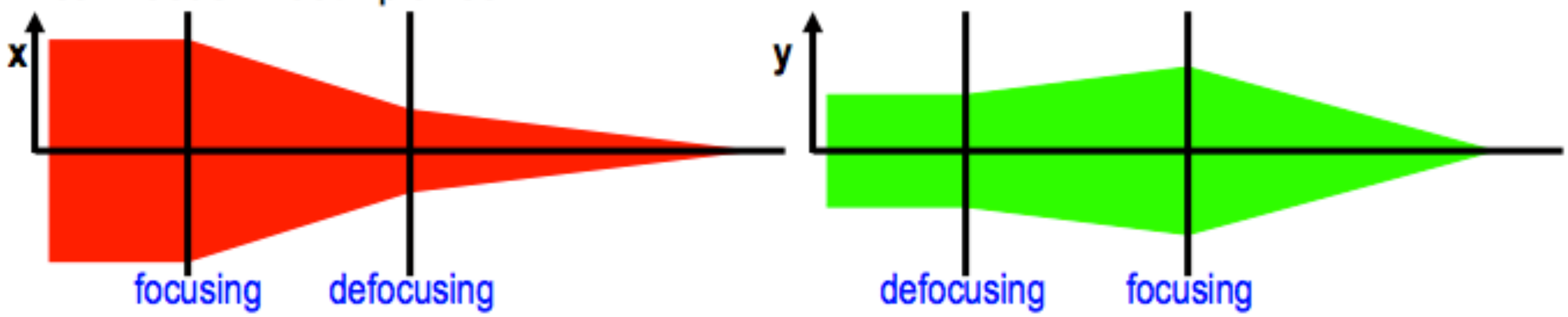
Horizontal focusing with $\Delta x' = -x\frac{\tan(\varepsilon)}{\rho}$

The longitudinal field above the enter plain defocuses, turns out to: $\Delta y' = y\frac{\tan(\varepsilon)}{\rho}$

Quadrupole effect: focusing in x and defocusing in y or defocusing in x and focusing in y.

Quadrupole Focusing

Transverse fields defocus in one plane if they focus in the other plane.
But two successive elements, one focusing the other defocusing,
can focus in both planes:



The Isocyclotron

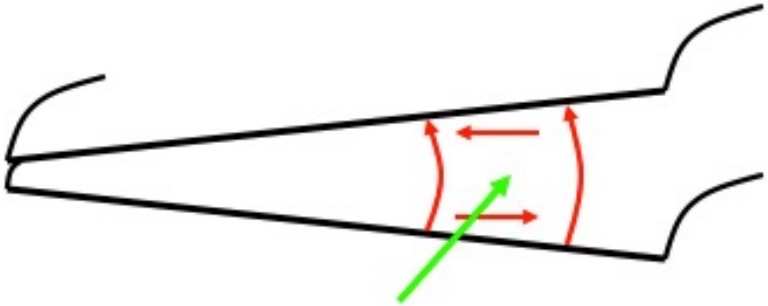
- The isocyclotron with constant

$$\omega_z = \frac{q}{m_0\gamma(E)} B_z(r(E))$$

Up to 600MeV but

this vertically defocuses the beam.

Edge focusing is therefore used.

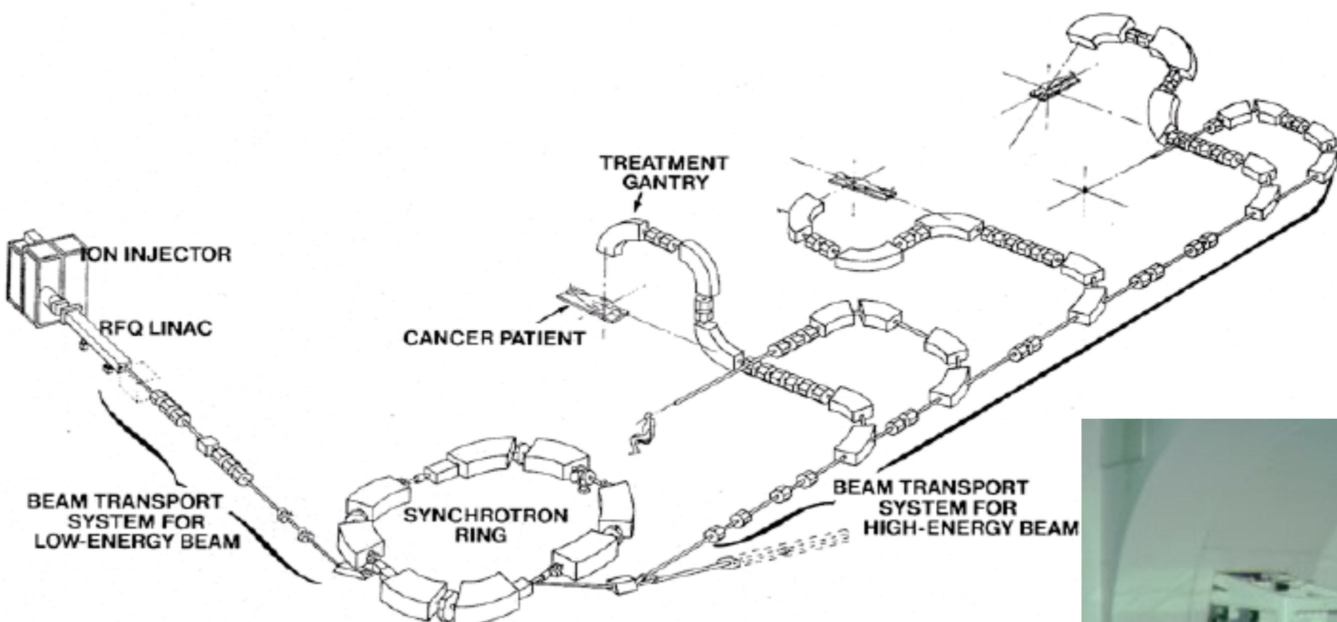


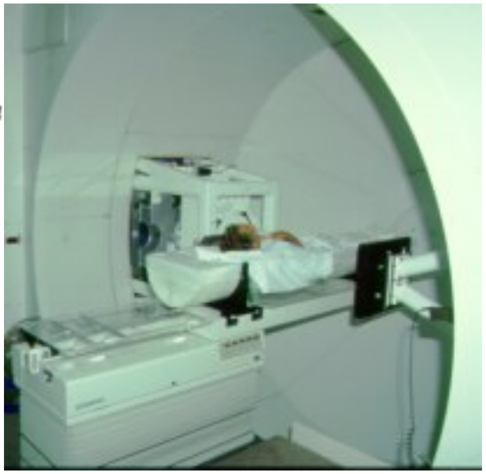
First Medical Accelerators

- 1939: Lawrence uses 60' cyclotron for 9MeV protons, 19MeV deuterons, and 35MeV 4He. First tests of tumor therapy with neutrons via $d + t \mapsto n + \alpha$ With 200-800keV d to get 10MeV neutrons.

Modern Nuclear Radiation Therapy

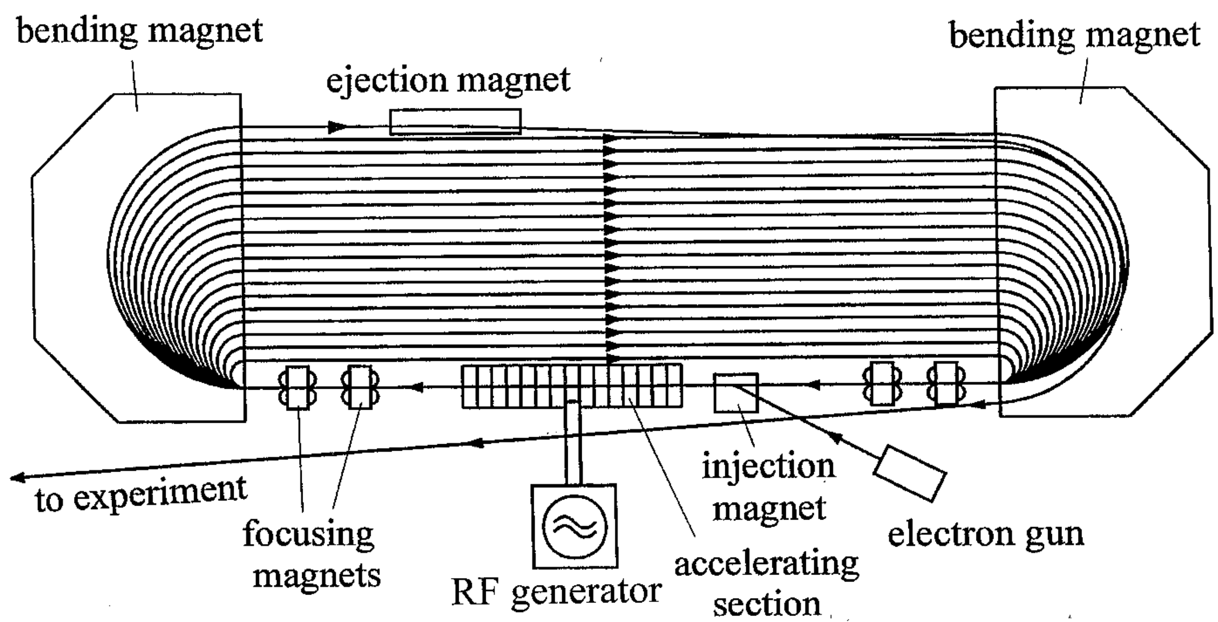
The Loma Linda proton therapy facility





The Microtron

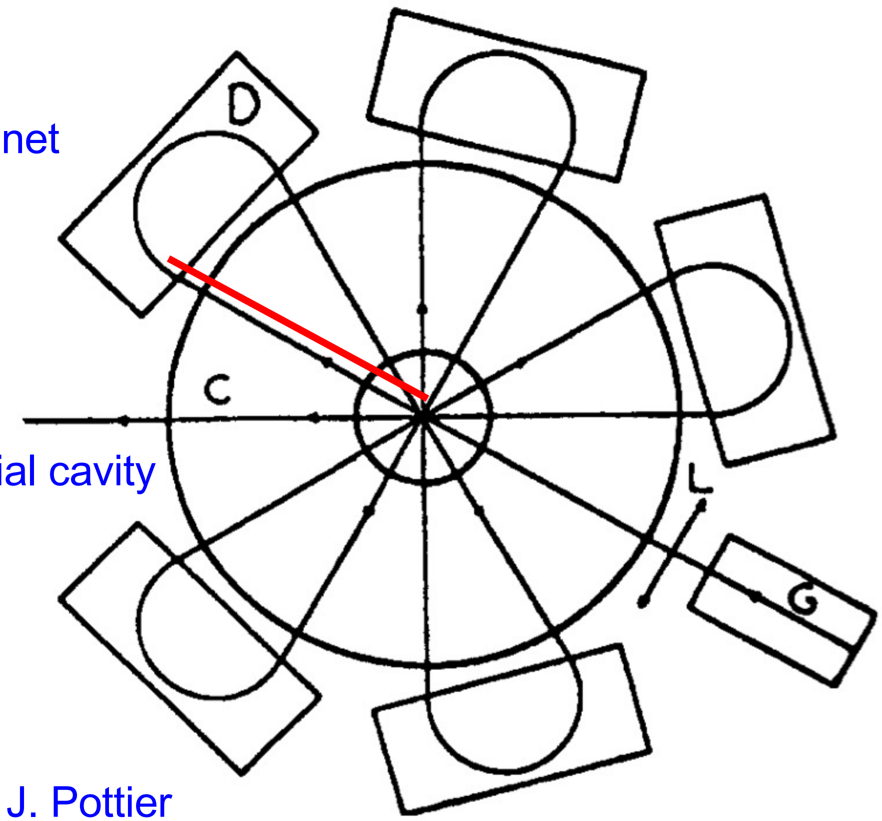
- Electrons are quickly relativistic and cannot be accelerated in a cyclotron.
- In a microtron the revolution frequency changes, but each electron misses an integer number of RF waves.



- Today: Used for medical applications with one magnet and 20MeV.
- Nuclear physics: MAMI designed for 820MeV as race track microtron.

The Rhodotron

Magnet

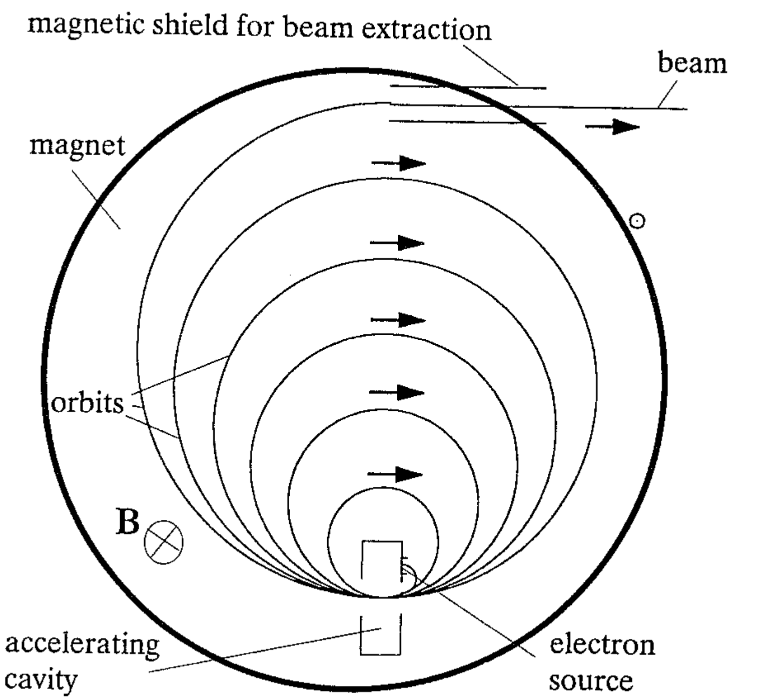


Coaxial cavity

- Invented in 1989 by J. Pottier (French, CEA)

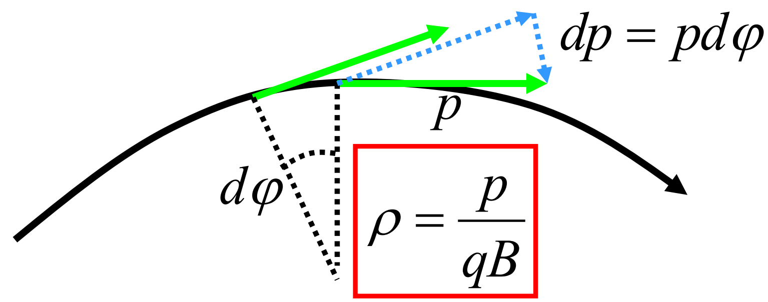
The Microtron Condition

- The extra time that each turn takes must be a multiple of the RF period.



B=1T, n=1, and f_{RF}=3GHz leads to 4.78MeV
This requires a small linear accelerator.

$$\frac{dp}{dt} = qvB \Rightarrow \rho = \frac{dl}{d\varphi} = \frac{vdt}{dp/p} = \frac{p}{qB}$$



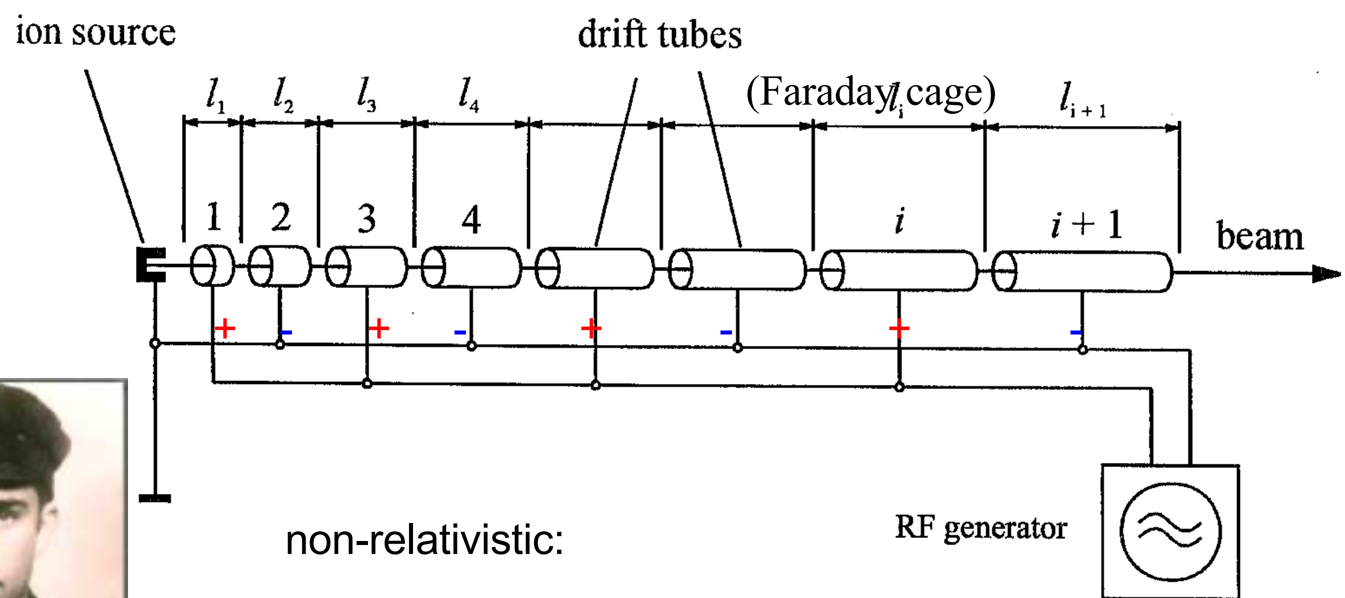
$$\rho = \frac{p}{qB}$$

$$\Delta t = 2\pi\left(\frac{\rho_{n+1}}{v_{n+1}} - \frac{\rho_n}{v_n}\right)$$

$$= \frac{2\pi}{qB}(m_0\gamma_{n+1} - m_0\gamma_n) = \frac{2\pi}{qBc^2}\Delta K$$

$$\Delta K = n\frac{qBc^2}{\omega_{RF}}$$ for an integer n

The Wideroe linear accelerator



Wideroe

non-relativistic:

$$K_n = nqU_{max}\sin\psi_0 = \tfrac{1}{2}mv_n^2$$

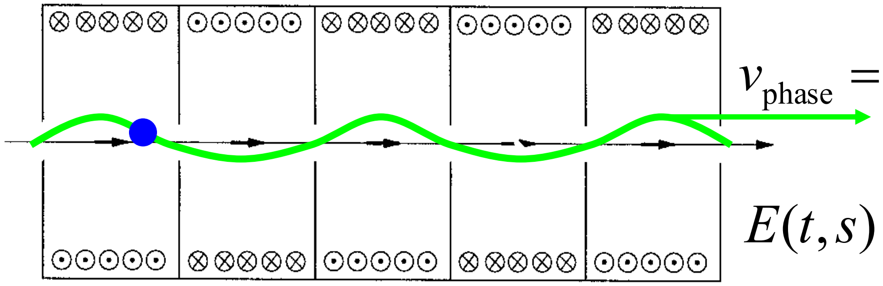
$$l_n = \tfrac{1}{2}v_nT_{RF} = \tfrac{1}{2}\beta_n\lambda_{RF} \propto \sqrt{n}$$

Called the π or the $1/2\beta\lambda$ mode

Accelerating Cavities

- 1933: J.W. Beams uses resonant cavities for acceleration

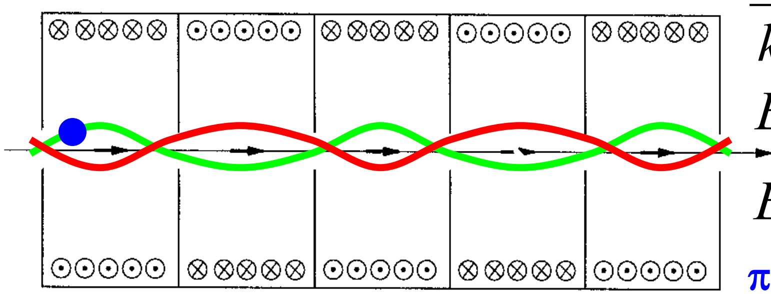
Traveling wave cavity:



$$v_{\text{phase}} = \frac{\omega}{k} = v_{\text{particle}}$$ Here v=c for electrons

$$E(t,s) \approx E_{\max} \sin(\omega t - ks)$$

Standing wave cavity:



$$\frac{\omega}{k} = v_{\text{particle}}$$

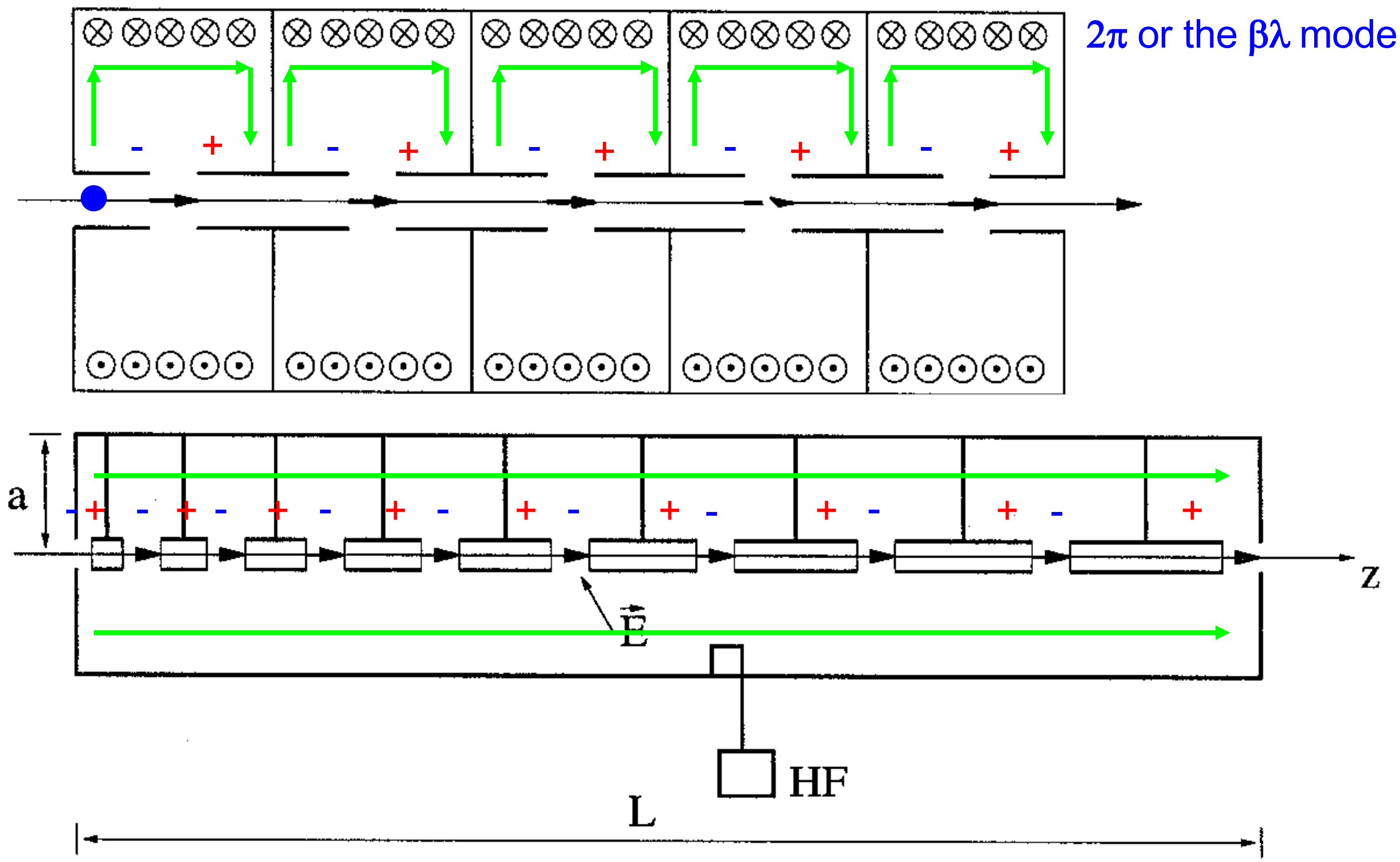
$$E(t,s) \approx E_{\max} \sin(\omega t) \sin(ks)$$

$$E(\tfrac{s}{v_{\text{particle}}}, s) \approx E_{\max} \sin^2(ks)$$

π or the 1/2βλ mode

Transit factor (for this example): $\langle E \rangle = \frac{1}{\lambda_{RF}} \int_0^{\lambda_{RF}} E(\tfrac{s}{v_{\text{particle}}}, s)\, ds \approx \tfrac{1}{2} E_{\max}$

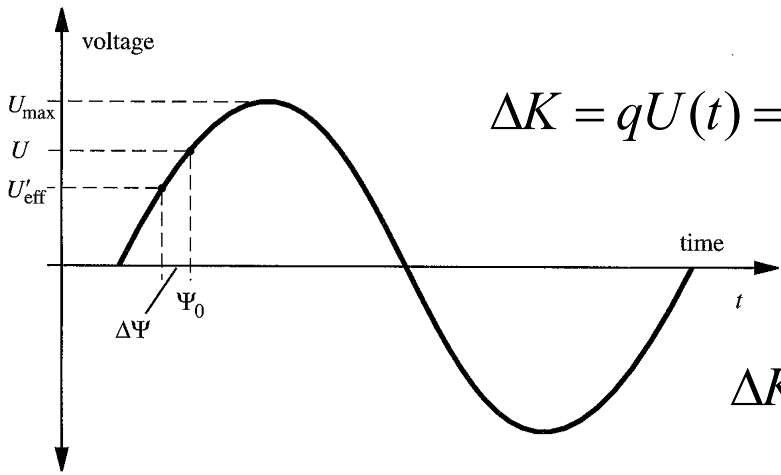
The Alvarez Linear Accelerator



Needs only one power input coupler and walls do not dissipate energy.

Phase Focusing

- 1945: Veksler (UDSSR) and McMillan (USA) realize the importance of phase focusing



$$\Delta K = qU(t) = qU_{\max}\sin(\omega(t-t_0)+\psi_0)$$

Longitudinal position in the bunch:

$$\sigma = s - s_0 = -v_0(t-t_0)$$

$$\Delta K(\sigma) = qU_{\max}\sin(-\tfrac{\omega}{v_0}(s-s_0)+\psi_0)$$

$$\Delta K(0) > 0$$ (Acceleration)

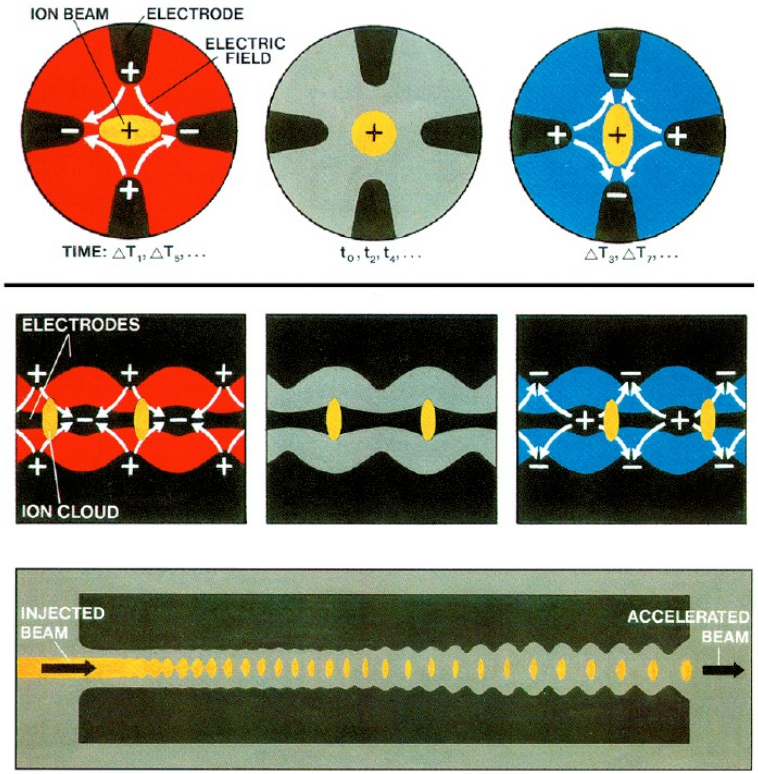
$$\Delta K(\sigma) < \Delta K(0) \text{ for } \sigma > 0 \Rightarrow \tfrac{d}{d\sigma}\Delta K(\sigma) < 0$$ (Phase focusing)

$$\left.\begin{array}{r} qU(t) > 0 \\ q\tfrac{d}{dt}U(t) > 0 \end{array}\right\} \quad \psi_0 \in (0, \frac{\pi}{2})$$

Phase focusing is required in any RF accelerator.

Radio Frequency Quadrupoles

- 1970: Kapchinskii and Teplyakov invent the RFQ



Three historic lines of accelerators

Direct Voltage Accelerators   Resonant Accelerators   Transformer Accelerator

- 1924: Wideroe invents the betatron
- 1940: Kerst and Serber build a betatron for 2.3MeV electrons and understand betatron (transverse) focusing (in 1942: 20MeV)

Betatron:

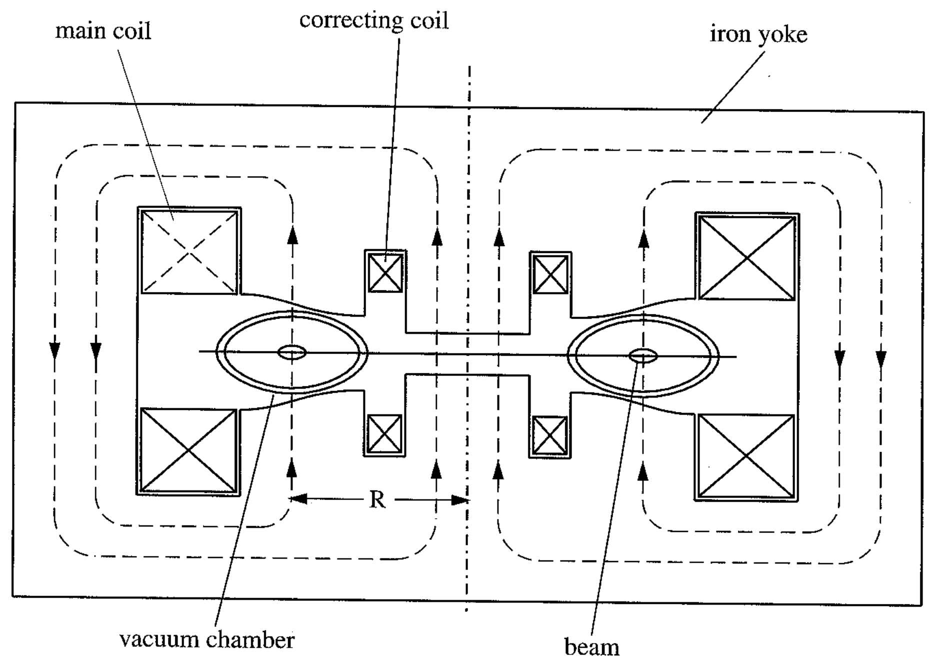
R=const, B=B(t)

Whereas for a cyclotron:

R(t), B=const

No acceleration section is needed since

$$\oint_{\partial A} \vec{E} \cdot d\vec{s} = -\iint_{A} \tfrac{d}{dt} \vec{B} \cdot d\vec{a}$$



The Betatron Condition

Condition: $R = \dfrac{-p_\varphi(t)}{qB_z(R,t)} = \text{const.}$ given $\oint_{\partial A} \vec{E}\cdot d\vec{s} = -\iint_A \frac{d}{dt}\vec{B}\cdot d\vec{a}$

$$E_\varphi(R,t) = -\tfrac{1}{2\pi R}\int \tfrac{d}{dt}B_z(r,t)\,r\,dr\,d\varphi = -\tfrac{R}{2}\left\langle \tfrac{d}{dt}B_z\right\rangle$$

$$\tfrac{d}{dt}p_\varphi(t) = qE_\varphi(R,t) = -q\,\tfrac{R}{2}\left\langle \tfrac{d}{dt}B_z\right\rangle$$

$$p_\varphi(t) = p_\varphi(0) - q\,\tfrac{R}{2}[\langle B_z\rangle(t) - \langle B_z\rangle(0)] = -RqB_z(R,t)$$

$$B_z(R,t) - B_z(R,0) = \frac{1}{2}[\langle B_z\rangle(t) - \langle B_z\rangle(0)]$$

Small deviations from this condition lead to transverse beam oscillations called betatron oscillations in all accelerators.

- Today: Betatrons with typically about 20MeV for medical applications

The Synchrotron

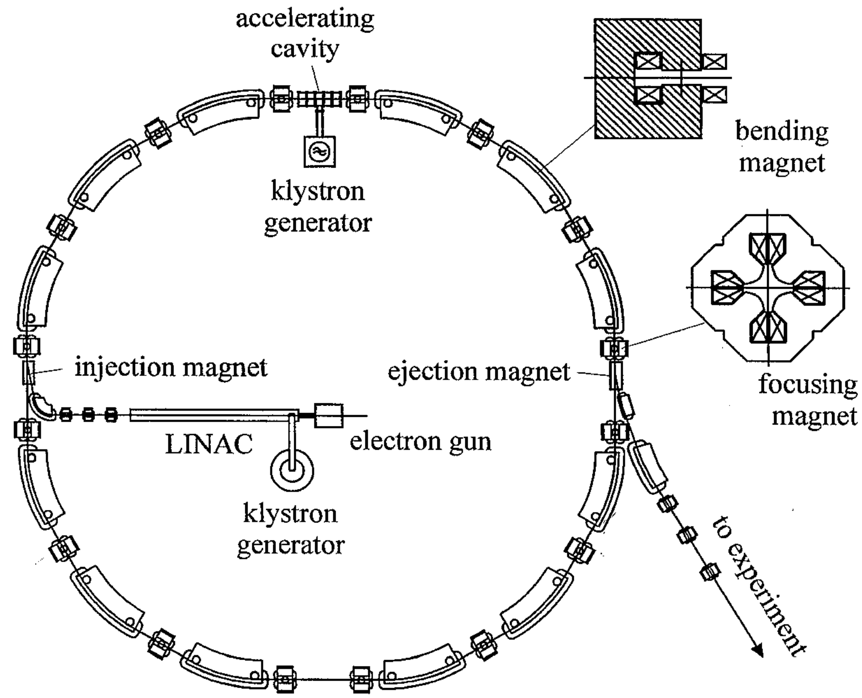
- 1945: Veksler (UDSSR) and McMillan (USA) invent the synchrotron
- 1946: Goward and Barnes build the first synchrotron (using a betatron magnet)
- 1949: Wilson et al. at Cornell are first to store beam in a synchrotron (later 300MeV, magnet of 80 Tons)
- 1949: McMillan builds a 320MeV electron synchrotron

- Many smaller magnets instead of one large magnet
- Only one acceleration section is needed, with

$$R = \frac{p(t)}{qB(R,t)} = \text{const.}$$

$$\omega = 2\pi \frac{v_{\text{particle}}}{L} n$$

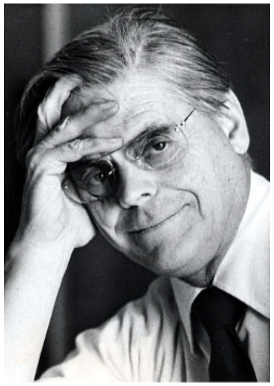
for an integer n called the harmonic number



Robert Wilson, Fermilab …

Science Ed Center, FNAL (1990)

Wilson Hall, FNAL

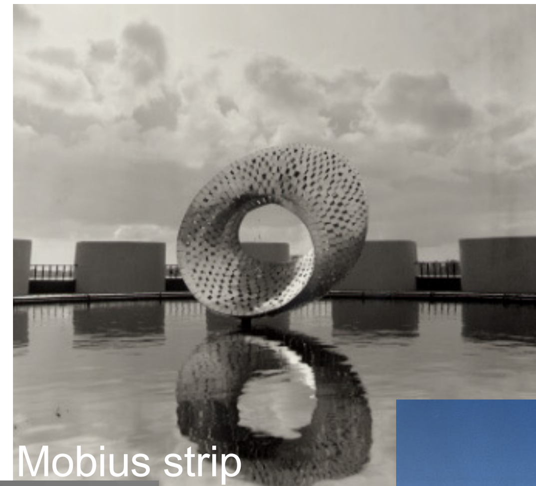


Robert R Wilson

USA 1914-2000

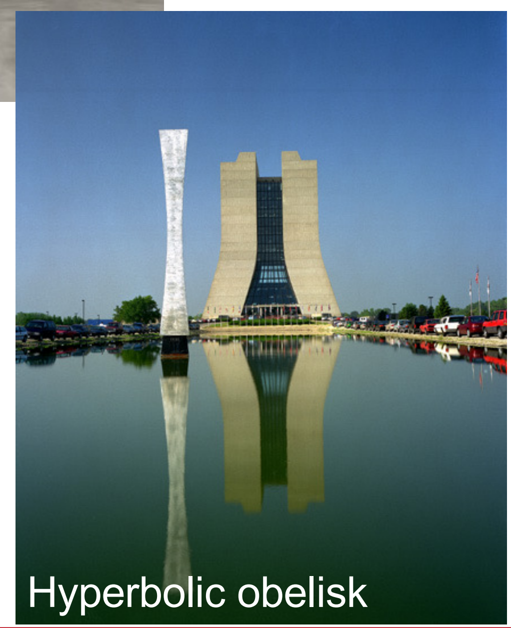
… art, architecture, and accelerators.


Broken symmetry


Mobius strip


Tractricious

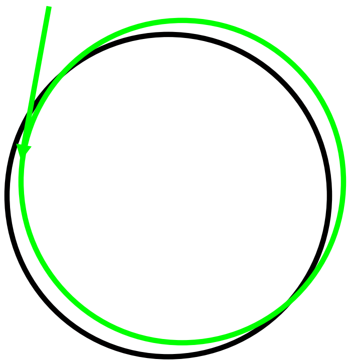

π lines

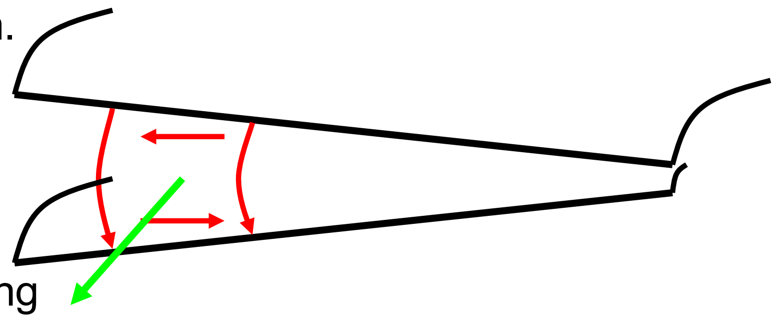

Hyperbolic obelisk

The weak focusing synchrotron

- 1952: Operation of the Cosmotron, 3.3 GeV proton synchrotron at Brookhaven: Beam pipe height: 15cm.

Natural ring focusing:



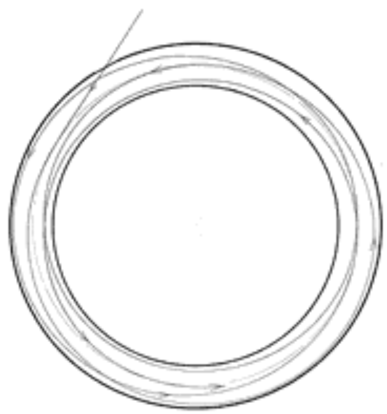


Vertical focusing

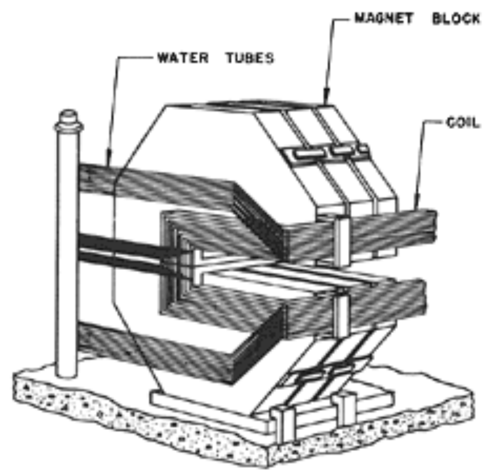
+ Horizontal defocusing + ring focusing

Focusing in both planes

The Cosmotron



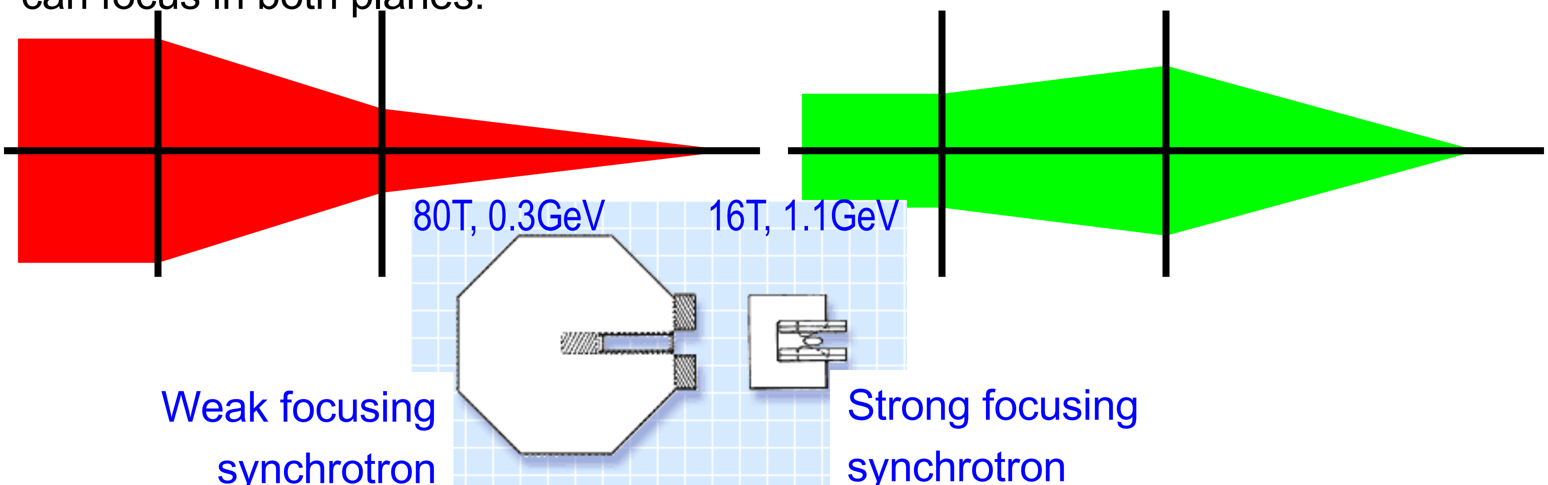
Weak focusing accelerator



Strong focusing synchrotrons

- 1952: Courant, Livingston, Snyder publish about strong focusing
- 1954: Wilson et al. build first synchrotron with strong focusing for 1.1MeV electrons at Cornell, 4cm beam pipe height, only 16 Tons of magnets.
- 1959: CERN builds the PS for 28GeV after proposing a 5GeV weak focusing accelerator for the same cost (still in use)

Transverse fields defocus in one plane if they focus in the other plane.
But two successive elements, one focusing the other defocusing,
can focus in both planes:



- Today: only strong focusing is used. Due to bad field quality at lower field excitations the injection energy is 20-500MeV from a linac or a microtron.

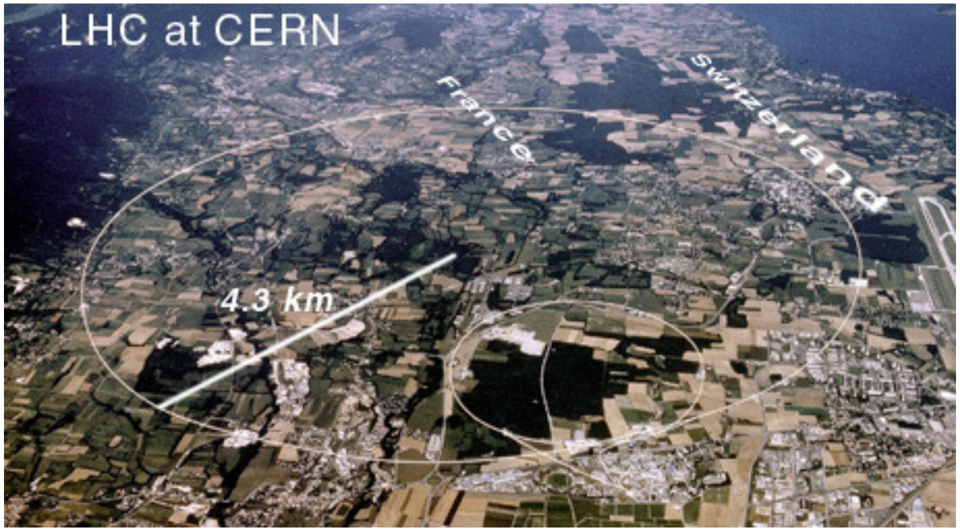
Limits of synchrotrons

$$\rho = \frac{p}{qB} \quad \Rightarrow \quad \text{The rings become too long}$$

Protons with p = 20 TeV/c , B = 6.8 T would require a 87 km SSC tunnel
Protons with p = 7 TeV/c , B = 8.4 T require CERN's 27 km LHC tunnel

$$P_{\text{radiation}} = \frac{c}{6\pi\varepsilon_0} N \frac{q^2}{\rho^2} \gamma^4 \quad \Downarrow$$

Energy needed to compensate
Radiation becomes too large



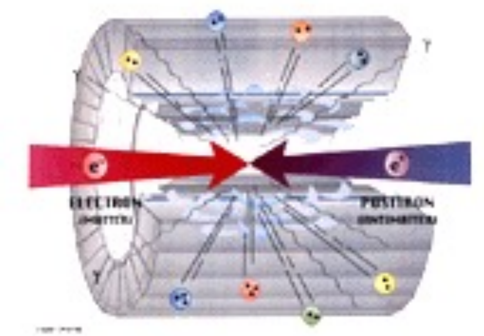
Electron beam with p = 0.1 TeV/c in CERN's 27 km LEP tunnel radiated 20 MW
Each electron lost about 4GeV per turn, requiring many RF accelerating sections.

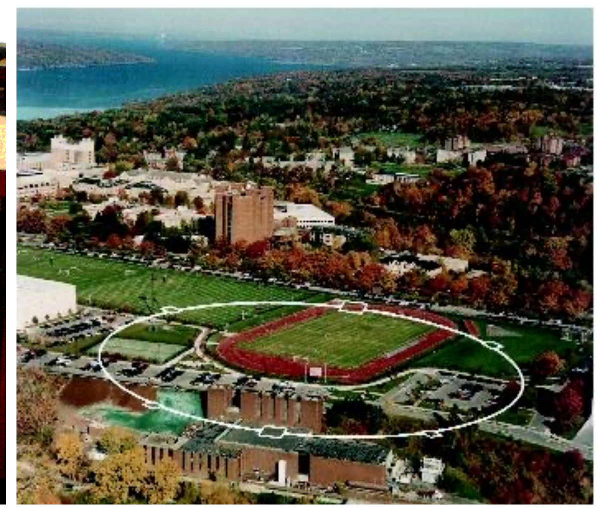
Colliding beam accelerators

- 1961: First storage ring for electrons and positrons (AdA) in Frascati for 250MeV
- 1972: SPEAR electron positron collider at 4GeV. Discovery of the J/Psi at 3.097GeV by Richter (SPEAR) and Ting (AGS) starts the November revolution and was essential for the quarkmodel and chromodynamics.
- 1979: 5GeV electron positron collider CESR (designed for 8GeV)

Advantage:

More center of mass energy





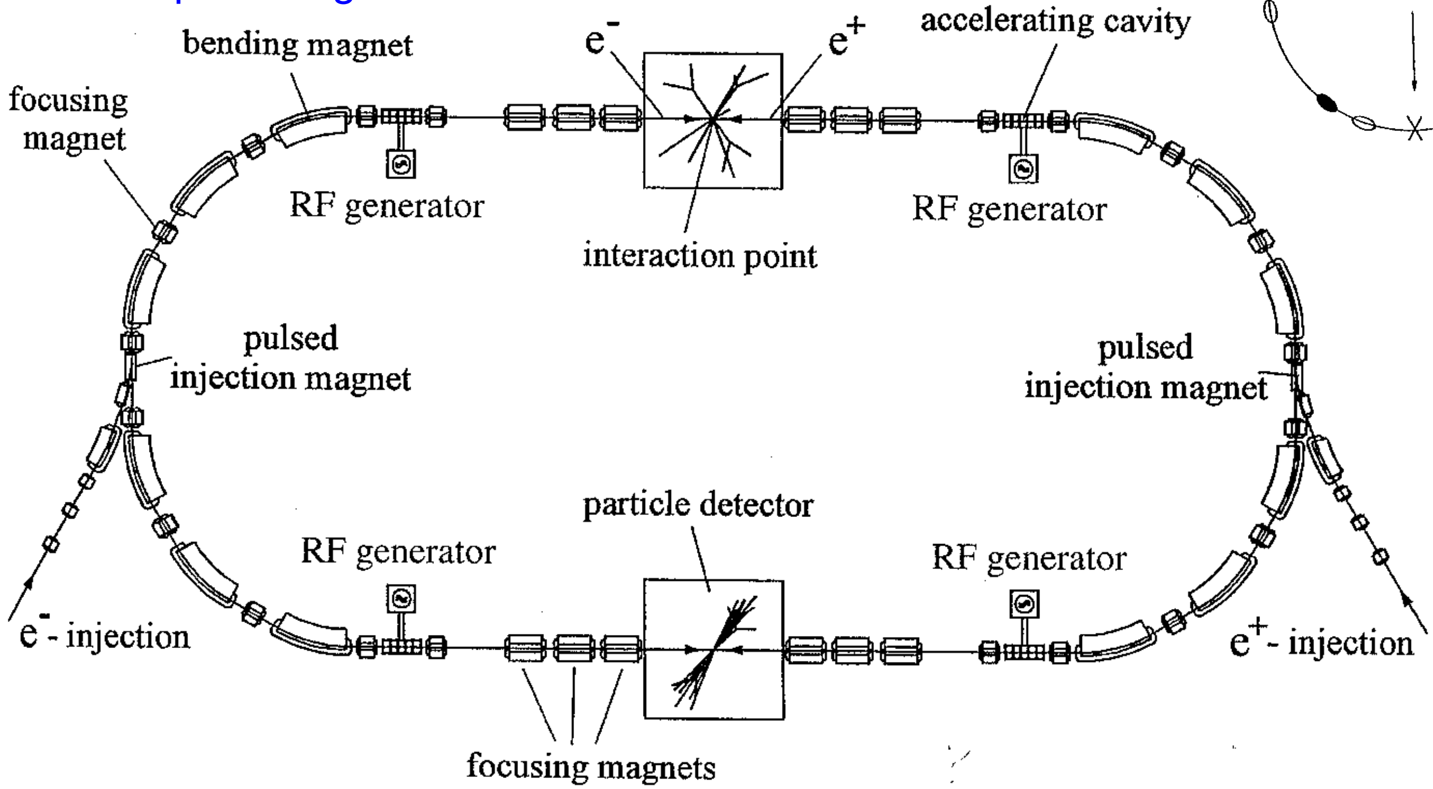
Drawback:

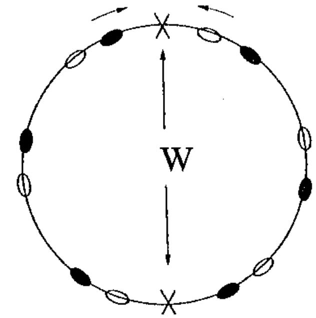
Less dense target

The beams therefore must be stored for a long time.

Elements of a collider

- Saving one beam while injection another
- Avoiding collisions outside the detectors.
- Compensating the forces between e^+ and e^- beams





Storage Rings

To avoid the loss of collision time during filling of a synchrotron, the beams in colliders must be stored for many millions of turns.

Challenges:

- Required vacuum of pressure below 10^{-7} Pa = 10^{-9} mbar, 3 orders of magnitude below that of other accelerators.
- Fields must be stable for a long time, often for hours.
- Field errors must be small, since their effect can add up over millions of turns.
- Even though a storage ring does not accelerate, it needs acceleration sections for phase focusing and to compensate energy loss due to the emission of radiation.

Further developments of Colliders

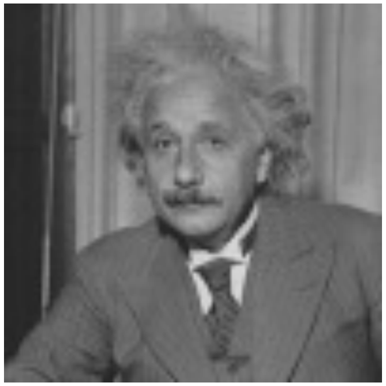
- 1981: Rubbia and van der Meer use stochastic cooling of anti-portons and discover W+,W- and Z vector bosons of the weak interaction
- 1987: Start of the superconducting TEVATRON at FNAL
- 1989: Start of the 27km long LEP electron positron collider
- 1990: Start of the first asymmetric collider, electron (27.5GeV) proton (920GeV) in HERA at DESY
- 1998: Start of asymmetric two ring electron positron colliders KEK-B / PEP-II
- Today: 27km, 6.8 TeV proton collider LHC; Higgs discovery in 2012

NP 1984
Carlo Rubbia
Italy 1934 -

NP 1984
Simon van der Meer
Netherlands 1925 -

Special Relativity

$$E = mc^2$$



Albert Einstein, 1879-1955
Nobel Prize, 1921
Time Magazine Man of the Century

Four-Vectors:
Quantities that transform according to the Lorentz transformation when viewed from a different inertial frame.

Examples:

$$X^\mu \in \{ct, x, y, z\}$$

$$P^\mu \in \{\tfrac{1}{c}E, p_x, p_y, p_z\}$$

$$\Phi^\mu \in \{\tfrac{1}{c}\phi, A_x, A_y, A_z\}$$

$$J^\mu \in \{c\rho, j_x, j_y, j_z\}$$

$$K^\mu \in \{\tfrac{1}{c}\omega, k_x, k_y, k_z\}$$

$$X^\mu \in \{ct, x, y, z\} \quad \Rightarrow \quad X^\mu X_\mu = (ct)^2 - \vec{x}^2 = \text{const.}$$

$$P^\mu \in \{\tfrac{1}{c}E, p_x, p_y, p_z\} \Rightarrow \quad P^\mu P_\mu = \left(\frac{E}{c}\right)^2 - \vec{p}^2 = (m_0 c)^2 = \text{const.}$$

Available Energy in Collisions

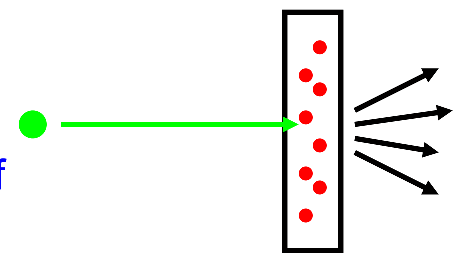
$$\frac{1}{c^2}E_{\rm cm}^2 = (P_1^\mu + P_2^\mu)_{\rm cm}(P_{1\mu} + P_{2\mu})_{\rm cm}$$

$$= (P_1^\mu + P_2^\mu)(P_{1\mu} + P_{2\mu})$$

$$= \frac{1}{c^2}(E_1 + E_2)^2 - (p_{z1} - p_{z2})^2$$

$$= 2(\frac{E_1 E_2}{c^2} + p_{z1}p_{z2}) + (m_{01}c)^2 + (m_{02}c)^2$$

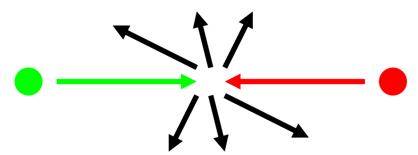
Operation of synchrotrons: fixed target experiments where some energy is in the motion of the center off mass of the scattering products



$$E_1 >> m_{01}c^2, m_{02}c^2; p_{z2} = 0; E_2 = m_{02}c^2 \quad \Rightarrow \quad E_{\rm cm} = \sqrt{2E_1 m_{02}c^2}$$

Operation of colliders:

the detector is in the center of mass system



$$E_1 >> m_{01}c^2; E_2 >> m_{02}c^2 \quad \Rightarrow \quad E_{\rm cm} = 2\sqrt{E_1 E_2}$$

Example: Fixed-target production of p-bar

- 1954: Operation of Bevatron, first proton synchrotron for 6.2GeV, production of the anti-porton by Chamberlain and Segrè

$$p+p \quad \mapsto \quad p+p+p+\bar{p}$$

$$\tfrac{1}{c^2}E_{\text{cm}}^2 = 2(\frac{E_1E_2}{c^2}+p_{z1}p_{z2})+(m_{01}c)^2+(m_{02}c)^2$$

$$(4m_{p0}c)^2 < \frac{E_{cm}^2}{c^2} = 2\,E_1m_{p0} + (m_{po}c)^2 + (m_{po}c)^2$$

$$7m_{p0}c^2 < E_1$$

$$K_1 = E_1 - m_0c^2 > 6m_{p0}c^2 = 5.628\,\text{GeV}$$

NP 1959
Emilio Gino Segrè
Italy 1905 – USA 1989

NP 1959
Owen Chamberlain
USA 1920 - 2006

Example: production of c / c-bar states

- 1974: Observation of $c-\bar{c}$ resonances (J/Ψ) at Ecm = 3095MeV at the e$^+$/e$^-$ collider SPEAR

$$\frac{1}{c^2}E_{\rm cm}^2 = 2(\frac{E_1E_2}{c^2} + p_{z1}p_{z2}) + (m_{01}c)^2 + (m_{02}c)^2$$

Resonance in c/c-bar creation when $E_{cm} = 2m_{c0}c^2$ $\quad E_1 = E_2 \quad \Rightarrow \quad E_{\rm cm}^2 = 4E^2$

Energy per beam: $K = E - m_0c = 1547\text{MeV}$

Beam energy needed for an equivalent fixed target experiment: $\frac{E_{cm}^2}{c^2} = 2[Em + (mc)^2]$

$$K = E - m_{0e}c^2 = \frac{E_{cm}^2}{2m_{0e}c^2} - 2m_{0e}c^2 = 9.4\text{TeV}$$

NP 1976
Burton Richter
USA 1931 -

NP 1976
Samuel CC Ting
USA 1936 -

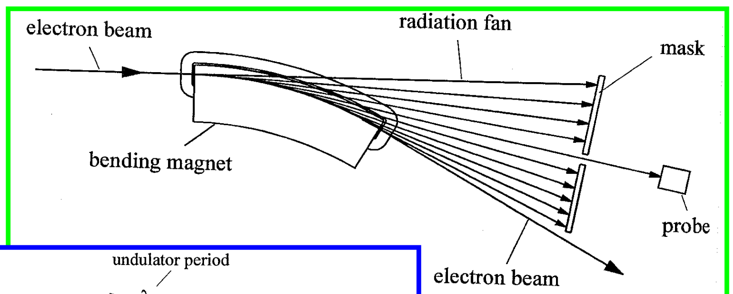
Rings for Synchrotron Radiation

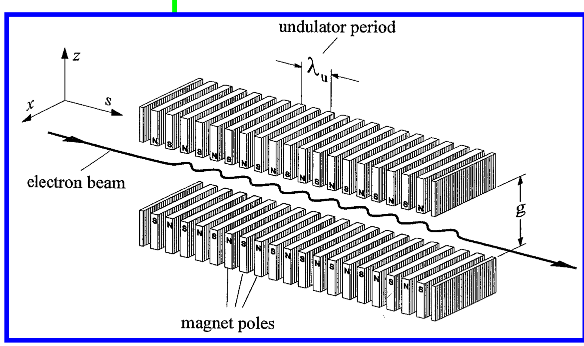
- 1947: First detection of synchrotron light at General Electrics.
- 1952: First accurate measurement of synchrotron radiation power by Dale Corson with the Cornell 300MeV synchrotron.
- 1968: TANTALOS (U of Wisconsin), first dedicated storage ring for synchrotron radiation

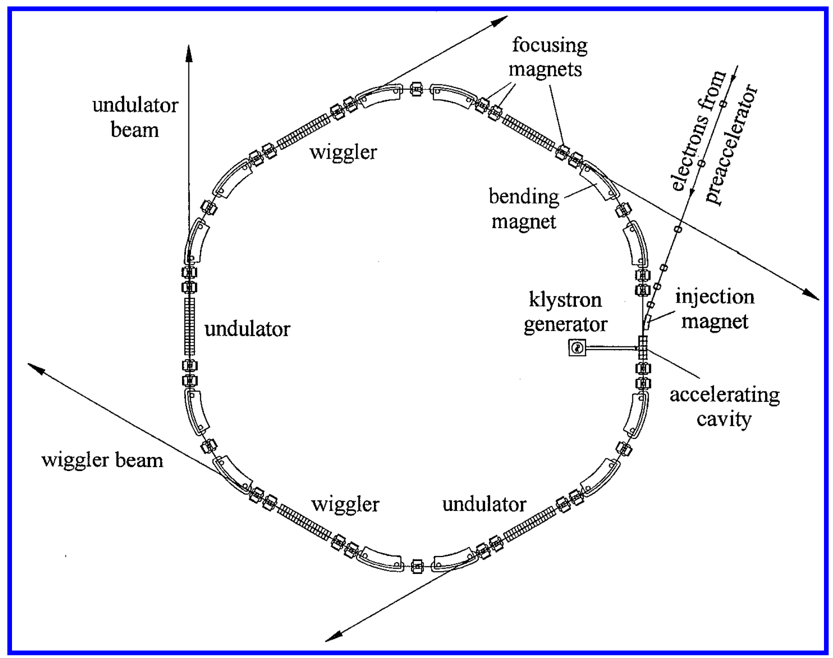
Dale Corson
Cornell's 8th president
USA 1914 –

4 Generations of Light Sources

- 1st Genergation (1970s): Many HEP rings are parasitically used for X-ray production
- 2nd Generation (1980s): Many dedicated X-ray sources (light sources)
- 3rd Generation (1990s): Several rings with dedicated radiation devices (wigglers and undulators)
- Today (4th Generation): Construction of Free Electron Lasers (FELs) driven by LINACs







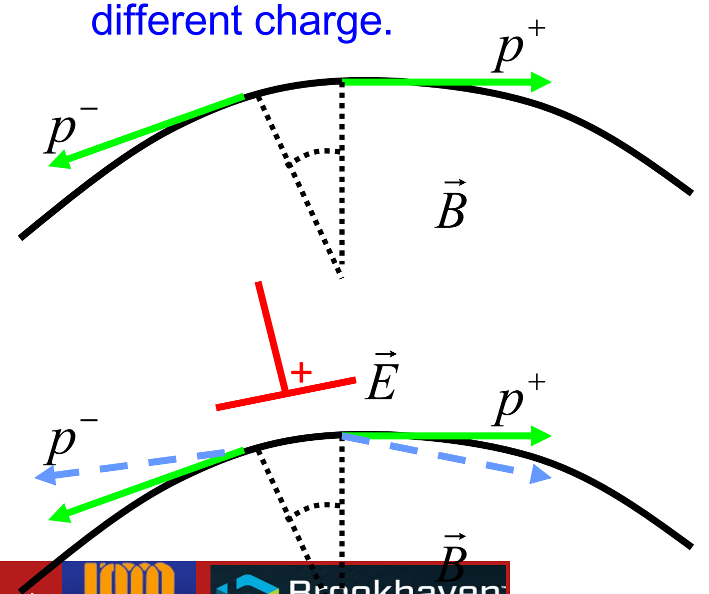
Cornell University

Macroscopic Fields in Accelerators

$$\frac{d}{dt}\vec{p} = q(\vec{E} + \vec{v} \times \vec{B})$$

E has a similar effect as v B.
For relativistic particles B = 1T has a similar effect as
E = cB = 3 10^8 V/m , such an
Electric field is beyond technical limits.

- Electric fields are only used for very low energies or
- For separating two counter rotating beams with different charge.




Electrostatic separators at CESR

Cornell University

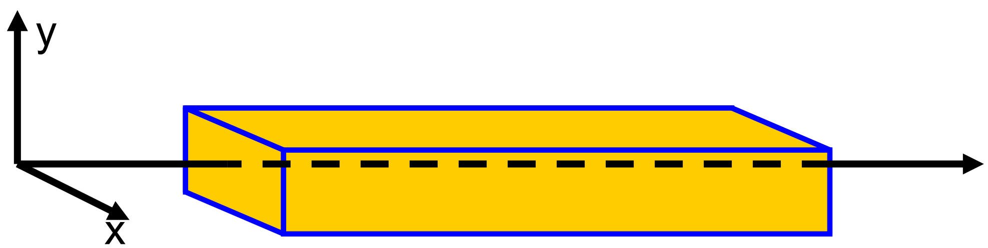
Brookhaven National Laboratory

Magnetic Fields in Accelerators

Static magnetic fileds: $\partial_t \vec{B} = 0\,; \quad \vec{E} = 0$ Charge free space: $\vec{j} = 0$

$$\vec{\nabla} \times \vec{B} = \mu_0(\vec{j} + \varepsilon_0 \partial_t \vec{E}) = 0 \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi(\vec{r})$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{\nabla}^2 \psi(\vec{r}) = 0$$

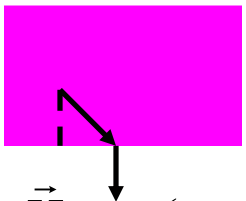


(x=0,y=0) is the beam's design curve

For finite fields on the design curve,
Ψ can be power expanded in x and y:

$$\psi(x,y,z) = \sum_{n,m=0}^{\infty} b_{nm}(z) x^n y^m$$

Surfaces of equal scalar magnetic potential



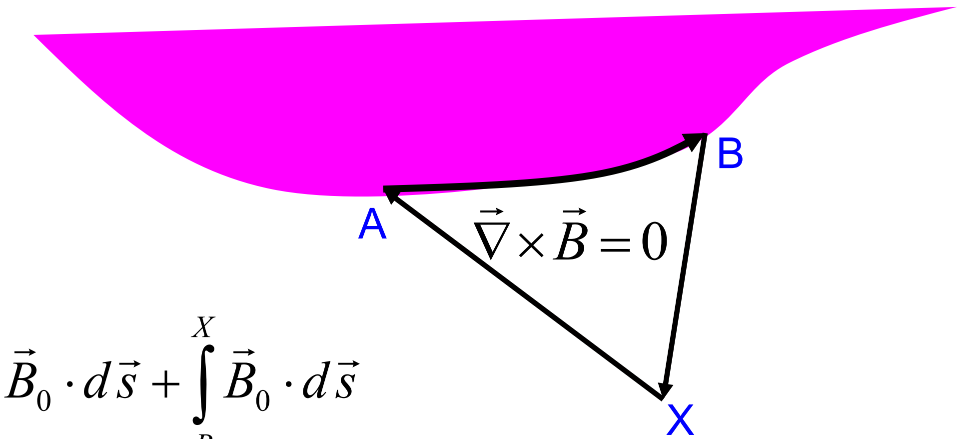
$$\vec{B}_\perp(\text{out}) = \vec{B}_\perp(\text{in})$$

$$\vec{H}_{\text{parallel}}(\text{out}) = \vec{H}_{\text{parallel}}(\text{in})$$

$$\vec{B}_{\text{parallel}}(\text{out}) = \frac{1}{\mu_r}\vec{B}_{\text{parallel}}(\text{in})$$

$$\vec{B}(\vec{r}) = -\vec{\nabla}\Psi(\vec{r})$$

For large permeability, H(out) is perpendicular to the surface.



$$0 = \oint \vec{B}\cdot d\vec{s} = \int_X^A \vec{B}_0\cdot d\vec{s} + \int_A^B \vec{B}_0\cdot d\vec{s} + \int_B^X \vec{B}_0\cdot d\vec{s}$$

$$= \int_X^A \vec{B}_0\cdot d\vec{s} + \frac{1}{\mu_r}\int_A^B \vec{B}_0\cdot d\vec{s} + \int_B^X \vec{B}_0\cdot d\vec{s}$$

$$\approx \int_X^A \vec{B}_0\cdot d\vec{s} + \int_B^X \vec{B}_0\cdot d\vec{s} = \Psi(A) - \Psi(B)$$

For highly permeable materials (like iron) surfaces have a constant potential.

Green's Theorem

$$\vec{\nabla}^2\psi = 0$$

Green function:

$$\vec{\nabla}_0^2 G(\vec{r},\vec{r}_0) = \delta(\vec{r}-\vec{r}_0)$$

$$\begin{aligned}
\psi(\vec{r}) &= \int_V \psi(\vec{r}_0)\delta(\vec{r}-\vec{r}_0)\,d^3\vec{r}_0 \\
&= \int_V \left[\psi(\vec{r}_0)\vec{\nabla}_0^2 G - G\vec{\nabla}_0^2\psi(\vec{r}_0)\right] d^3\vec{r}_0 \\
&= \int_V \vec{\nabla}_0\left[\psi(\vec{r}_0)\vec{\nabla}_0 G - G\vec{\nabla}_0\psi(\vec{r}_0)\right] d^3\vec{r}_0 \\
&= \int_{\partial V} \left[\psi(\vec{r}_0)\vec{\nabla}_0 G - G\vec{\nabla}_0\psi(\vec{r}_0)\right]\cdot d^2\vec{r}_0 \\
&= \int_{\partial V} \left[\psi(\vec{r}_0)\vec{\nabla}_0 G + \vec{B}(\vec{r}_0)G\right]\cdot d^2\vec{r}_0
\end{aligned}$$

Knowledge of the field and the scalar magnetic potential on a closed surface inside a magnet determines the magnetic field for the complete volume which is enclosed.

Expansion of the scalar potential

If field data in a plane (for example the midplane of a cyclotron or of a beam line magnet) is known, the complete filed is determined:

$$\psi(x,y,z)=\sum_{n=0}^{\infty} b_n(x,z)y^n \quad \Rightarrow \quad \vec{B}(x,0,z)=-\begin{pmatrix}\partial_x b_0(x,z)\\ b_1(x,z)\\ \partial_z b_0(x,z)\end{pmatrix}$$

$$0=\vec{\nabla}^2\psi=\sum_{n=0}^{\infty}(\partial_x^2+\partial_z^2)\,b_n\,y^n+\sum_{n=2}^{\infty}n(n-1)\,b_n\,y^{n-2}$$

$$=\sum_{n=0}^{\infty}\left[(\partial_x^2+\partial_z^2)\,b_n+(n+2)(n+1)\,b_{n+2}\right]y^n$$

$$b_{n+2}(x,z)=-\frac{1}{(n+2)(n+1)}(\partial_x^2+\partial_y^2)\,b_n(x,z)$$

Data of the magnetic field in the plane y=0 is used to determine $b_0(x,z)$ and $b_1(x,z)$.

Complex expansion of the potential

$$w = x + iy \quad , \quad \overline{w} = x - iy$$

$$\partial_x = \partial_w + \partial_{\overline{w}} \; , \quad \partial_y = i\partial_w - i\partial_{\overline{w}} = i(\partial_w - \partial_{\overline{w}})$$

$$\vec{\nabla}^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 = (\partial_w + \partial_{\overline{w}})^2 - (\partial_w - \partial_{\overline{w}})^2 + \partial_z^2 = 4\partial_w \partial_{\overline{w}} + \partial_z^2$$

$$\psi = \mathrm{Im}\{\sum_{\nu,\lambda=0}^{\infty} a_{\nu\lambda}(z) \cdot (w\overline{w})^{\lambda}\, \overline{w}^{\nu}\}$$

$$\vec{\nabla}^2 \psi = \mathrm{Im}\{ \sum_{\nu=0,\lambda=1}^{\infty} 4a_{\nu\lambda}(\lambda+\nu)\lambda(w\overline{w})^{\lambda-1}\,\overline{w}^{\nu} + \sum_{\nu=0,\lambda=0}^{\infty} a''_{\nu\lambda}(w\overline{w})^{\lambda}\,\overline{w}^{\nu}\}$$

$$= \mathrm{Im}\{\sum_{\nu,\lambda=0}^{\infty} [4(\lambda+1+\nu)(\lambda+1)a_{\nu\lambda+1} + a''_{\nu\lambda}](w\overline{w})^{\lambda}\,\overline{w}^{\nu}\} = 0$$

Iteration equation: $a_{\nu\lambda+1} = \dfrac{-1}{4(\lambda+1+\nu)(\lambda+1)} a''_{\nu\lambda} \quad , \quad a_{\nu 0} = \Psi_\nu(z)$

The functions $\Psi_\nu(z)$ along a line determine the complete field inside a magnet.

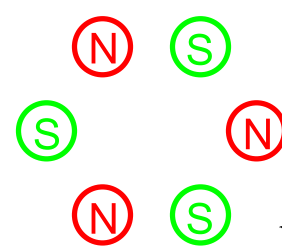
Multipole coefficients

$\Psi_\nu(z)$ are called the z-dependent multipole coefficients

$$\psi(x,y,z)=\mathrm{Im}\{\sum_{\nu,\lambda=0}^{\infty}\frac{(-1)^\lambda \nu!}{(\lambda+\nu)!\lambda!}\left(\frac{w\overline{w}}{4}\right)^\lambda \overline{w}^\nu \Psi_\nu^{[2\lambda]}(z)\}$$

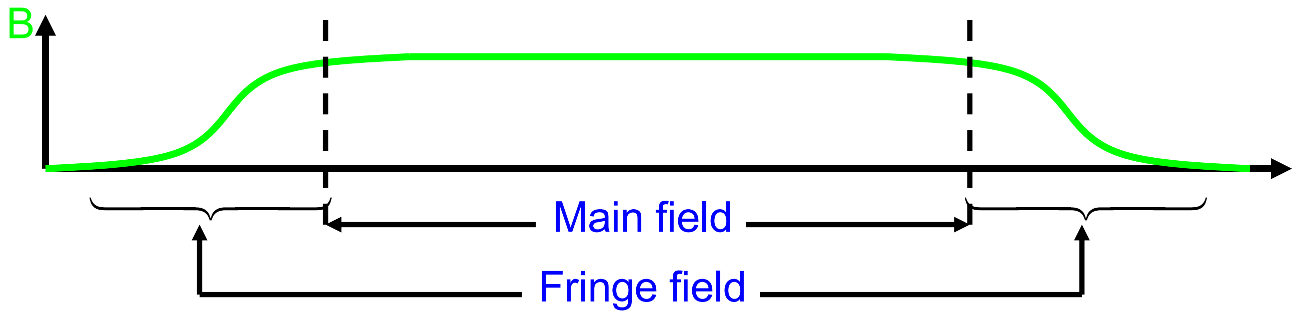
$$\psi(r,\varphi,z)=\sum_{\nu,\lambda=0}^{\infty}\frac{(-1)^\lambda \nu!}{(\lambda+\nu)!\lambda!}\left(\frac{r}{2}\right)^{2\lambda} r^\nu\,\mathrm{Im}\{\Psi_\nu^{[2\lambda]}(z)e^{-i\nu\varphi}\}$$

The index ν describes C_ν Symmetry around the z-axis $\vec{e}_z$ due to a sign change after $\Delta\varphi=\frac{\pi}{\nu}$



$\nu=3$

Fringe Fields and Main fields



Only the fringe field region has terms with $\lambda \neq 0$ and $\partial_z^2 \psi \neq 0$

Main fields in accelerator physics: $\lambda = 0\,,\quad \partial_z^2 \psi = 0$

$$\Psi_\nu = \begin{cases} e^{i\nu\vartheta_\nu} |\Psi_\nu| & \text{for} \quad \nu \neq 0 \\ i \quad |\Psi_0| & \text{for} \quad \nu = 0 \end{cases}$$

$$\psi(r,\varphi) = \sum_{\nu=1}^{\infty} r^\nu |\Psi_\nu| \, \mathrm{Im}\{e^{-i\nu(\varphi-\vartheta_\nu)}\} + |\Psi_0|$$

Main-Field Potential

Main field potential: $$\psi = |\Psi_0| - \sum_{\nu=1}^{\infty} r^{\nu} |\Psi_{\nu}| \sin[\nu(\varphi - \vartheta_{\nu})]$$

The isolated multipole: $$\psi = -r^{\nu} |\Psi_{\nu}| \sin(\nu\varphi)$$

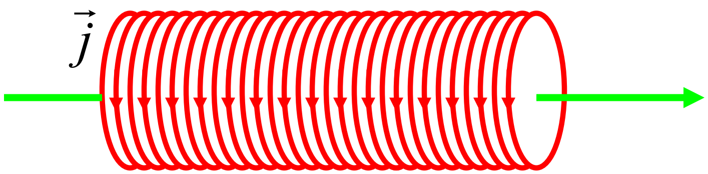
Where the rotation ϑ_{ν} of the coordinate system is set to 0

The potentials produced by different multipole components Ψ_{ν} have

a) Different rotation symmetry C_{ν}

b) Different radial dependence r^{ν}

Multipoles in Accelerators: ν=0, Solenoids



$$m\gamma\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = q\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}\times\begin{pmatrix}-\frac{x}{2}B_z'\\ -\frac{y}{2}B_z'\\ B_z\end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \frac{qB_z}{m\gamma}\begin{pmatrix}\dot{y}\\ -\dot{x}\end{pmatrix} + \frac{qB_z'\dot{z}}{2m\gamma}\begin{pmatrix}y\\ -x\end{pmatrix}$$

$$\Downarrow$$

$$\ddot{w} = -i\frac{qB_z}{m\gamma}\dot{w} - i\frac{q\dot{B}_z}{2m\gamma}w$$

$$\psi = \Psi_0(z) - \frac{w\bar{w}}{4}\Psi_0''(z) \pm \dots$$

$$\vec{B} = \begin{pmatrix}\frac{x}{2}\Psi_0''\\ \frac{y}{2}\Psi_0''\\ -\Psi_0'\end{pmatrix} \Rightarrow \vec{\nabla}\cdot\vec{B} = 0$$

$$g = \frac{qB_z}{2m\gamma}\ , \quad w_0 = w\, e^{i\int_0^t g\,dt}$$

$$\ddot{w}_0 = (\ddot{w} + i2g\dot{w} + i\dot{g}w - g^2 w)\, e^{i\int_0^t g\,dt} = -g^2 w_0$$

$$\ddot{x}_0 = -g^2 x_0$$
$$\ddot{y}_0 = -g^2 y_0$$

Focusing in a rotating coordinate system

Strong vs. Solenoid Focusing

If the solenoids field was perpendicular to the particle's motion,

its bending radius would be $\rho_z = \frac{p}{qB_z}$

$$\ddot{r} = -\left(\frac{qB_z}{2m\gamma}\right)^2 r = -\frac{qv_z}{m\gamma} B_z \frac{r}{4\rho_z}$$

Solenoid focusing is weak compared to the deflections created by a transverse magnetic field.

Transverse fields: $\vec{B} = B_x \vec{e}_x + B_y \vec{e}_y$

$$m\gamma \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \times \begin{pmatrix} B_x \\ B_y \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \frac{qv_z}{m\gamma} \begin{pmatrix} -B_y \\ B_x \end{pmatrix}$$

Strong focusing

$$\ddot{x} = -\frac{q\,v_z}{m\,\gamma}\frac{\partial B_y}{\partial x} x$$

Weak focusing < Strong focusing by about magnet aperture / bending radius

Natural Ring vs. Solenoid Focusing

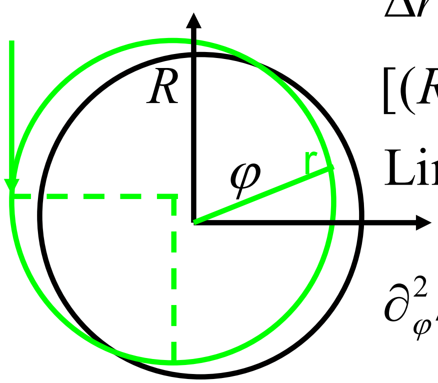
Solenoid magnets are used in detectors for particle identification via $\rho = \frac{p}{qB}$

The solenoid's rotation $\dot{\varphi} = -\frac{qB_z}{2m\gamma}$ of the beam is often compensated by a reversed solenoid called compensator.

Solenoid or Weak Focusing:

Solenoids are also used to focus low γ beams: $\ddot{w} = -\left(\frac{qB_z}{2m\gamma}\right)^2 w$

Weak focusing from natural ring focusing:



$$\Delta r = r - R$$

$$[(R+\Delta r)\cos\varphi - \Delta x_0]^2 + [(R+\Delta r)\sin\varphi - \Delta y_0]^2 = R^2$$

$$\text{Linearization in } \Delta: \quad \Delta r = (\cos\varphi\,\Delta x_0 + \sin\varphi\,\Delta y_0)$$

$$\partial_\varphi^2 \Delta r = -\Delta r \quad \Rightarrow \quad \Delta\ddot{r} = -\dot{\varphi}^2\Delta r = -\left(\frac{v}{\rho}\right)^2 \Delta r = -\left(\frac{qB}{m\gamma}\right)^2 \Delta r$$

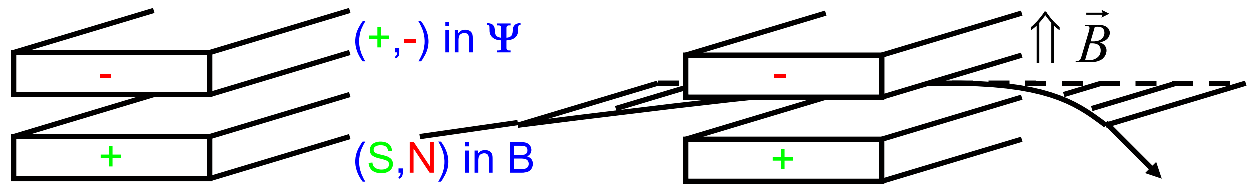
$$\frac{d^2\Delta r}{ds^2} = \frac{d^2\Delta r}{dt^2}\frac{1}{v^2} = \frac{1}{\rho^2}\Delta r$$

Focusing strength: $\frac{1}{\rho^2}$

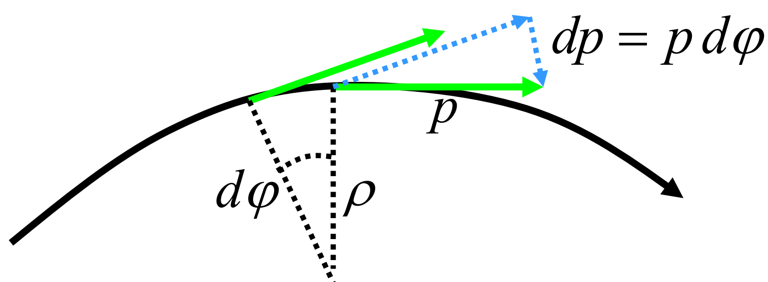
Multipoles in Accelerators: ν=1, Dipoles

$$\psi = \Psi_1 \,\mathrm{Im}\{x - iy\} = -\Psi_1 \cdot y \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_1 \,\vec{e}_y$$

Equipotential
$y = \text{const.}$



Dipole magnets are used for steering the beams direction

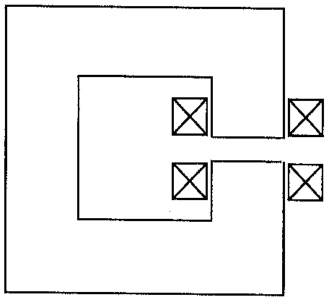


$$\frac{d\vec{p}}{dt} = q\,\vec{v} \times \vec{B} \quad \Rightarrow \quad \frac{dp}{dt} = qvB_\perp \quad \Rightarrow \quad \rho = \frac{dl}{d\varphi} = \frac{vdt}{dp/p} = \frac{p}{qB_\perp}$$

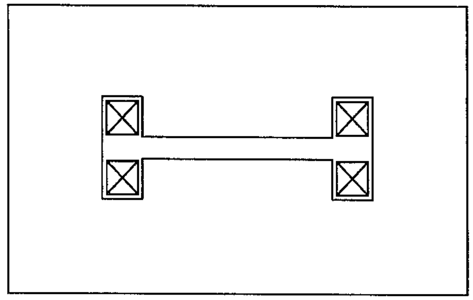
Bending radius: $\rho = \dfrac{p}{qB}$

Types of iron-dominated Dipoles

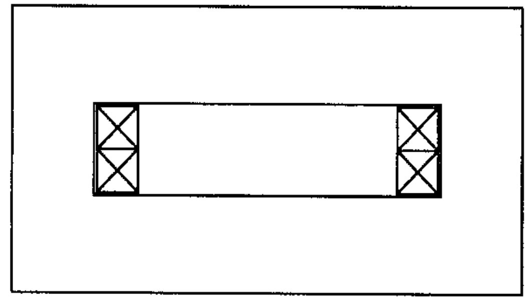
C-shape magnet:

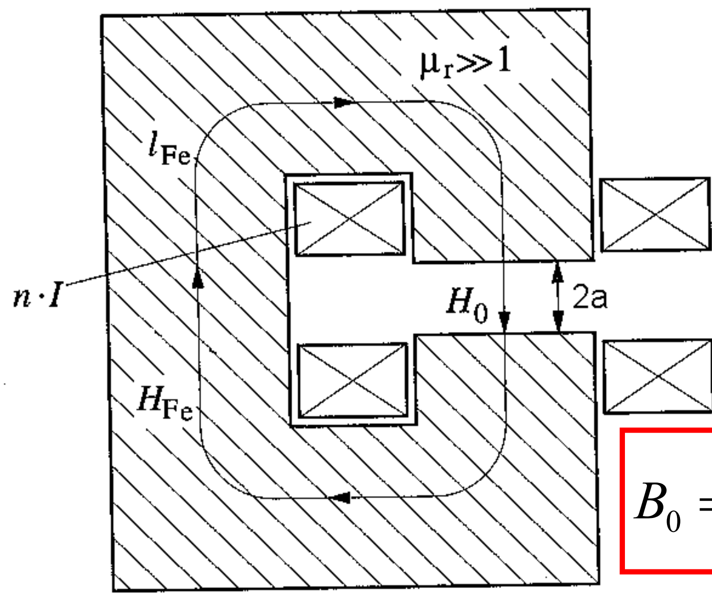


H-shape magnet:



Window frame magnet:





$$\vec{B}_\perp(\text{out}) = \vec{B}_\perp(\text{in})$$

$$\vec{H}_\perp(\text{out}) = \mu_r \vec{H}_\perp(\text{in})$$

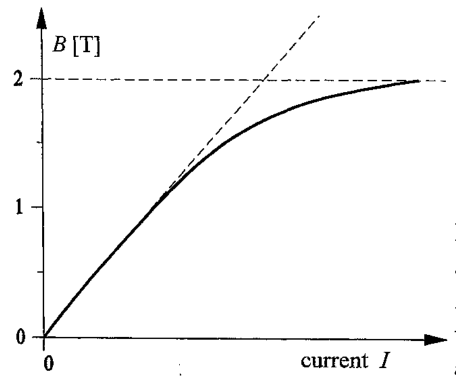
$$2nI = \oint \vec{H}\cdot d\vec{s} = H_{Fe} l_{Fe} + H_0 2a$$

$$= \tfrac{1}{\mu_r} H_0 l_{Fe} + H_0 2a \approx H_0 2a$$

$$B_0 = \mu_0 \frac{nI}{a}$$

Dipole strength: $\frac{1}{\rho} = \frac{q\mu_0}{p}\frac{nI}{a}$

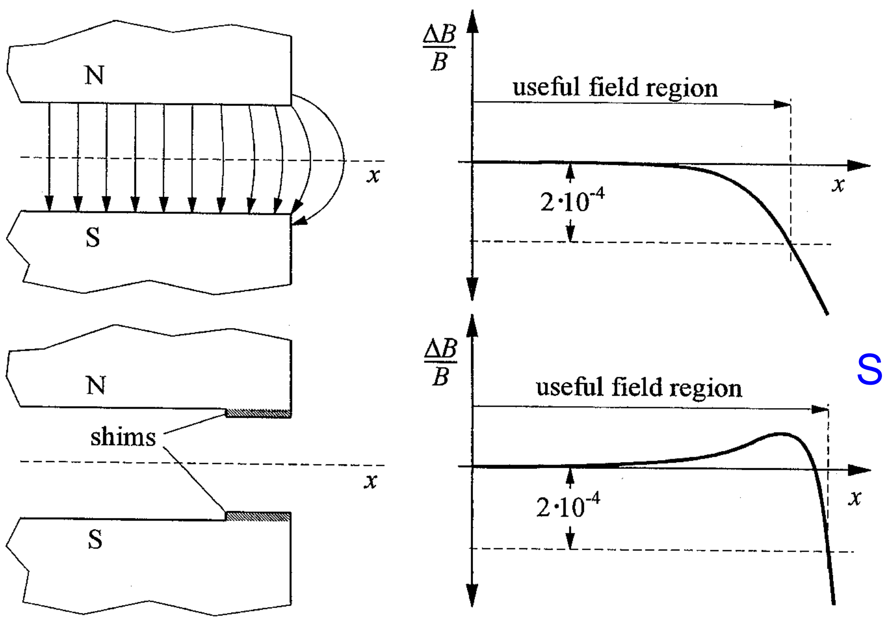
Iron-dominated Dipoles Fields



B = 2 T: Typical limit, since the field becomes dominated by the coils, not the iron. Limiting j for Cu is about 100A/mm²

B < 1.5 T: Typically used region

B < 1 T: Region in which $B_0 = \mu_0 \frac{nI}{a}$

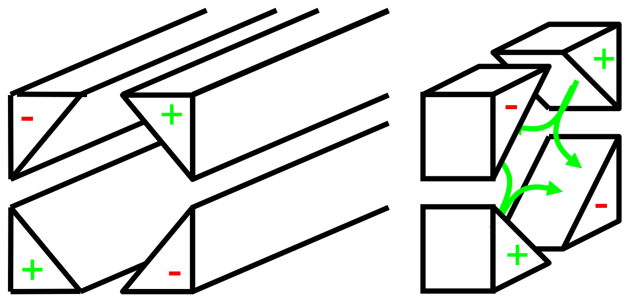


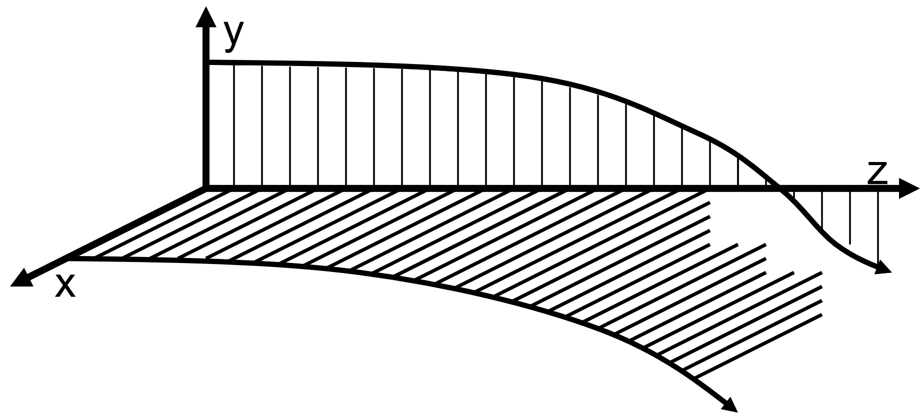
Shims reduce the space that is open to the beam, but they also reduce the fringe field region.

Multipoles in Accelerators: ν=2, Quadrupoles

$$\psi = \Psi_2 \operatorname{Im}\{(x - iy)^2\} = -\Psi_2 \cdot 2xy \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_2\, 2\begin{pmatrix} y \\ x \end{pmatrix}$$

C_2 Symmetry

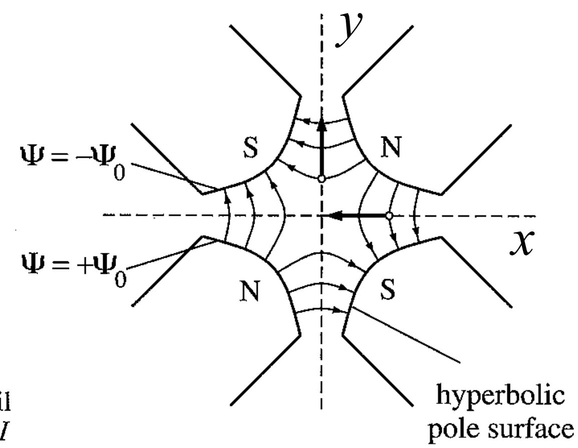


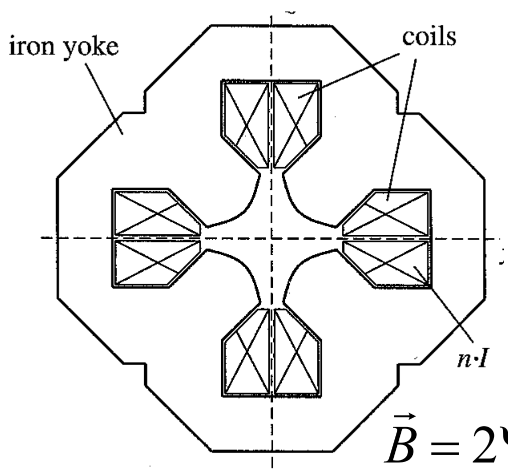


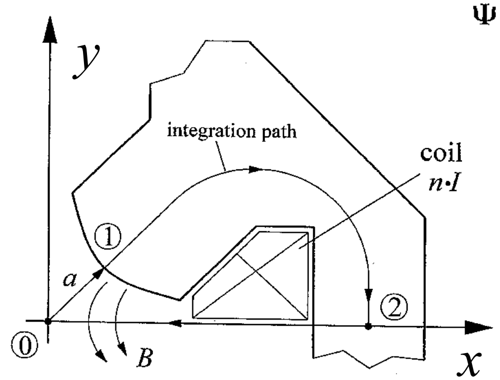
In a quadrupole particles are focused in one plane and defocused in the other plane. Other modes of strong focusing are not possible.

Iron-dominated quadrupoles fields

$$\psi = -\Psi_2 \cdot 2xy \quad \Rightarrow \quad \text{Equipotential: } x = \frac{\text{const.}}{y}$$





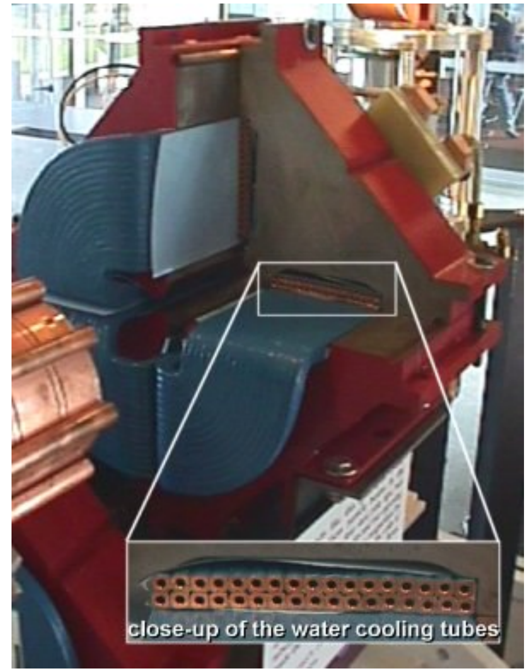


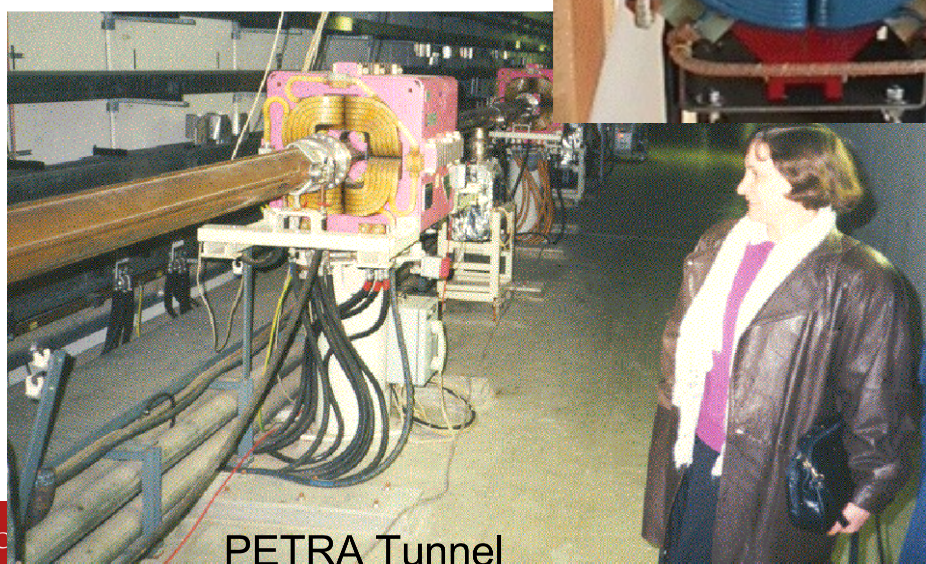
$$\vec{B} = 2\Psi_2 \begin{pmatrix} y \\ x \end{pmatrix} \quad \Rightarrow \quad \vec{B}(0 \mapsto 1) = 2\Psi_2 r\,\vec{e}_r$$

Quadrupole strength:

$$nI = \oint \vec{H} \cdot d\vec{s} \approx \int_0^a H_r dr = \Psi_2 \frac{a^2}{\mu_0} \qquad k_1 = \frac{q}{p} \partial_x B_y \Big|_0 = \frac{q\mu_0}{p} \frac{2nI}{a^2}$$

Real Quadrupoles



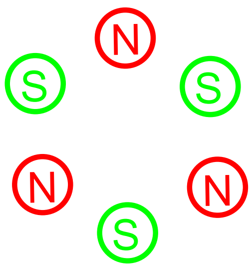


The coils show that this is an upright quadrupole not a rotated or skew quadrupole.

Multipoles in Accelerators: ν=3, Sextupoles

$$\psi = \Psi_3 \,\mathrm{Im}\{(x-iy)^3\} = \Psi_3 \cdot (y^3 - 3x^2 y) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

C_3 Symmetry



i) Sextupole fields hardly influence the particles close to the center, where one can linearize in x and y.

ii) In linear approximation a by Δx shifted sextupole has a quadrupole field.

iii) When Δx depends on the energy, one can build an energy dependent quadrupole.

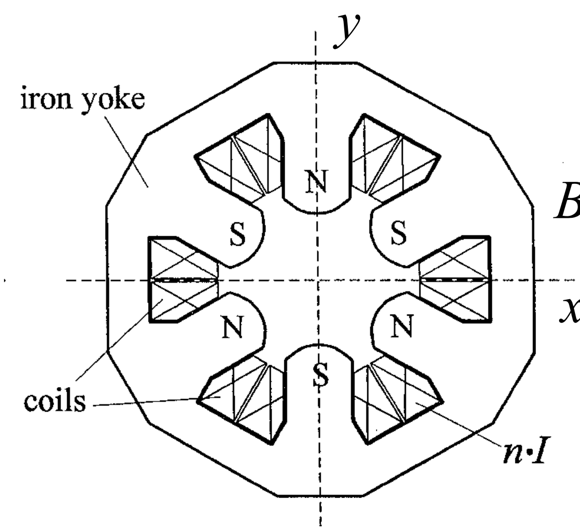
$$\vec{B} = -\vec{\nabla}\psi = \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

$$x \mapsto \Delta x + x$$

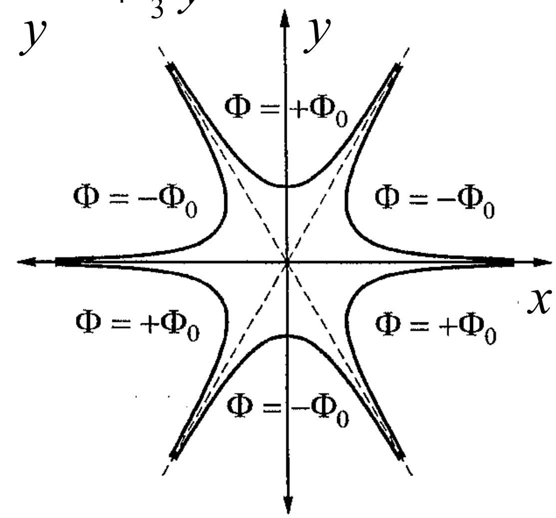
$$\vec{B} \approx \Psi_3 \, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix} + 6\Psi_3 \Delta x \begin{pmatrix} y \\ x \end{pmatrix} + O(\Delta x^2)$$

Iron-dominated sextupole fields

$$\psi = \Psi_2 \cdot (y^3 - 3x^2 y) \quad \Rightarrow$$ Equipotential: $$x = \sqrt{\frac{\text{const.}}{y} + \tfrac{1}{3} y^2}$$



$$B_y\big|_{x=0} = -\Psi_3 3y^2$$



$$nI = \oint \vec{H} \cdot d\vec{s} \approx \int_0^a H_r dr = \Psi_3 \frac{a^3}{\mu_0}$$

Quadrupole strength:

$$k_2 = \frac{q}{p} \partial_x^2 B_y\big|_0 = \frac{q\mu_0}{p} \frac{6nI}{a^3}$$

Real Sextupoles

Higher-order multipoles

$$\psi = \Psi_n \operatorname{Im}\{(x - iy)^n\} = \Psi_n \cdot (\ldots - i\, n\, x^{n-1} y) \quad \Rightarrow \quad \vec{B}(y=0) = \Psi_n\, n \begin{pmatrix} 0 \\ x^{n-1} \end{pmatrix}$$

Multipole strength: $k_n = \frac{q}{p} \partial_x^n B_y \Big|_{x,y=0} = \frac{q}{p} \Psi_{n+1}\,(n+1)!$ units: $\frac{1}{\mathrm{m}^{n+1}}$

p/q is also called Bρ and used to describe the energy of multiply charge ions

Names: dipole, quadrupole, sextupole, octupole, decapole, duodecapole, …

Higher order multipoles come from

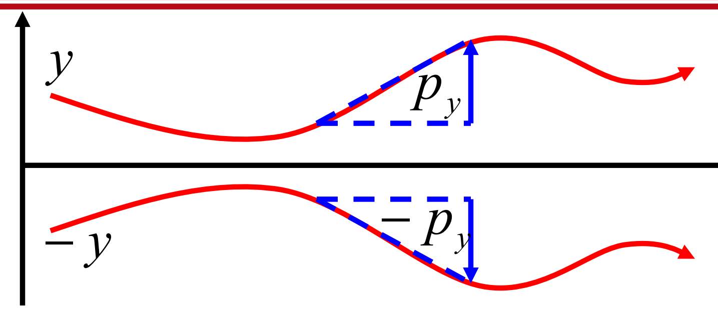
- Field errors in magnets
- Magnetized materials
- From multipole magnets that compensate such erroneous fields
- To compensate nonlinear effects of other magnets
- To stabilize the motion of many particle systems
- To stabilize the nonlinear motion of individual particles

Midplane-symmetric motion

$$\vec{r}^{\oplus} = (x, -y, z)$$

$$\vec{p}^{\oplus} = (p_x, -p_y, p_z)$$

$$\tfrac{d}{dt}\vec{p} = \vec{F}(\vec{r}, \vec{p}) \quad \Rightarrow \quad \tfrac{d}{dt}\vec{p}^{\oplus} = \vec{F}(\vec{r}^{\oplus}, \vec{p}^{\oplus})$$



$$v_y B_z - v_z B_y = -v_y B_z(x,-y,z) - v_z B_y(x,-y,z)$$
$$v_z B_x - v_x B_z = -v_z B_x(x,-y,z) + v_x B_z(x,-y,z) \Rightarrow$$
$$v_x B_y - v_y B_x = \ \ v_x B_y(x,-y,z) + v_y B_x(x,-y,z)$$

$$B_x(x,-y,z) = -B_x(x,y,z)$$
$$B_y(x,-y,z) = \ \ B_y(x,y,z)$$
$$B_z(x,-y,z) = -B_z(x,y,z)$$

$$\boxed{\psi(x,-y,z) = -\psi(x,y,z)}$$

$$\Psi_n \operatorname{Im}\{e^{in\vartheta_n}(x+iy)^n\} = -\Psi_n \operatorname{Im}\{e^{in\vartheta_n}(x-iy)^n\}$$

$$\Rightarrow \quad \Psi_n \operatorname{Im}\left[e^{in\vartheta_n} 2\operatorname{Re}\{(x+iy)^n\}\right] = 0 \quad \Rightarrow \quad \boxed{\vartheta_n = 0}$$

The discussed multipoles

produce midplane symmetric motion. When the field is rotated by π/2, i.e $\vartheta_n = \pi/2n$, one speaks of a skew multipole.

Superconducting magnets

Above 2T the field from the bare coils dominate over the magnetization of the iron.

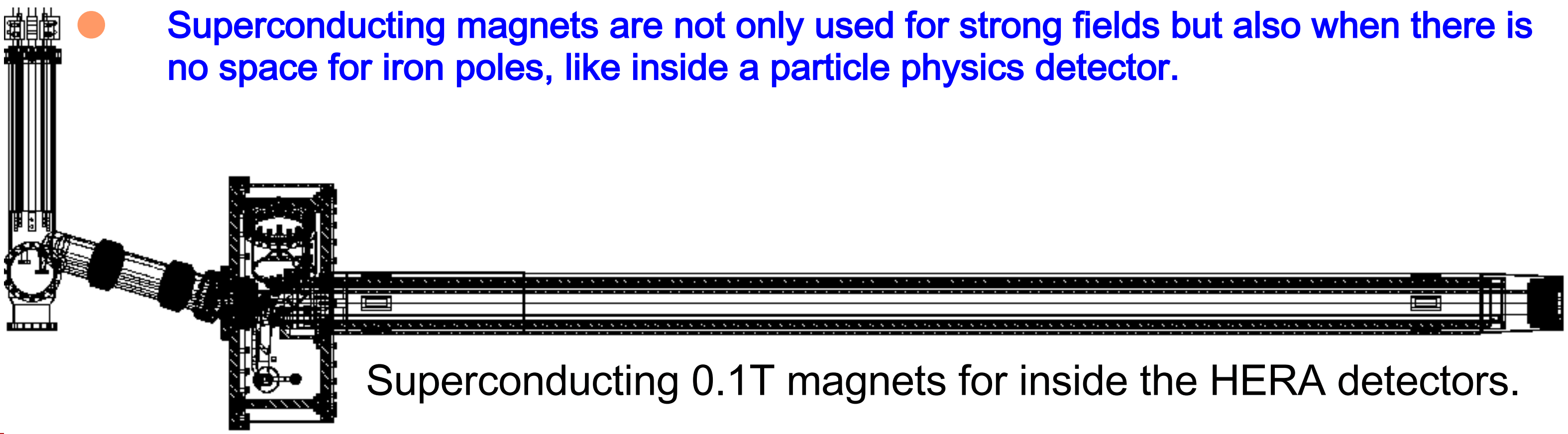
But Cu wires cannot create much filed without iron poles:

5T at 5cm distance from a 3cm wire would require a current density of

$$j = \frac{I}{d^2} = \frac{1}{d^2}\frac{2\pi\, r\, B}{\mu_0} = 1389\,\frac{\mathrm{A}}{\mathrm{mm}^2}$$

Cu can only support about 100A/mm^2.

- Superconducting cables routinely allow current densities of 1500A/mm^2 at 4.6 K and 6T. Materials used are usually Nb aloys, e.g. NbTi, Nb_3Ti or Nb_3Sn.
- Superconducting magnets are not only used for strong fields but also when there is no space for iron poles, like inside a particle physics detector.

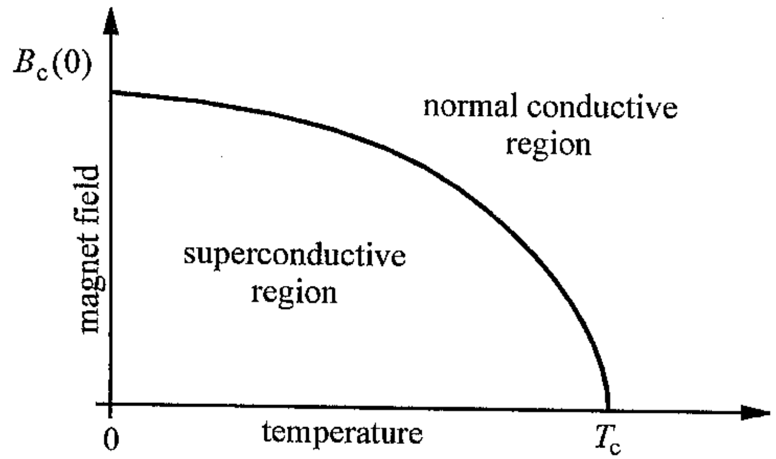


Superconducting 0.1T magnets for inside the HERA detectors.

Superconducting cables

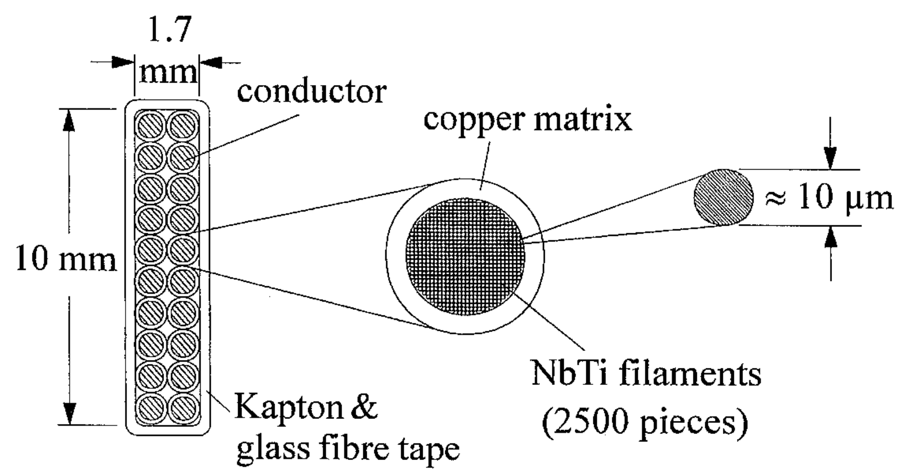
Problems:

- Superconductivity brakes down for too large fields
- Due to the Meissner-Ochsenfeld effect superconductivity current only flows on a thin surface layer.



Remedy:

- Superconducting cable consists of many very thin filaments (about 10μm).



Complex scalar magnetic potential of a wire

Straight wire at the origin: $\vec{\nabla}\times\vec{B}=\mu_0\,\vec{j} \quad\Rightarrow\quad \vec{B}(r)=\frac{\mu_0\,I}{2\pi\,r}\vec{e}_\varphi=\frac{\mu_0\,I}{2\pi\,r}\begin{pmatrix}-y\\ x\end{pmatrix}$

Wire at $\vec{a}$:

$$\vec{B}(x,y)=\frac{\mu_0\,I}{2\pi\,(\vec{r}-\vec{a})^2}\begin{pmatrix}-[y-a_y]\\ x-a_x\end{pmatrix}$$

This can be represented by complex multipole coefficients Ψ_ν

$$\vec{B}(x,y)=-\vec{\nabla}\Psi \quad\Rightarrow\quad B_x+iB_y=-\left(\partial_x+i\partial_y\right)\psi=-2\partial_{\overline{w}}\psi$$

$$B_x+iB_y=\frac{\mu_0 I}{2\pi}\frac{-i(w_a-w)}{(w_a-w)(\overline{w}_a-\overline{w})}=i\,\frac{\mu_0 I}{2\pi}\frac{-\frac{w_a}{a^2}}{1-\frac{\overline{w}w_a}{a^2}}$$

$$=i\,\frac{\mu_0 I}{2\pi}\partial_{\overline{w}}\ln(1-\tfrac{\overline{w}w_a}{a^2})=-2\partial_{\overline{w}}\,\mathrm{Im}\left\{\tfrac{\mu_0 I}{2\pi}\ln(1-\tfrac{\overline{w}w_a}{a^2})\right\}$$

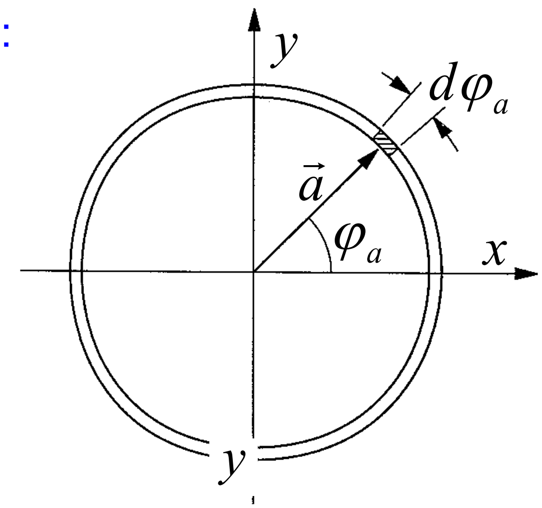
$$\psi=\mathrm{Im}\left\{\tfrac{\mu_0 I}{2\pi}\ln(1-\tfrac{\overline{w}w_a}{a^2})\right\}=-\mathrm{Im}\left\{\tfrac{\mu_0 I}{2\pi}\sum_{\nu=1}^{\infty}\tfrac{1}{\nu}\left(\tfrac{w_a}{a^2}\right)^{\nu}\overline{w}^{\nu}\right\} \quad\Rightarrow\quad \Psi_\nu=\tfrac{\mu_0 I}{2\pi}\tfrac{1}{\nu}\tfrac{1}{a^\nu}e^{i\nu\varphi_a}$$

Air-coil multipoles

Creating a multipole be created by an arrangement of wires:

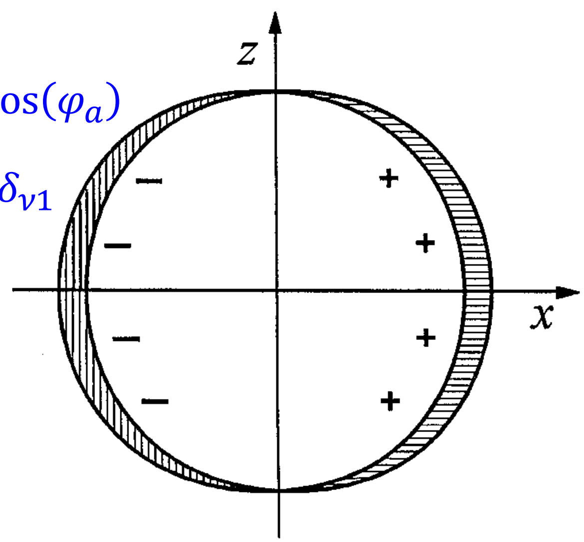
$$\Psi_\nu = \int_0^{2\pi} \frac{\mu_0}{2\pi}\frac{1}{\nu}\frac{1}{a^\nu} e^{i\nu\varphi_a} \frac{dI}{d\varphi_a} d\varphi_a$$

The Multipole coefficient Ψ_ν is the **Fourier coefficient** of the angular charge distribution $\frac{\mu_0}{\nu a^\nu}\frac{dI}{d\varphi_a}$.

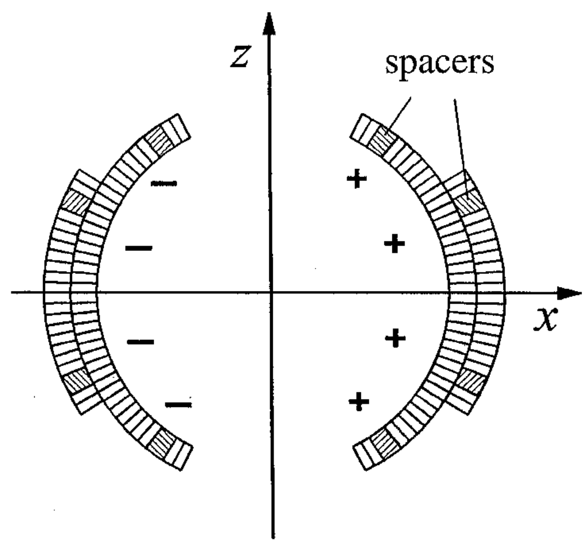


$\frac{dI}{d\varphi_a} \propto \cos(\varphi_a)$

➔ $\Psi_\nu \propto \delta_{\nu 1}$

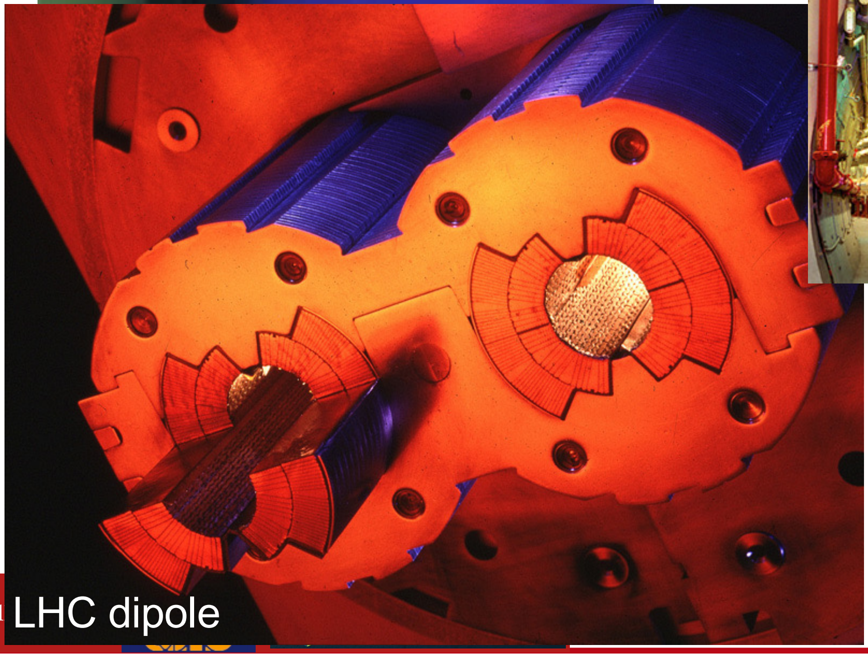


Ideal multipole



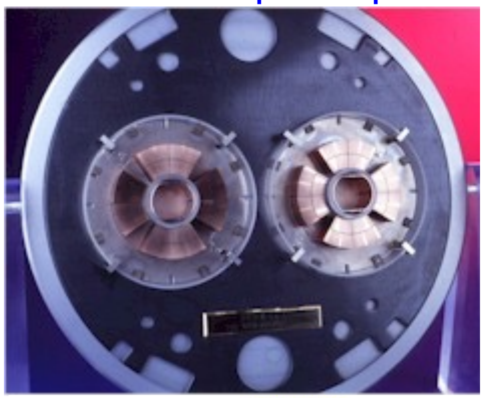
Approximate multipole

Real Air-coil multipoles

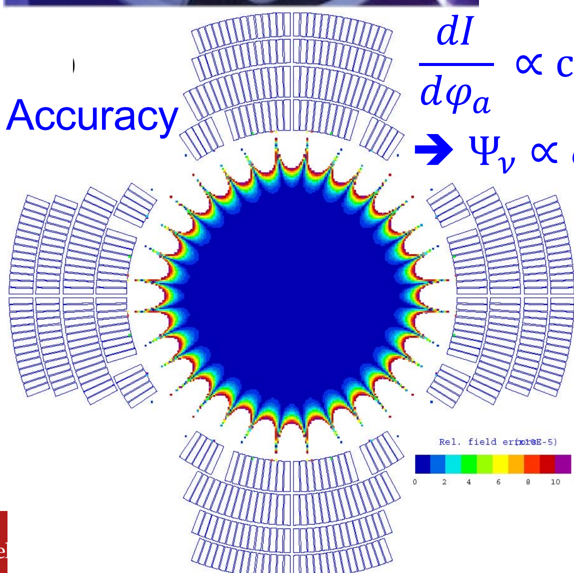


Special super-conducting Air-coil magnets

LHC double quadrupole

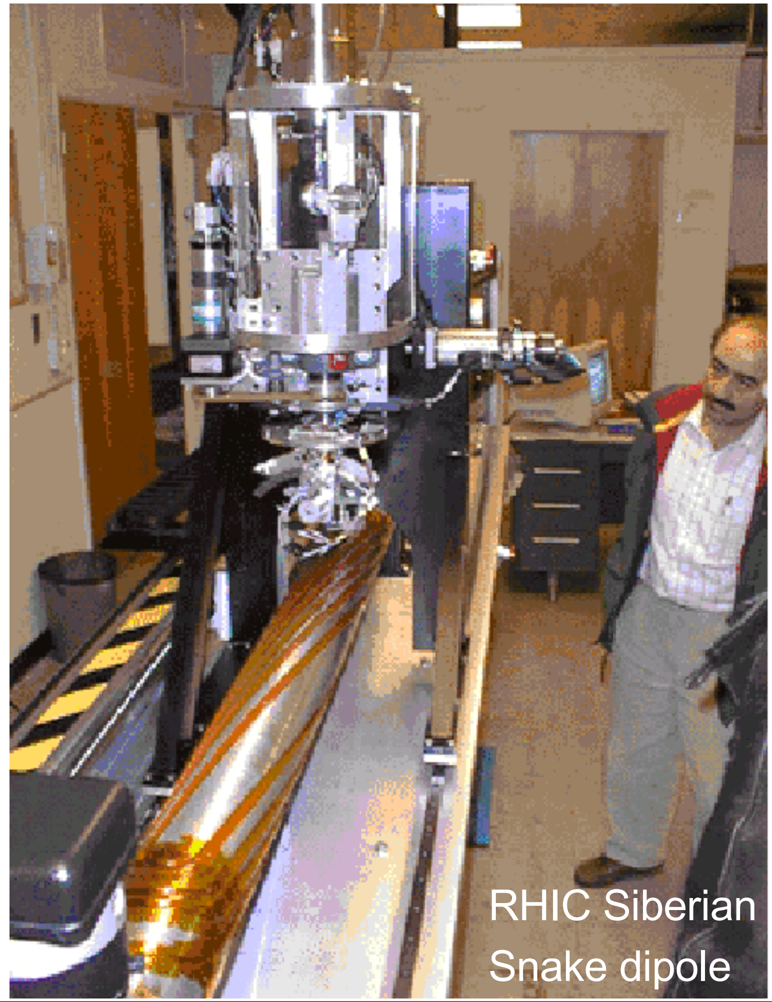


Accuracy

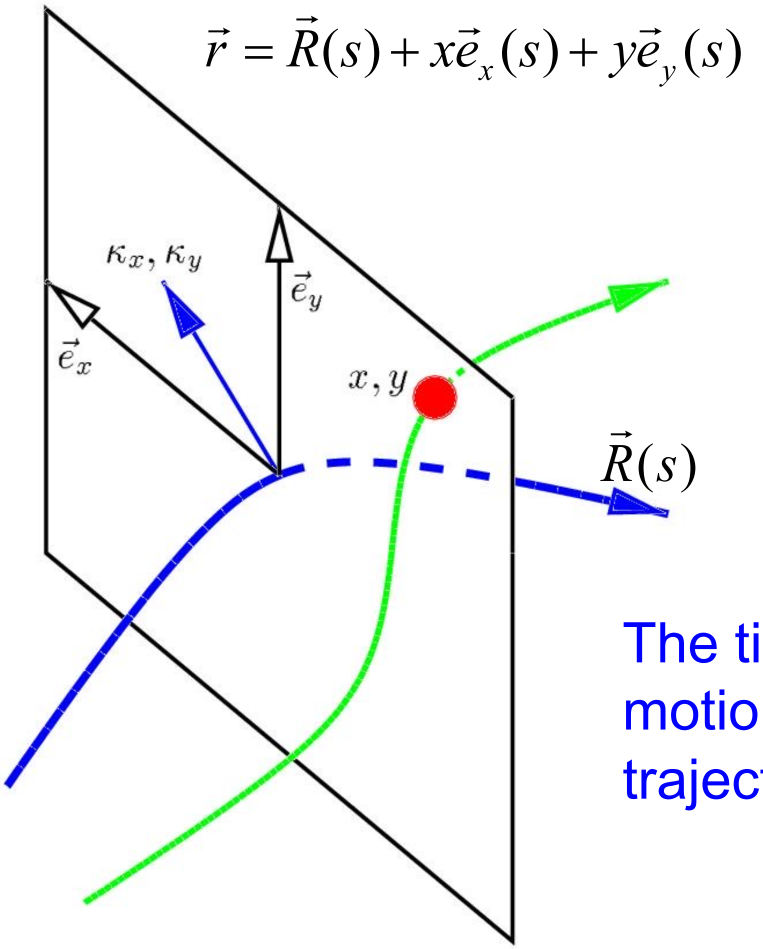


$$\frac{dI}{d\varphi_a} \propto \cos(2\varphi_a)$$

$$\rightarrow \Psi_\nu \propto \delta_{\nu 2}$$



The comoving coordinate system



$$\left|d\vec{R}\right| = ds$$

$$\vec{e}_s \equiv \tfrac{d}{ds}\vec{R}(s)$$

The time dependence of a particle's motion is often not as interesting as the trajectory along the accelerator length "s".

The 4-dimensional equation of motion

$$\frac{d^2}{dt^2}\vec{r} = \vec{f}_r(\vec{r}, \tfrac{d}{dt}\vec{r}, t)$$

3 dimensional ODE of 2nd order can be changed to a
6 dimensional ODE of 1st order:

$$\left.\begin{aligned}\frac{d}{dt}\vec{r} &= \frac{1}{m\gamma}\vec{p} = \frac{c}{\sqrt{p^2-(mc)^2}}\vec{p}\\ \frac{d}{dt}\vec{p} &= \vec{F}(\vec{r},\vec{p},t)\end{aligned}\right\}\quad \frac{d}{dt}\vec{Z} = \vec{f}_Z(\vec{Z},t)\,,\quad \vec{Z} = (\vec{r},\vec{p})$$

If the force does not depend on time, as in a typical beam line magnet, the energy is conserved so that one can reduce the dimension to 5. The equation of motion is then autonomous.

Furthermore, the time dependence is often not as interesting as the trajectory along the accelerator length "s". Using "s" as the independent variable reduces the dimensions to 4. The equation of motion is then no longer autonomous.

$$\frac{d}{ds}\vec{z} = \vec{f}_z(\vec{z},s)\,,\quad \vec{z} = (x, y, p_x, p_y)$$

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6D equation of motion

Usually one prefers to compute the trajectory as a function of "s" along the accelerator even when the energy is not conserved, as when accelerating cavities are in the accelerator.

Then the energy "E" and the time "t" at which a particle arrives at the cavities are important. And the equations become 6 dimensional again:

$$\frac{d}{ds}\vec{z} = \vec{f}_z(\vec{z},s)\ , \quad \vec{z} = (x, y, p_x, p_y, -t, E)$$

But: $\vec{z} = (\vec{r},\vec{p})$ is an especially suitable variable, since it is a phase space vector so that its equation of motion comes from a Hamiltonian, or by variation principle from a Lagrangian.

$$\delta\int\Big[p_x\dot{x} + p_y\dot{y} + p_s\dot{s} - H(\vec{r},\vec{p},t)\Big]dt = 0 \quad \Rightarrow \quad \text{Hamiltonian motion}$$

$$\delta\int\Big[p_x x' + p_y y' - H\,t' + p_s(x,y,p_x,p_y,t,H)\Big]ds = 0 \quad \Rightarrow \quad \text{Hamiltonian motion}$$

The new canonical coordinates are: $\vec{z} = (x, y, p_x, p_y, -t, E) \quad \text{with} \quad \mathrm{E} = \mathrm{H}$

The new Hamiltonian is: $K = -p_s(\vec{z},s)$

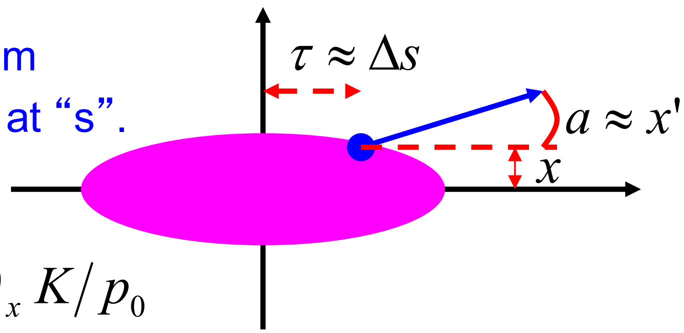
6D phase space motion

Using a reference momentum p_0 and a reference time t_0:

$$\vec{z} = (x, a, y, b, \tau, \delta)$$

$$a = \frac{p_x}{p_0}, \quad b = \frac{p_y}{p_0}, \quad \delta = \frac{E - E_0}{E_0}, \quad \tau = (t_0 - t)\frac{c^2}{v_0} = (t_0 - t)\frac{E_0}{p_0}$$

Usually p_0 is the design momentum of the beam
And t_0 is the time at which the bunch center is at “s”.



$$\left.\begin{aligned} x' &= \partial_{p_x} K \\ p_x' &= -\partial_x K \end{aligned}\right\} \Rightarrow \begin{cases} x' = \partial_a K/p_0\,, & a' = -\partial_x K/p_0 \\ y' = \partial_b K/p_0\,, & b' = -\partial_y K/p_0 \end{cases}$$

$$-t' = \partial_E K \quad \Rightarrow \quad \tau' = \tfrac{c^2}{v_0} \partial_\delta K/E_0 = \partial_\delta K/p_0$$

$$E' = -\partial_{-t} K \quad \Rightarrow \quad \delta' = -\tfrac{1}{E_0} \partial_\tau K \tfrac{c^2}{v_0} = -\partial_\tau K/p_0$$

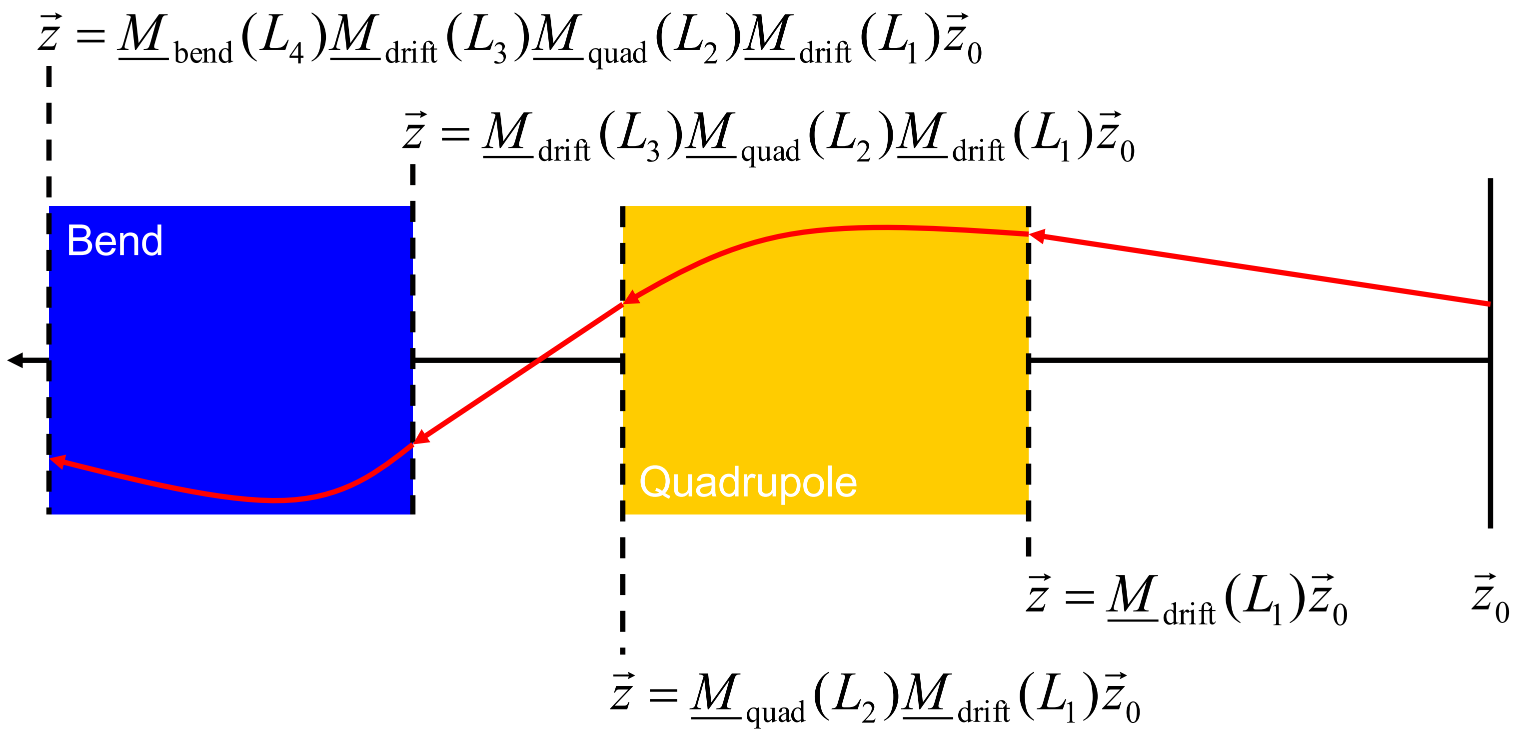
New Hamiltonian:

$$\widetilde{H} = K/p_0$$

The matrix solution of linear equations of motions

Linear equation of motion: $\vec{z}' = \underline{F}(s)\vec{z}$ ➔ $\vec{z}(s) = \underline{M}(s)\,\vec{z}_0$

Matrix solution of the starting condition $\vec{z}(0) = \vec{z}_0$



Simplest example: motion through an empty drift

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\ddot{x} = 0 \quad \Rightarrow \quad x'' = 0 \quad \Rightarrow \quad a = x', a' = 0$$

Linear solution:

$$x(s) = x_0 + x_0' s$$

$$\begin{pmatrix} x \\ a \\ y \\ b \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} x_0 + s a_0 \\ a \\ y_0 + s b_0 \\ b_0 \\ \tau_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \underline{0} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \vec{z}_0$$

Significance of the Hamiltonian

The equations of motion can be determined by one function:

$$\tfrac{d}{ds}x = \partial_{p_x} H(\vec{z},s)\,,\quad \tfrac{d}{ds}p_x = -\partial_x H(\vec{z},s)\,,\quad \dots$$

$$\tfrac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial} H(\vec{z},s) = \vec{F}(\vec{z},s) \quad \text{with} \quad \underline{J} = \mathrm{diag}(\underline{J}_2)\,,\quad \underline{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The force has a Hamiltonian Jacobi Matrix:

A linear force: $\vec{F}(\vec{z},s) = \underline{F}(s)\cdot\vec{z}$

The Jacobi Matrix of a linear force: $\underline{F}(s)$

The general Jacobi Matrix : $F_{ij} = \partial_{z_j} F_i$ or $\underline{F} = \left(\vec{\partial}\,\vec{F}^T\right)^T$

Hamiltonian Matrices: $\underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = 0$

Prove : $F_{ij} = \partial_{z_j} F_i = \partial_{z_j} J_{ik}\partial_{z_k} H = J_{ik}\,\partial_k \partial_j H \quad \Rightarrow \quad \underline{F} = \underline{J}\,\underline{D}\underline{H}$

$$\underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = \underline{J}\,\underline{D}\underline{J}H + \underline{J}\,\underline{D}^T\,\underline{J}^T H = 0$$

Hamiltonian → Symplectic Flow

The flow of a Hamiltonian equation of motion has a symplectic Jacobi Matrix

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

A linear flow: $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0$

The Jacobi Matrix of a linear flow: $\underline{M}(s)$

The general Jacobi Matrix : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = \left(\vec{\partial}_0 \vec{M}^T\right)^T$

The Symplectic Group SP(2N) : $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

$$\tfrac{d}{ds}\vec{z} = \tfrac{d}{ds}\vec{M}(s, \vec{z}_0) = \underline{J}\vec{\nabla}H = \vec{F}$$

$$\tfrac{d}{ds}M_{ij} = \partial_{z_{0j}} F_i(\vec{z}, s) = \partial_{z_{0j}} M_k \partial_{z_k} F_i(\vec{z}, s)$$

$$\tfrac{d}{ds}\underline{M}(s, \vec{z}_0) = \underline{F}(\vec{z}, s)\underline{M}(s, \vec{z}_0)$$

$$K = \underline{M}\,\underline{J}\,\underline{M}^T$$

$$\tfrac{d}{ds}\underline{K} = \tfrac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T + \underline{M}\,\underline{J}\tfrac{d}{ds}\underline{M}^T = \underline{F}\,\underline{M}\,\underline{J}\,\underline{M}^T + \underline{M}\,\underline{J}\,\underline{M}^T\underline{F}^T = \underline{F}\,\underline{K} + \underline{K}\,\underline{F}^T$$

$\underline{K} = \underline{J}$ is a solution. Since this is a linear ODE , $\underline{K} = \underline{J}$ is the unique solution.

Symplectic Flow → Hamiltonian

For every symplectic transport map there is a Hamilton function

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

Force vector: $\vec{h}(\vec{z}, s) = -\underline{J}\left[\frac{d}{ds}\vec{M}(s, \vec{z}_0)\right]_{\vec{z}_0 = \vec{M}^{-1}(\vec{z}, s)}$

Since then: $\frac{d}{ds}\vec{z} = \underline{J}\vec{h}(\vec{z}, s)$

There is a Hamilton function H with: $\vec{h} = \vec{\partial} H$

If and only if: $\partial_{z_j} h_i = \partial_{z_i} h_j \quad \Rightarrow \quad \underline{h} = \underline{h}^T$

$$\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J} \quad \Rightarrow \quad \begin{cases} \frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T = -\underline{M}\,\underline{J}\,\frac{d}{ds}\underline{M}^T \\ \underline{M}^{-1} = -\underline{J}\,\underline{M}^T\underline{J} \end{cases}$$

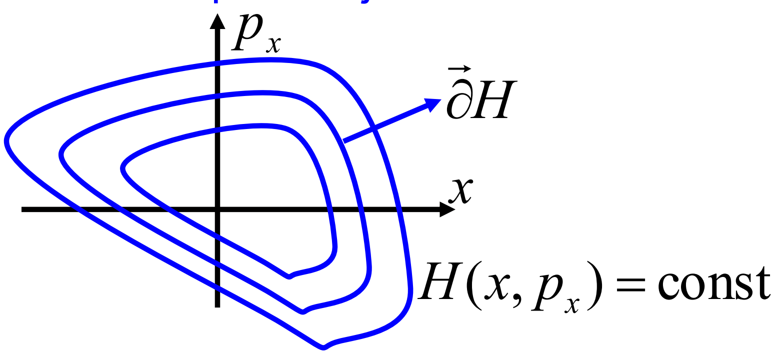
$$\vec{h} \circ \vec{M} = -\underline{J}\,\frac{d}{ds}\vec{M}$$

$$\underline{h}(\vec{M})\underline{M} = -\underline{J}\,\frac{d}{ds}\underline{M}$$

$$\underline{h}(\vec{M}) = -\underline{J}\,\frac{d}{ds}\underline{M}\,\underline{M}^{-1} = \underline{J}\,\frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T\underline{J} = -\underline{J}\,\underline{M}\,\underline{J}\,\frac{d}{ds}\underline{M}^T\underline{J} = \underline{M}^{-T}\,\frac{d}{ds}\underline{M}^T\underline{J} = \underline{h}^T$$

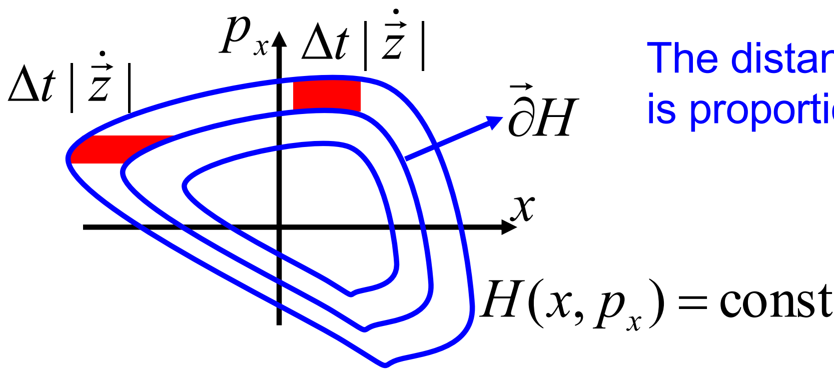
Phase space densities in 2D

- Phase space trajectories move on surfaces of constant energy



$$\frac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial}H \quad \Rightarrow \quad \frac{d}{ds}\vec{z} \perp \vec{\partial}H$$

- A phase space volume does not change when it is transported by Hamiltonian motion.

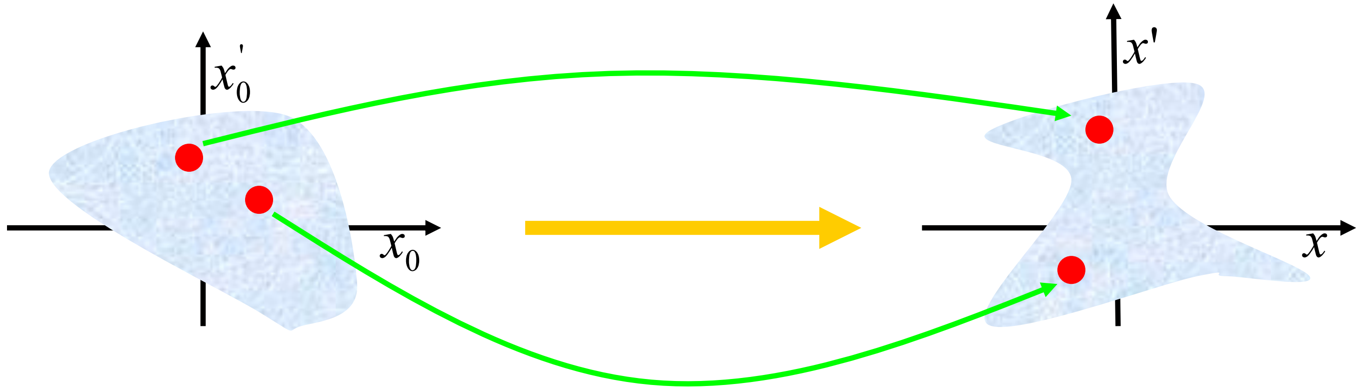


The distance d of lines with equal energy is proportional to $1/|\vec{\partial}H| \propto |\dot{\vec{z}}\,|^{-1}$

$$\boxed{d * \Delta t\,|\dot{\vec{z}}\,| = \mathrm{const}}$$

Liouville's Theorem

- A phase space volume does not change when it is transported by Hamiltonian motion. $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0 \quad \text{with} \quad \det[\underline{M}(s)] = +1$



Volume = $V = \iint_V d^n\vec{z} = \iint_{V_0} \left|\frac{\partial \vec{z}}{\partial \vec{z}_0}\right| d^n\vec{z}_0 = \iint_{V_0} |\underline{M}| d^n\vec{z}_0 = \iint_{V_0} d^n\vec{z}_0 = V_0$

Hamiltonian Motion → $V = V_0$

But Hamiltonian requires symplecticity, which is much more than just $\det[\underline{M}(s)] = +1$

Symplectic representations

Hamiltonian

$$\vec{z}\,' = \underline{J}\,\vec{\partial} H(\vec{z}, s)$$

ODE

$$\vec{z}\,' = \vec{F}\,, \quad \underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = 0$$

Symplectic transport map

$$\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$$

Generating Functions

$$(\vec{p}, \vec{p}_0) = -\underline{J}\,\vec{\partial} F_1(\vec{q}, \vec{q}_0, s)$$

[not shown in these notes]

Eigenvalues of symplectic matrices

For matrices with real coefficients:

If there is an eigenvector and eigenvalue: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$

then the complex conjugates are also eigenvector and eigenvalue: $\underline{M}\vec{v}_i^* = \lambda_i^* \vec{v}_i^*$

For symplectic matrices:

If there are eigenvectors and eigenvalues: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$ with $\underline{J} = \underline{M}^T \underline{J}\, \underline{M}$

then $\vec{v}_i^T \underline{J}\vec{v}_j = \vec{v}_i^T \underline{M}^T \underline{J}\, \underline{M}\vec{v}_j = \lambda_i \lambda_j \vec{v}_i^T \underline{J}\vec{v}_j \quad \Rightarrow \quad \vec{v}_i^T \underline{J}\vec{v}_j(\lambda_i\lambda_j - 1) = 0$

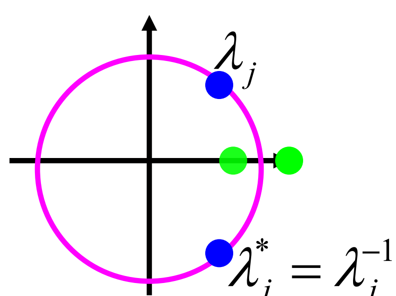
Therefore $\underline{J}\vec{v}_j$ is orthogonal to all eigenvectors with eigenvalues that are not $1/\lambda_j$. Since it cannot be orthogonal to all eigenvectors, there is at least one eigenvector with eigenvalue $1/\lambda_j$

Two dimensions: λ_j is eigenvalue

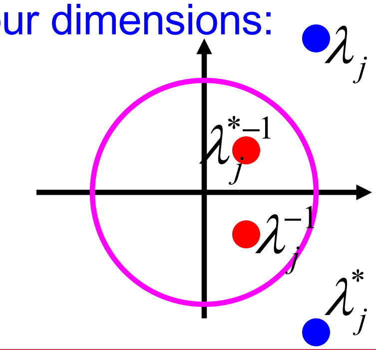
Then $1/\lambda_j$ and λ_j^* are eigenvalues

$\underline{\lambda_2 = 1/\lambda_1 = \lambda_1^*} \quad \Rightarrow \quad |\lambda_j| = 1$

$\lambda_2 = 1/\lambda_1 = \lambda_2^*$



Four dimensions:



Time of flight from symplecticity

$$\underline{M} = \begin{pmatrix} \underline{M}_4 & \vec{0} & \vec{D} \\ \vec{T}^T & 1 & M_{56} \\ \vec{0}^T & 0 & 1 \end{pmatrix}$$ is in SU(6) and therefore $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

$$\begin{pmatrix} \underline{M}_4 \underline{J}_4 & -\vec{D} & \vec{0} \\ \vec{T}^T \underline{J}_4 & -M_{56} & 1 \\ \vec{0}^T & -1 & 0 \end{pmatrix} \begin{pmatrix} \underline{M}_4^T & \vec{T} & \vec{0} \\ \vec{0}^T & 1 & 0 \\ \vec{D}^T & M_{56} & 1 \end{pmatrix} = \begin{pmatrix} \underline{J}_4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \underline{M}_4 \underline{J}_4 \underline{M}_4^T & \underline{M}_4 \underline{J}_4 \vec{T} - \vec{D} & \vec{0} \\ \vec{T}^T \underline{J}_4 \underline{M}_4^T + \vec{D}^T & 0 & 1 \\ \vec{0}^T & -1 & 0 \end{pmatrix} = \begin{pmatrix} \underline{J}_4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\vec{T} = -\underline{J}_4 \underline{M}_4^{-1} \vec{D}$$

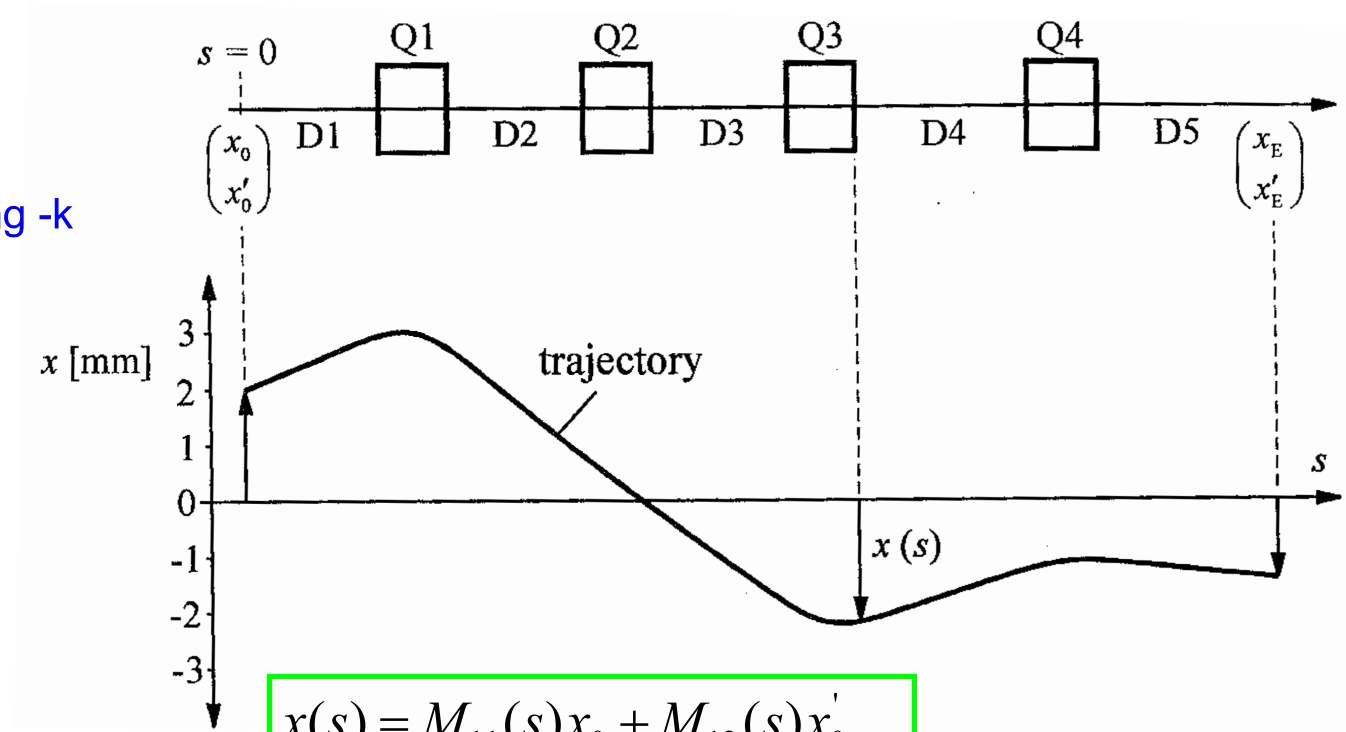
It is sufficient to compute the 4D map $\underline{M}_4$, the Dispersion $\vec{D}$ and the time of flight term M_{56}

Betatron formalism for linear motion

$$x'' = -x\,K$$
$$y'' = \;\; y\,k$$

In y: quadrupole defocusing -k

In x: $K = k + \frac{1}{\rho^2}$



$$x(s) = M_{11}(s)x_0 + M_{12}(s)x_0'$$
$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s) + \phi_0)$$

Twiss parameters

$$x'' = -k\,x$$

$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s)+\phi_0)$$

$$x'(s) = \sqrt{\tfrac{2J}{\beta}}[\beta\psi'\cos(\psi(s)+\phi_0) - \alpha\sin(\psi(s)+\phi_0)] \quad \text{with} \quad \alpha = -\tfrac{1}{2}\beta'$$

$$x''(s) = \sqrt{\tfrac{2J}{\beta}}[(\beta\psi'' - 2\alpha\psi')\cos(\psi(s)+\phi_0) - (\alpha' + \tfrac{\alpha^2}{\beta} + \beta\psi'^2)\sin(\psi(s)+\phi_0)]$$

$$= \sqrt{\tfrac{2J}{\beta}}[-k\beta\sin(\psi(s)+\phi_0)]$$

$$\beta\psi'' - 2\alpha\psi' = \beta\psi'' + \beta'\psi' = (\beta\psi')' = 0 \quad \Rightarrow \quad \psi' = \tfrac{I}{\beta}$$

$$\alpha' + \gamma = k\beta \quad \textit{with} \quad \gamma = \tfrac{I^2+\alpha^2}{\beta}$$

Universal choice: I=1!

$$\beta' = -2\alpha$$

$$\alpha' = k\beta - \gamma$$

$$\psi = \int_0^s \frac{I}{\beta(s')}\,ds'$$

$\alpha, \beta, \gamma, \psi$ are called Twiss parameters.

What are the initial conditions?

The phase ellipse

Particles with a common J and different ϕ all lie on an ellipse in phase space:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{I}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

(Linear transform of a circle)

$$x_{\max} = \sqrt{2J\beta} \text{ at } x' = -\alpha\sqrt{\frac{2J}{\beta}}$$

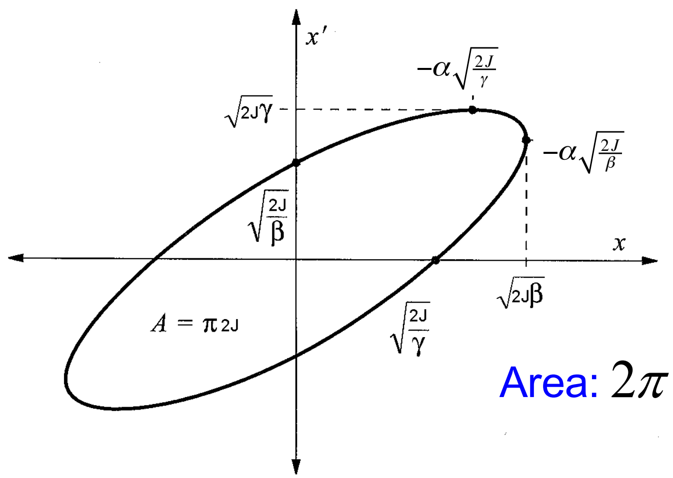
$$(x, x')\begin{pmatrix} \frac{I}{\sqrt{\beta}} & \frac{\alpha}{\sqrt{\beta}} \\ 0 & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} \frac{I}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = (x, x')\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = 2J$$

(Quadratic form)

$$\beta\gamma - \alpha^2 = I^2$$

Area: $2\pi J / I$

I=1 is therefore a useful choice!

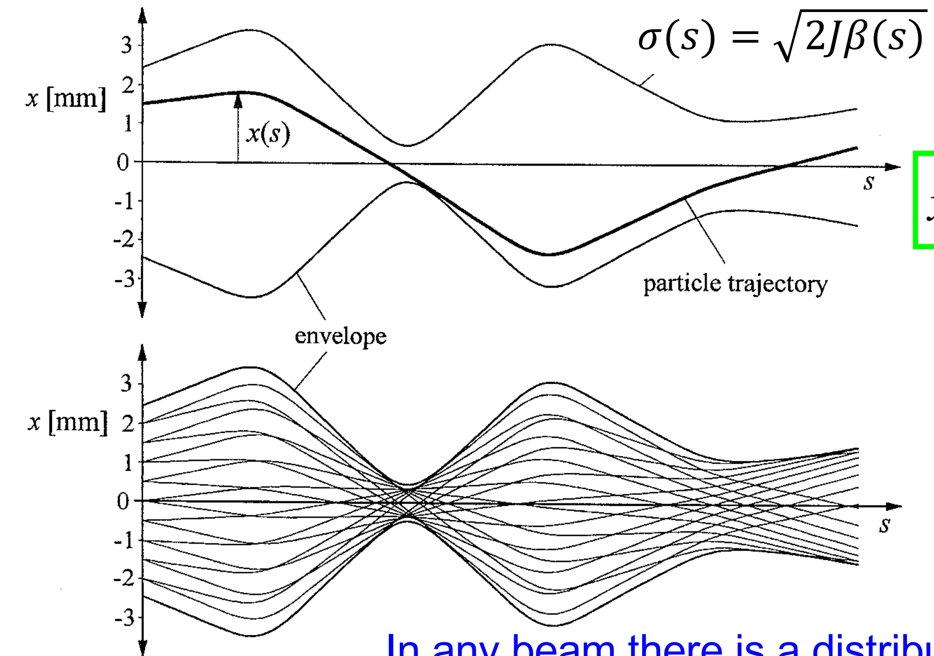


What β is for x, γ is for x'

$$x'_{\max} = \sqrt{2J\gamma} \text{ at } x = -\alpha\sqrt{\frac{2J}{\gamma}}$$

Area: $2\pi J$ → $\int_0^{2\pi}\int_0^{J} dJ\,d\phi = 2\pi J = \iint dx\,dx'$

The beam envelope



$$x(s) = \sqrt{2J\beta(s)}\sin(\psi(s) + \phi_0)$$

$$\sigma' = -\alpha\sqrt{\frac{2J}{\beta}}$$

$$\sigma'' = -(k\beta - \gamma)\sqrt{\frac{2J}{\beta}} - \alpha^2\sqrt{\frac{2J}{\beta^3}}$$

$$\sigma'' = -(k\beta^2 - 1)\sqrt{\frac{2J}{\beta^3}}$$

In any beam there is a distribution of initial parameters. If the particles with the largest J are distributed in ϕ over all angles, then the envelope of the beam is described by $\sigma = \sqrt{2J_{max}\beta(s)}$.

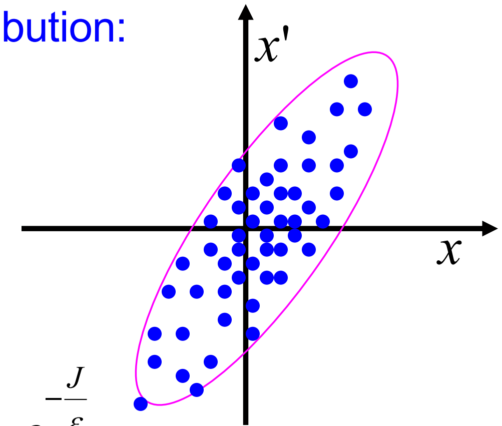
The initial conditions of β and α are chosen so that this is approximately the case.

The envelope equation: $\sigma'' = -k\sigma + \frac{(2J)^2}{\sigma^3}$

The phase space distribution

Often one can fit a Gauss distribution to the particle distribution:

$$\rho(x,x') = \frac{1}{2\pi\varepsilon} e^{-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{2\varepsilon}}$$



The equi-density lines are then ellipses. And one chooses the starting conditions for β and α according to these ellipses!

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix} \qquad \rho(J,\phi_0) = \frac{1}{2\pi\varepsilon} e^{-\frac{J}{\varepsilon}}$$

$$\langle 1 \rangle = \frac{1}{2\pi\varepsilon} \int_0^{2\pi}\int_0^{\infty} e^{-J/\varepsilon} dJ d\phi_0 = 1$$

Initial beam distribution ⟶ initial α, β, γ

$$\langle x^2 \rangle = \frac{1}{2\pi\varepsilon} \iint 2J\beta \sin\phi_0^2 e^{-J/\varepsilon} dJ d\phi_0 = \varepsilon\beta \longrightarrow \langle x'^2 \rangle = \varepsilon\gamma$$

$$\langle xx' \rangle = -\frac{1}{2\pi\varepsilon} \iint 2J\alpha \sin\varphi_0^2 e^{-J/\varepsilon} dJ d\varphi_0 = -\varepsilon\alpha$$

$$\varepsilon = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$ is called the emittance.

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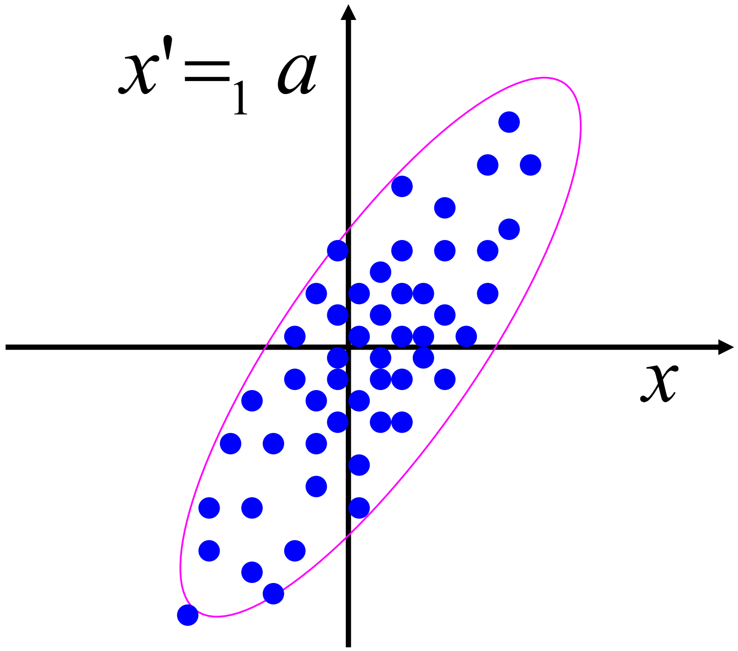
The normalized emittance



with $a = p_x/p_0$
and the beam's reference momentum p_0.

Remarks:

(1) The phase space area that a beam fills in (x , a) phase space shrinks during acceleration by the factor p_0/p. This area is the emittance ε.

(2) The phase space area that a beam fills in (x , p_x) phase space is conserved. This area (divided by mc) is the normalized emittance ε_n.

$$\varepsilon = \varepsilon_n / \beta_r \gamma_r$$

Invariant of motion

$$x(s) = \sqrt{2J\beta(s)}\sin(\psi(s)+\phi_0)$$

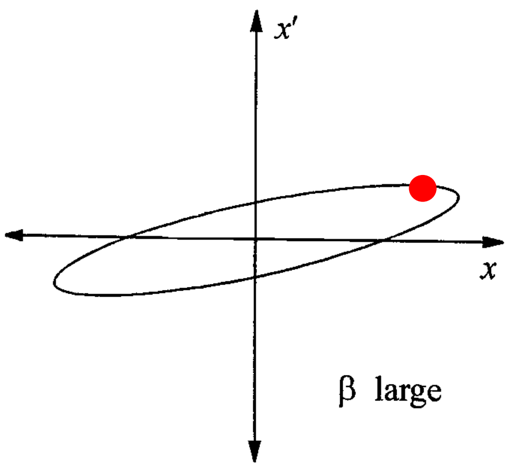
Where J and ϕ are given by the starting conditions x_0 and x'_0.

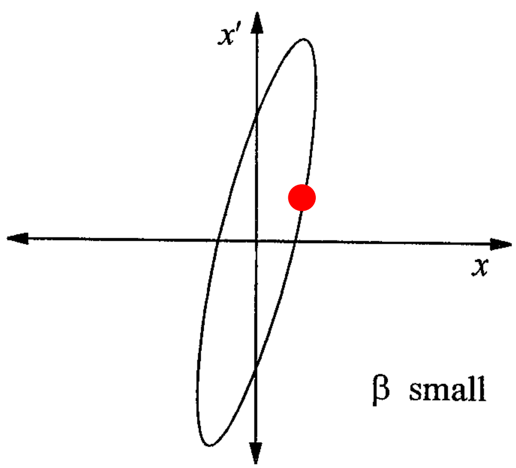
$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = 2J$$

Leads to the invariant of motion:

$$f(x,x',s) = \gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 \quad \Rightarrow \quad \tfrac{d}{ds} f = 0$$

It is called the Courant-Snyder invariant.





Twiss differential equation ➔ usually too hard

$$\gamma = \frac{1+\alpha^2}{\beta}$$
$$\beta' = -2\alpha$$
$$\alpha' = k\beta - \gamma$$

$$\beta'' = 2\gamma = 2\frac{1+\frac{1}{4}\beta'^2}{\beta} = \frac{d\beta'}{d\beta}\frac{d\beta}{ds}$$
$$\frac{\beta'}{1+\frac{1}{4}\beta'^2}d\beta' = 2\frac{d\beta}{\beta}$$
$$\log(1+\tfrac{1}{4}\beta'^2) = \log(\beta/\beta_0)$$
$$\beta' = 2\sqrt{\beta/\beta_0 - 1}$$
$$\frac{d\beta}{2\sqrt{\beta/\beta_0 - 1}} = ds$$
$$\beta_0\sqrt{\beta/\beta_0 - 1} = s - s_0$$
$$\beta(s) = \beta_0\left(1+\left(\tfrac{s-s_0}{\beta_0}\right)^2\right)$$

Propagation of Twiss parameters

$$(x_0, x_0')\begin{pmatrix}\gamma_0 & \alpha_0\\ \alpha_0 & \beta_0\end{pmatrix}\begin{pmatrix}x_0\\ x_0'\end{pmatrix} = 2J$$

$$(x, x')\begin{pmatrix}\gamma & \alpha\\ \alpha & \beta\end{pmatrix}\begin{pmatrix}x\\ x'\end{pmatrix} = 2J = (x_0, x_0')\underline{M}^T\begin{pmatrix}\gamma & \alpha\\ \alpha & \beta\end{pmatrix}\underline{M}\begin{pmatrix}x_0\\ x_0'\end{pmatrix}$$

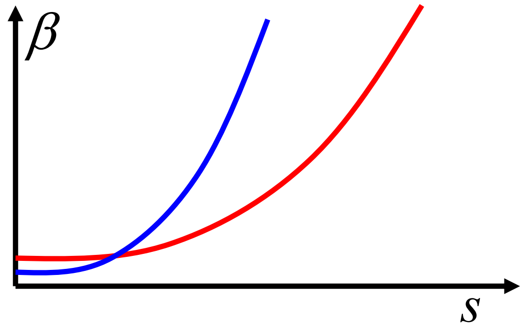
$$\begin{pmatrix}\gamma & \alpha\\ \alpha & \beta\end{pmatrix} = \underline{M}^{-T}\begin{pmatrix}\gamma_0 & \alpha_0\\ \alpha_0 & \beta_0\end{pmatrix}\underline{M}^{-1}$$

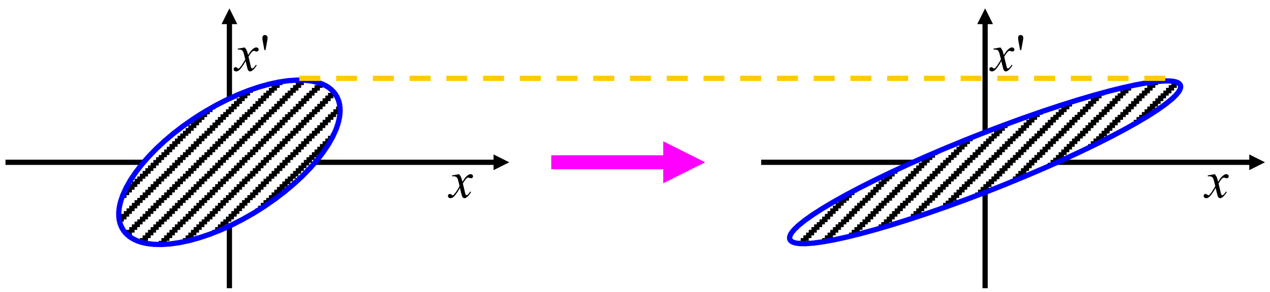
$$\begin{pmatrix}\beta & -\alpha\\ -\alpha & \gamma\end{pmatrix} = \underline{M}\begin{pmatrix}\beta_0 & -\alpha_0\\ -\alpha_0 & \gamma_0\end{pmatrix}\underline{M}^T$$

Twiss parameters in a drift

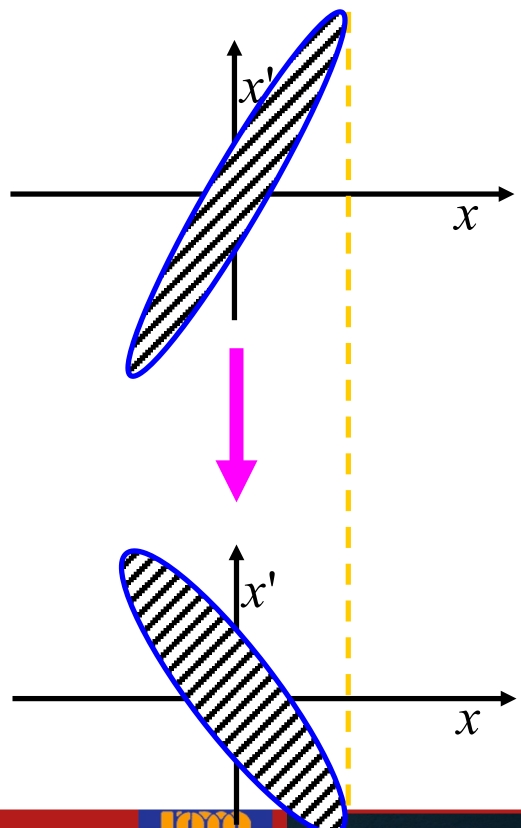
$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} \beta_0 - 2\alpha_0 s + \gamma_0 s^2 & \gamma_0 s - \alpha_0 \\ \gamma_0 s - \alpha_0 & \gamma_0 \end{pmatrix}$$

$$\beta = \beta_0^* [1 + (\tfrac{s}{\beta_0^*})^2] \quad \text{for} \quad \alpha_0^* = 0$$



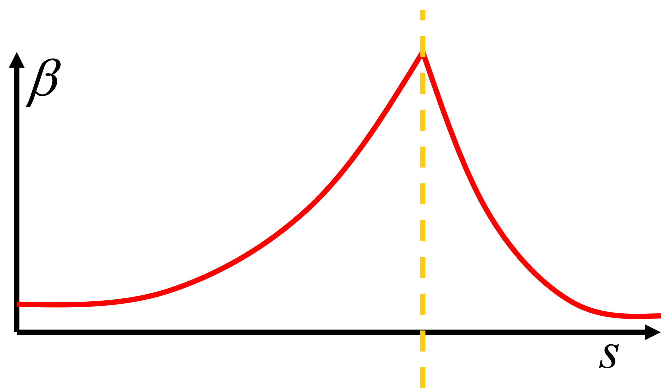


Twiss parameters in a thin quadrupole



$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\alpha = \alpha_0 + k\beta_0}$$



From Twiss parameter to Transfer Marix

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta_0} & 0 \\ -\frac{\alpha_0}{\sqrt{\beta_0}} & \frac{1}{\sqrt{\beta_0}} \end{pmatrix} \begin{pmatrix} \sin(\phi_0) \\ \cos(\phi_0) \end{pmatrix}$$

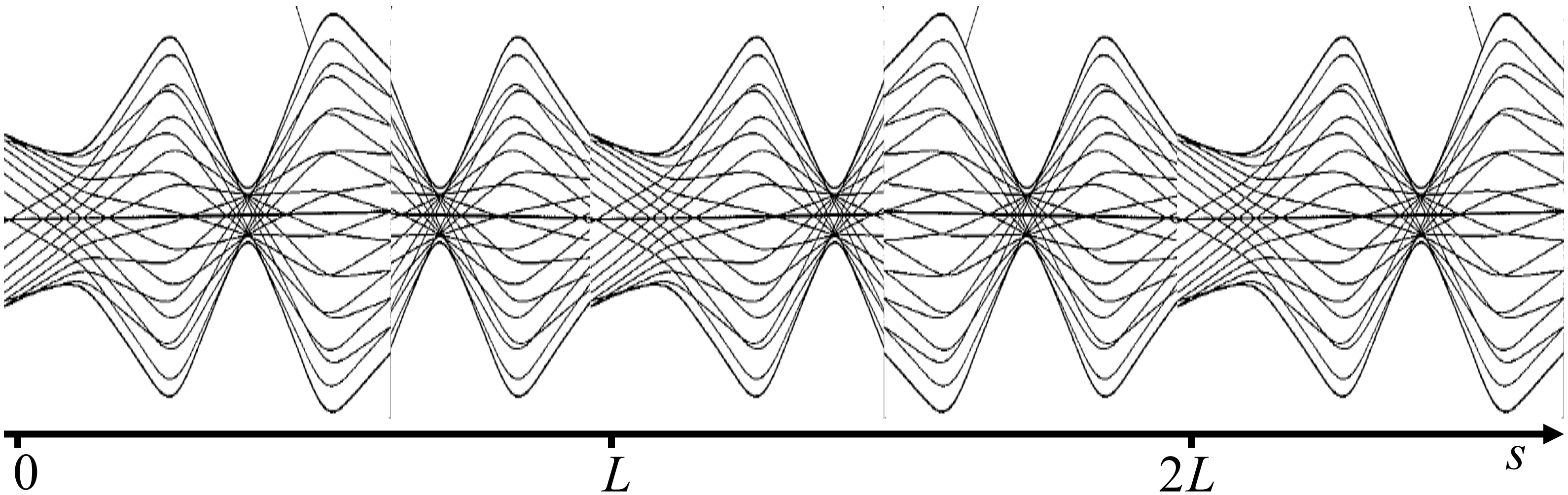
$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

$$= \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\psi(s) & \sin\psi(s) \\ -\sin\psi(s) & \cos\psi(s) \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix}$$

$$\underline{M}(s) = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\psi(s) & \sin\psi(s) \\ -\sin\psi(s) & \cos\psi(s) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}[\cos\psi + \alpha_0 \sin\psi] & \sqrt{\beta_0\beta}\sin\psi \\ \sqrt{\frac{1}{\beta_0\beta}}[(\alpha_0 - \alpha)\cos\psi - (1+\alpha_0\alpha)\sin\psi] & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi - \alpha\sin\psi] \end{pmatrix}$$

Periodic solutions in a periodic accelerator



$$\vec{z}(s) = \underline{M}(s,0)\vec{z}(0)$$

$$\vec{z}(L) = \underline{M}(L,0)\vec{z}(0)$$

$$\vec{z}(s+L) = \underline{M}_0(s)\vec{z}(s) \quad , \quad \underline{M}_0 = \underline{M}(s+L,s)$$

$$\vec{z}(s+nL) = \underline{M}_0^n(s)\vec{z}(s)$$

Periodic beta functions

If the particle distribution in a ring or any other periodic structure is stable, it is periodic from turn to turn.

$$\rho(x,x',s+L)=\rho(x,x',s)$$

To be matched to such a beam, the Twiss parameters α, β, γ must be the same after every turn.

$$\underline{M}(s,0)=\begin{pmatrix}\sqrt{\frac{\beta}{\beta_0}}[\cos\psi+\alpha_0\sin\psi] & \sqrt{\beta_0\beta}\sin\psi \\ \sqrt{\frac{1}{\beta_0\beta}}[(\alpha_0-\alpha)\cos\psi-(1+\alpha_0\alpha)\sin\psi] & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi-\alpha\sin\psi]\end{pmatrix}$$

$$\underline{M}_p(s)=\begin{pmatrix}\cos\mu+\alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu-\alpha\sin\mu\end{pmatrix}=\underline{1}\cos\mu+\begin{pmatrix}\alpha & \beta \\ -\gamma & -\alpha\end{pmatrix}\sin\mu$$

$$\mu=\psi(s+L)-\psi(s)$$

One turn matrix to periodic Twiss parameters

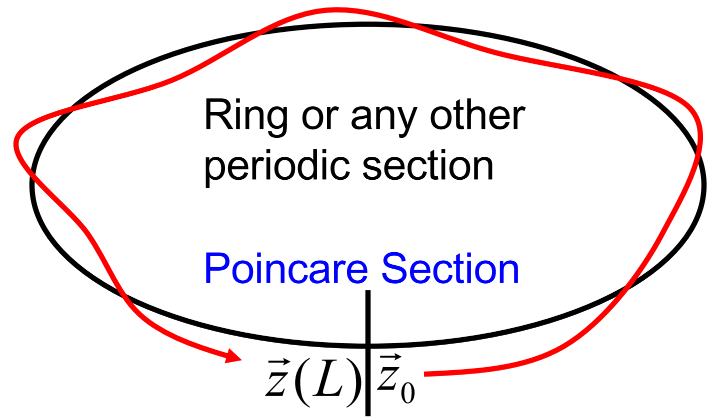
The periodic Twiss parameters are the solution of a nonlinear differential equation with periodic boundary conditions:

$$\beta' = -2\alpha \quad \text{with} \quad \beta(L) = \beta(0)$$

$$\alpha' = k\beta - \frac{1+\alpha^2}{\beta} \quad \text{with} \quad \alpha(L) = \alpha(0)$$

$$\mu = \int_0^L \frac{1}{\beta(\hat{s})} d\hat{s}$$

Note: $\beta(s) > 0$



$$\underline{M}_0(s) = \underline{1}\cos\mu + \underline{\beta}\sin\mu \; ; \; \underline{\beta} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\cos\mu = \tfrac{1}{2}\mathrm{Tr}[\underline{M}_0(s)]$$

$$\beta = \underline{M}_{0,12} \frac{1}{\sin\mu}$$

$$\alpha = (\underline{M}_{0,11} - \underline{M}_{0,22}) \frac{1}{2\sin\mu}$$

$$\gamma = \frac{1+\alpha^2}{\beta}$$

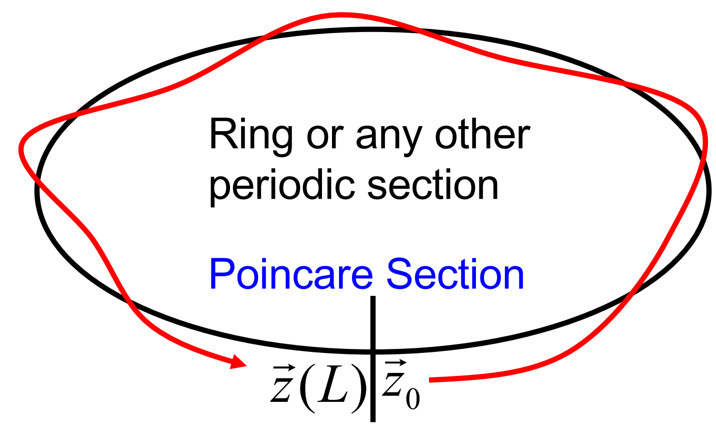
Stable beam motion and thus a periodic beta function can only exist when |Tr[M]| < 2.

The tune of a particle accelerator

The betatron phase advance per turn devided by 2π is called the TUNE.

$$\mu = 2\pi\nu = \psi(s+L) - \psi(s)$$

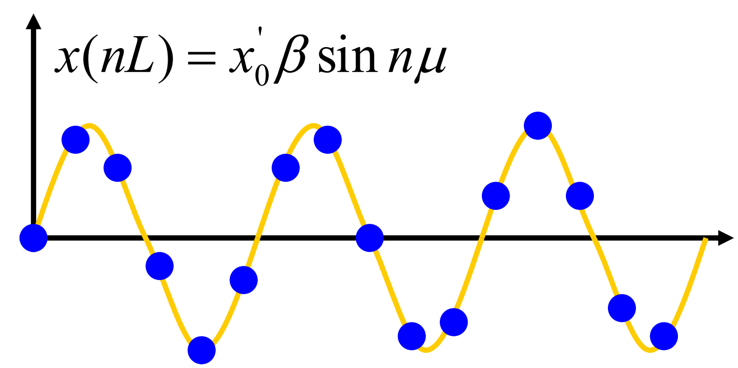
It is a property of the ring and does not depend on the azimuth s.



$$\underline{M}_0(s) = \underline{1}\cos\mu + \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} \sin\mu$$

$$2\cos\mu(s) = \mathrm{Tr}[\underline{M}_0(s)] = \mathrm{Tr}[\underline{M}(s,0)\underline{M}_0(0)\underline{M}^{-1}(s,0)]$$

$$= \mathrm{Tr}[\underline{M}_0(0)] = 2\cos\mu(0)$$



$$\underline{M}_0^n = \underline{1}\cos n\mu + \underline{\beta}\sin n\mu$$

The drift

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Note that in nonlinear expansion $x' \neq a$ so that the drift does not have a linear transport map even though $x(s) = x_0 + x_0' s$ is completely linear.

$$\begin{pmatrix} x \\ a \\ y \\ b \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} x_0 + s a_0 \\ a \\ y_0 + s b_0 \\ b_0 \\ \tau_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \underline{0} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \vec{z}_0$$

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The quadrupole

$$x'' = -x\,k$$

$$y'' = \;\; y\,k$$

$$\underline{M}_4 = \begin{pmatrix} \cos(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}\,s) & & \\ -\sqrt{k}\sin(\sqrt{k}\,s) & \cos(\sqrt{k}\,s) & & \underline{0} \\ & & \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ & \underline{0} & \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix}$$

As for a drift, the energy does not change, i.e. $\delta = \delta_0$.

The time of flight only depends on energy, i.e. $\tau = \tau_0 + M56\,\delta$.

For k<0 one has to take into account that

$$\cos(\sqrt{k}\,s) = \cosh(\sqrt{|k|}\,s)\,, \quad \sin(\sqrt{k}\,s) = i\sinh(\sqrt{|k|}\,s)$$

$$\cosh(\sqrt{k}\,s) = \cos(\sqrt{|k|}\,s)\,, \quad \sinh(\sqrt{k}\,s) = i\sin(\sqrt{|k|}\,s)$$

Variation of constants

$$\vec{z}\,' = \vec{f}(\vec{z},s)$$

$$\vec{z}\,' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z},s)$$

Field errors, nonlinear fields, etc can lead to $\Delta\vec{f}(\vec{z},s)$

$$\vec{z}_H' = \underline{L}(s)\vec{z}_H \quad\Rightarrow\quad \vec{z}_H(s) = \underline{M}(s)\vec{z}_{H0} \quad \text{with} \quad \underline{M}'(s)\vec{a} = \underline{L}(s)\underline{M}(s)\vec{a}$$

$$\vec{z}(s) = \underline{M}(s)\vec{a}(s) \quad\Rightarrow\quad \vec{z}\,'(s) = \underline{M}'(s)\vec{a} + \underline{M}(s)\vec{a}\,'(s) = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z},s)$$

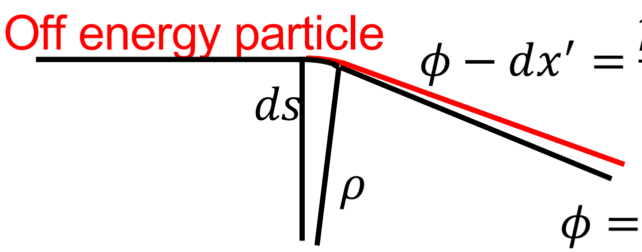
$$\vec{a}(s) = \vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vec{z}(s) = \underline{M}(s)\left\{\vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}\right\}$$

$$= \vec{z}_H(s) + \int_0^s \underline{M}(s-\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}$$

Perturbations are propagated from s to s'

The dipole equation of motion



$$\phi - dx' = \frac{p_0}{p} ds/\rho = \phi(1 - \frac{dp}{p})$$

$$\phi = ds/\rho$$

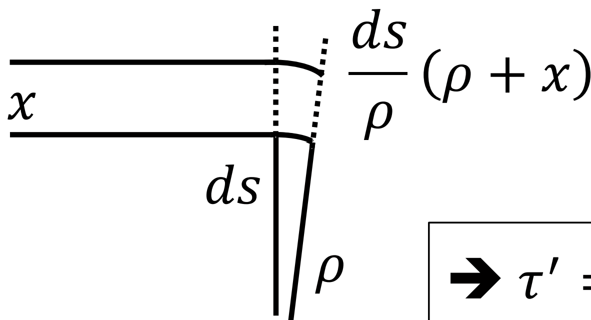
$$\Rightarrow x'' = \frac{1}{\rho}\frac{dp}{p} = \frac{1}{\rho}\frac{dp}{dE}\frac{E}{p}\delta = \frac{1}{\rho}\frac{1}{\beta^2}\delta$$

$x'' + x\,\kappa^2 = \frac{\kappa}{\beta^2}\delta$ with $\kappa = \frac{1}{\rho}$ or $\begin{aligned} x' &= a \\ a' &= -\kappa^2 x + \frac{\kappa}{\beta^2}\delta \end{aligned}$

$$x_H'' + x_H\,\kappa^2 = 0 \;\Rightarrow\; \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) \\ -\kappa\sin(\kappa s) & \cos(\kappa s) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \underline{M}(s)\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} - \underline{M}(s)\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \int_0^s \underline{M}(s-\zeta)\begin{pmatrix} 0 \\ \frac{\kappa}{\beta^2}\delta \end{pmatrix} d\zeta = \int_0^s \begin{pmatrix} \frac{1}{\kappa}\sin(\kappa s) \\ \cos(\kappa s) \end{pmatrix} d\zeta\,\frac{\kappa}{\beta^2}\delta = \begin{pmatrix} \frac{1}{\kappa}(1-\cos(\kappa s)) \\ \sin(\kappa s) \end{pmatrix}\frac{1}{\beta^2}\delta$$

The dipole transport matrix



$$\Rightarrow \tau' = -\frac{x}{\rho}\frac{1}{v}\frac{E_0}{p_0} = -\frac{x}{\rho}\frac{1}{\beta^2} = -\frac{\kappa}{\beta^2}x$$

$$\underline{M} = \begin{pmatrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) & 0 & 0 & 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ -\kappa\sin(\kappa s) & \cos(\kappa s) & 0 & 0 & 0 & \sin(\kappa s) \\ 0 & 0 & 1 & s & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] & 0 & 0 & 1 & \kappa^{-1}[\sin(\kappa s)-s\kappa] \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(for $\beta = 1$)

$$M_{56} = -\kappa \int_0^s M_{16} ds$$

The combined function bend

$$x'' = -x\,(\underbrace{\kappa^2 + k}_{K}) + \delta\kappa$$

$$y'' = y\,k \quad , \quad \tau' = -\kappa\, x$$

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0}\,\vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{T} & \underline{0} & \underline{M}_\tau \end{pmatrix}$$

$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K}\,s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\,s) \\ -\sqrt{K}\sin(\sqrt{K}\,s) & \cos(\sqrt{K}\,s) \end{pmatrix}$$

$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \frac{\kappa}{K}[1-\cos(\sqrt{K}s)] \\ \frac{\kappa}{\sqrt{K}}\sin(\sqrt{K}s) \end{pmatrix}$$

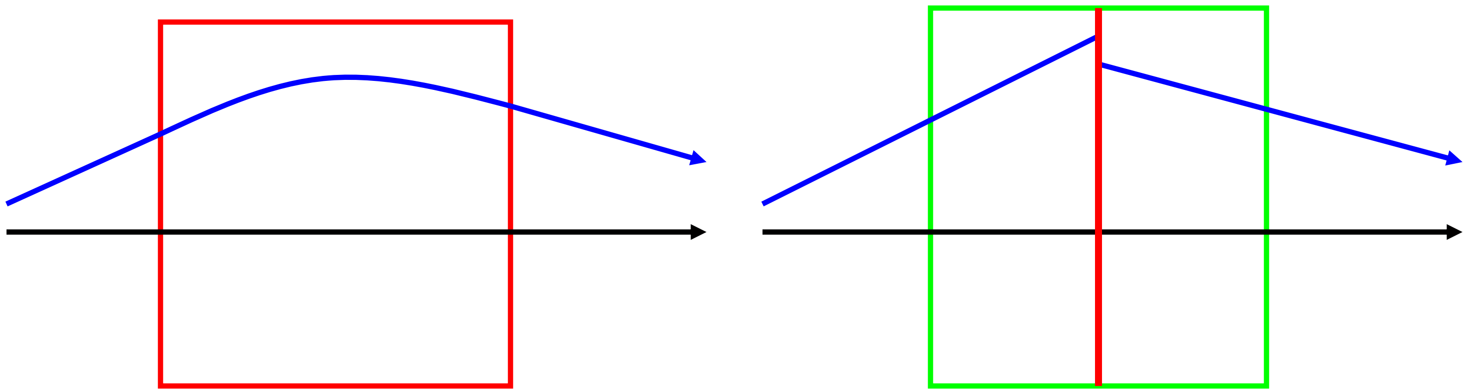
Options:

- For k>0:
  focusing in x, defocusing in y.
- For k<0, K<0:
  defocusing in x, focusing in y.
- For k<0, K>0:
  weak focusing in both planes.

$$\underline{M}_\tau = \begin{pmatrix} 1 & -\kappa\int_0^s M_{16}\,ds \\ 0 & 1 \end{pmatrix}$$

$\underline{T}$ from symplecticity

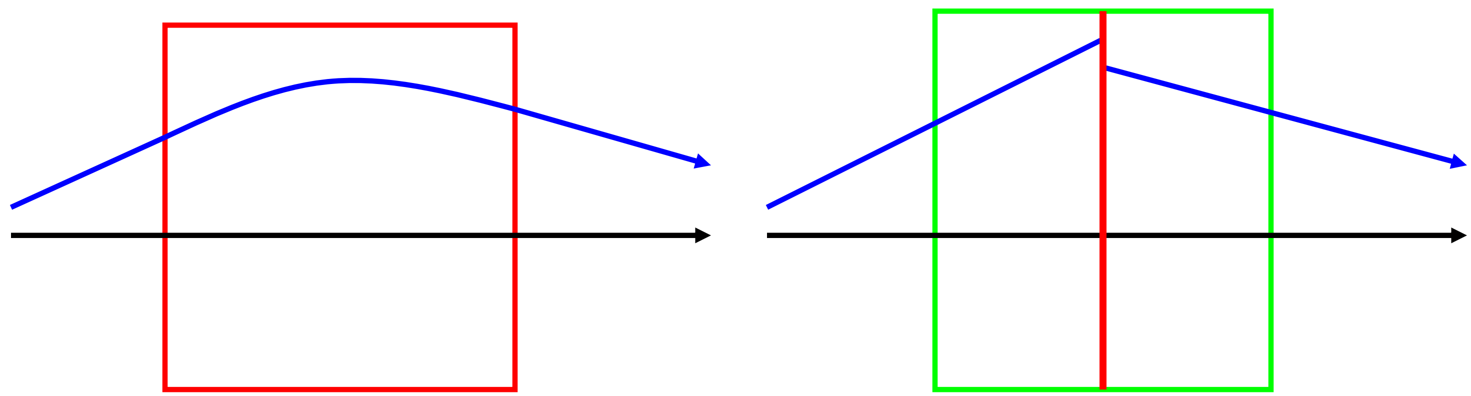
Thin lens approximation



$$\vec{z}(s) = \underline{M}(s)\,\vec{z}_0 = \underline{D}(\tfrac{s}{2})\underline{D}^{-1}(\tfrac{s}{2})\underline{M}(s)\underline{D}^{-1}(\tfrac{s}{2})\underline{D}(\tfrac{s}{2})\vec{z}_0$$

Drift: $\underline{M}^{\text{thin}}_{\text{drift}}(s) = \underline{D}^{-1}(\tfrac{s}{2})\underline{M}(s)\underline{D}^{-1}(\tfrac{s}{2}) = \underline{1}$

The thin lens quadrupole



$$\underline{M}^{\text{thin}}_{\text{quad,x}}(s) = \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\sqrt{k}s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}s) \\ -\sqrt{k}\sin(\sqrt{k}s) & \cos(\sqrt{k}s) \end{pmatrix} \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & s \\ -ks & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{s}{2} \\ -ks & 1+\frac{ks^2}{2} \end{pmatrix}$$

Weak magnet limit: $\sqrt{k}s << 1$

$$\underline{M}^{\text{thin}}_{\text{quad,x}}(s) \approx \begin{pmatrix} 1 & 0 \\ -ks & 1 \end{pmatrix}$$

The thin lens dipole

$$\underline{M} = \begin{pmatrix} \begin{matrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) \\ -\kappa\sin(\kappa s) & \cos(\kappa s) \end{matrix} & \underline{0} & \begin{matrix} 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ 0 & \sin(\kappa s) \end{matrix} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \begin{matrix} -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] \\ 0 & 0 \end{matrix} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix}$$

Weak magnet limit: $\kappa s << 1$

$$\underline{M}^{\text{thin}}_{\text{bend},x\tau}(s) = \underline{D}(-\tfrac{s}{2})\underline{M}_{\text{bend},x\tau}\underline{D}(-\tfrac{s}{2}) \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\kappa^2 s & 1 & 0 & \kappa s \\ -\kappa s & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The thin lens combined function bend

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0}\,\vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{T} & \underline{0} & \underline{1} \end{pmatrix}$$

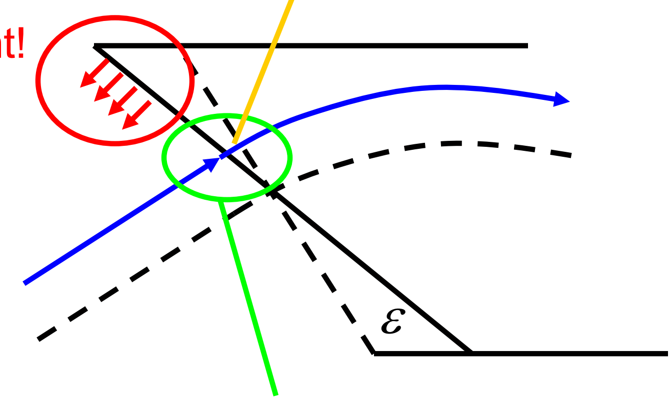
Weak magnet limit: $\kappa s << 1$

$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K}\,s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\,s) \\ -\sqrt{K}\sin(\sqrt{K}\,s) & \cos(\sqrt{K}\,s) \end{pmatrix} \qquad \underline{M}_x^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ -K\,s & 1 \end{pmatrix}$$

$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix} \longrightarrow \underline{M}_y^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ k\,s & 1 \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \frac{\kappa}{K}[1-\cos(\sqrt{K}s)] \\ \frac{\kappa}{\sqrt{K}}\sin(\sqrt{K}s) \end{pmatrix} \qquad \vec{D} = \begin{pmatrix} 0 \\ \kappa\,s \end{pmatrix}$$

Edge focusing



Horizontal focusing with $\Delta x' = -x\frac{\tan(\varepsilon)}{\rho}$

$$B_x = \partial_y B_s\Big|_{y=0} y\tan(\varepsilon) = \partial_s B_y\Big|_{y=0} y\tan(\varepsilon)$$

$$y'' = \tfrac{q}{p}\partial_s B_y\Big|_{y=0} y\tan(\varepsilon)$$

$$\Delta y' = \int y''ds = \tfrac{q}{p}B_y y\tan(\varepsilon) = y\frac{\tan(\varepsilon)}{\rho}$$

Quadrupole effect with

$$kl = \frac{\tan(\varepsilon)}{\rho}$$

$$\vec{z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\tan(\varepsilon)}{\rho} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\tan(\varepsilon)}{\rho} & 1 \end{pmatrix}\vec{z}_0$$

Orbit distortions for a one-pass accelerator

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \Delta f$$

The extra force can for example come from an erroneous dipole field or from a correction coil: $\Delta f = \frac{q}{p}\Delta B_y = \Delta\kappa$

Variation of constants: $\vec{z} = \underline{M}\vec{z}_0 + \Delta\vec{z}$ with $\Delta\vec{z} = \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \Delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$

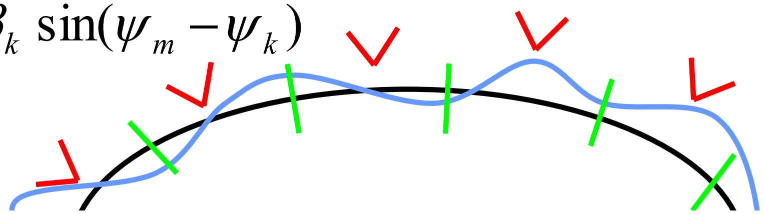
$$\Delta x(s) = \sum_k \Delta\vartheta_k \sqrt{\beta(s)\beta_k}\sin(\psi(s) - \psi_k)$$

Orbit correction for a one-pass accelerator

When the closed orbit $x_{\mathrm{co}}^{\mathrm{old}}(s_m)$ is measured at beam position monitors (BPMs, index m) and is influenced by corrector magnets (index k), then the monitor readings before and after changing the kick angles created in the correctors by $\Delta\vartheta_k$ are related by

$$x_{\mathrm{co}}^{\mathrm{new}}(s_m) = x_{\mathrm{co}}^{\mathrm{old}}(s_m) + \sum_k \Delta\vartheta_k \sqrt{\beta_m \beta_k}\,\sin(\psi_m - \psi_k)$$

$$= x_{\mathrm{co}}^{\mathrm{old}}(s_m) + \sum_k O_{mk} \Delta\vartheta_k$$



$$\vec{x}_{\mathrm{co}}^{\mathrm{new}} = \vec{x}_{\mathrm{co}}^{\mathrm{old}} + \underline{O}\Delta\vec{\vartheta}$$

$$\Delta\vec{\vartheta} = -\underline{O}^{-1}\vec{x}_{\mathrm{co}}^{\mathrm{old}} \quad \Rightarrow \quad \vec{x}_{\mathrm{co}}^{\mathrm{new}} = 0$$

It is often better not to try to correct the closed orbit at the the BPMs to zero in this way since

1. computation of the inverse can be numerically unstable, so that small errors in the old closed orbit measurement lead to a large error in the corrector coil settings.
2. A zero orbit at all BPMs can be a bad orbit in-between BPMs

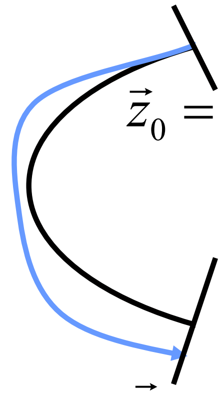
Dispersion of one-pass accelerators

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \kappa\delta$$

$$\vec{z} = \underline{M}\vec{z}_0 + \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix}0\\ \delta\kappa(\hat{s})\end{pmatrix}d\hat{s}$$

$$\Rightarrow \quad \vec{D}(s) = \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix}0\\ \kappa(\hat{s})\end{pmatrix}ds'$$



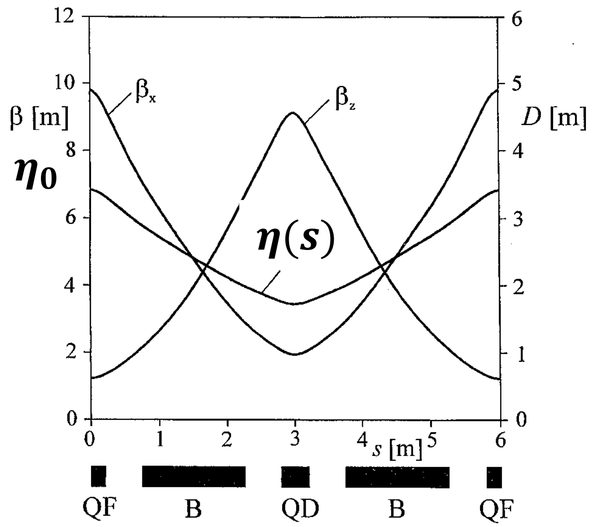
$$\Delta\kappa = \delta\kappa$$

$$D(s) = \sqrt{\beta(s)}\int_0^s \kappa(\hat{s})\sqrt{\beta(\hat{s})}\sin(\psi(s)-\psi(\hat{s}))d\hat{s}$$

Alternatively, one can multiply the 6x6 matrices and take $\boldsymbol{D(s) = M_{16}(s)}$

Fodo Cells and periodic dispersion

Alternating gradients allow focusing in both transverse plains. Therefore, focusing and defocusing quadrupoles are usually alternated and interleaved with bending magnets.



The dispersion that starts with 0 is called $D(s)$, the dispersion that is periodic in a section is called $\eta(s)$.

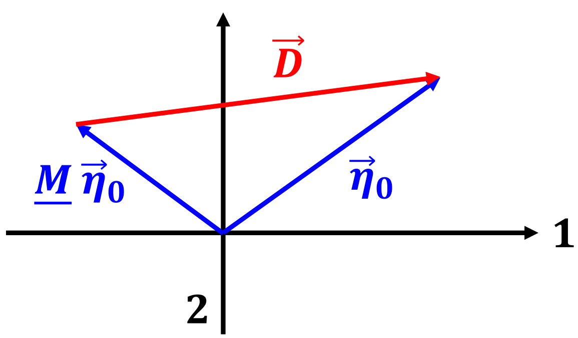
$$\underline{M}_0 = \underline{M}_{FoDo}^N$$

The periodic beta function and dispersion for each FODO is also periodic for an accelerator section that consists of many FODO cells. Often large sections of an accelerator consist of FODOs.

$$\vec{\eta}(s) = \begin{pmatrix} \eta(s) \\ \eta'(s) \end{pmatrix}, \vec{D}(s) = \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}$$

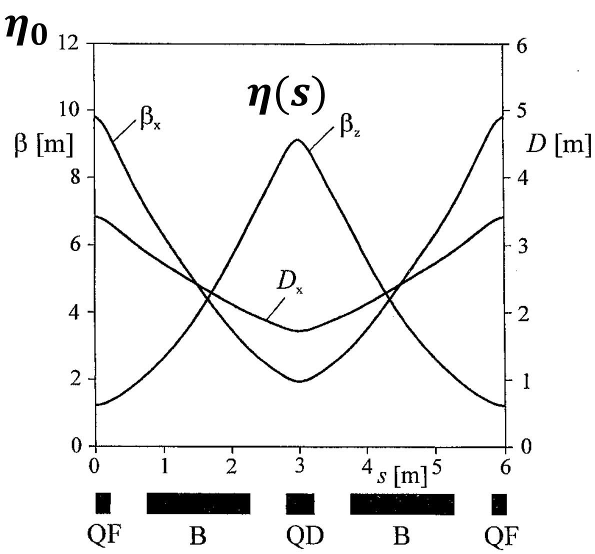
$$\vec{\eta}(s) = \underline{M}(s)\,\vec{\eta}_0 + \vec{D}(s)$$

After the FoDo: $\vec{\eta}_0 = \underline{M}(s)\,\vec{\eta}_0 + \vec{D}$ ➔ $\vec{\eta}_0 = (\underline{1} - \underline{M})^{-1}\,\vec{D}$

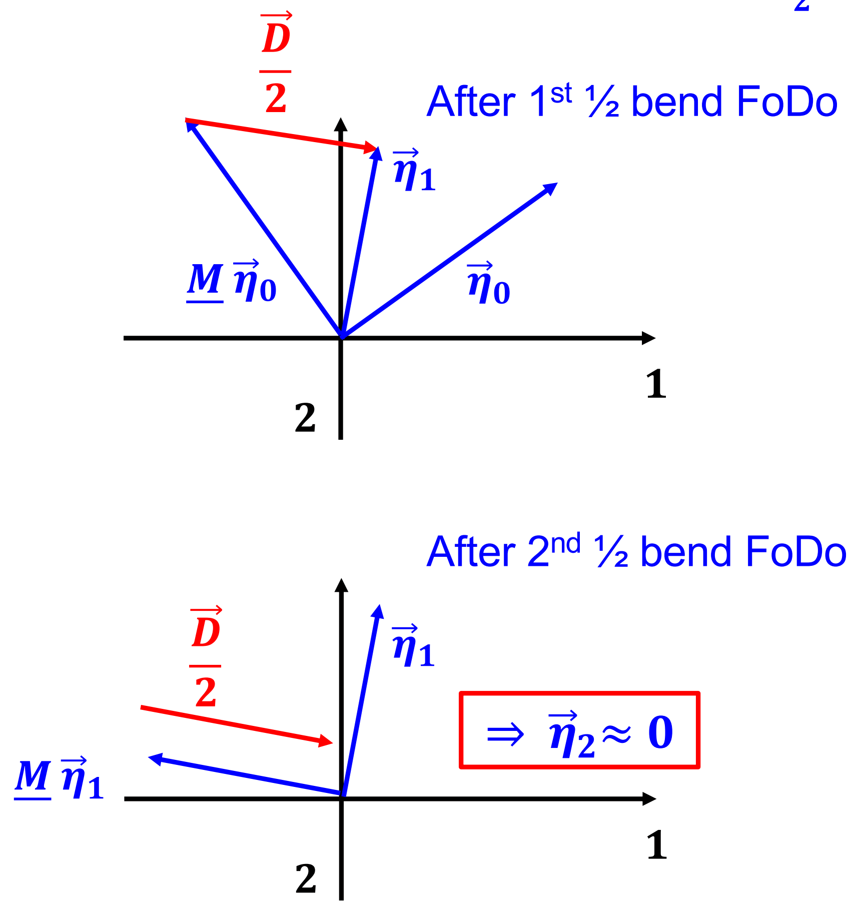


Dispersion suppression by missing bends

After an arc of periodic FoDo cells one would often like to suppress the dispersion to 0 while not changing the betas much from the periodic cells. This can be done by having one FoDo with bends reduced by a factor α followed by one reduced by (1- α).



Example: **90 degrees** FoDo cell and $\alpha = \frac{1}{2}$:



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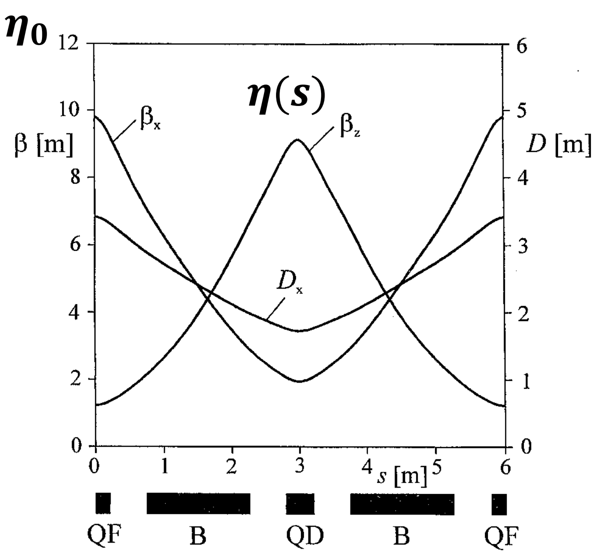
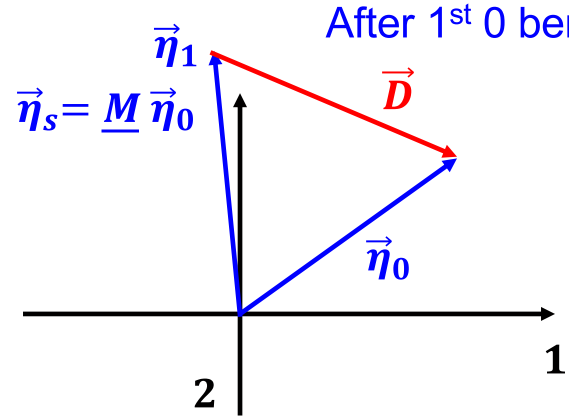
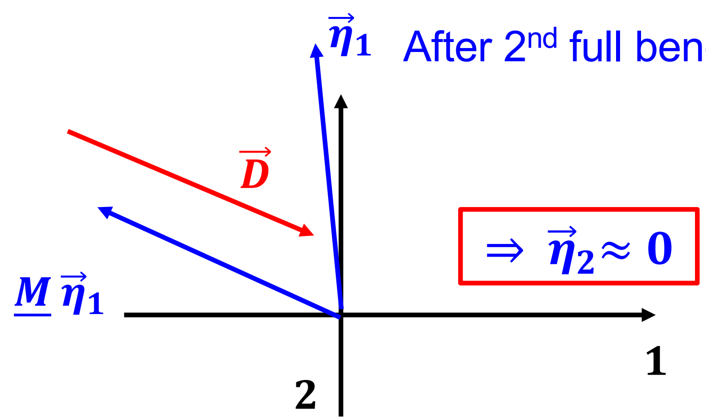


Example: **60 degrees** FoDo cell and $\boldsymbol{\alpha = 0}$:

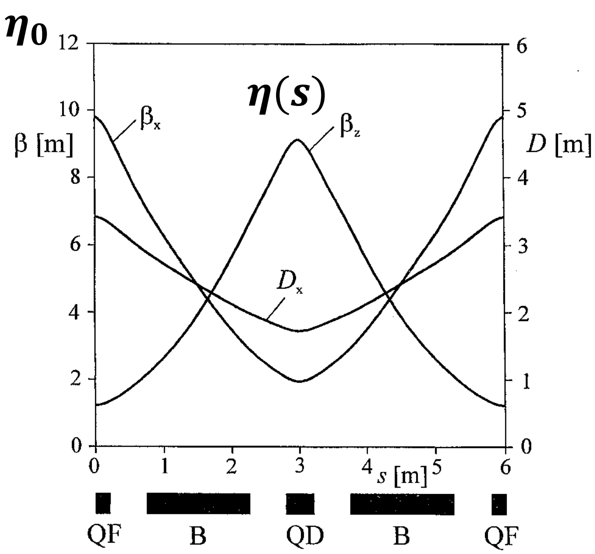




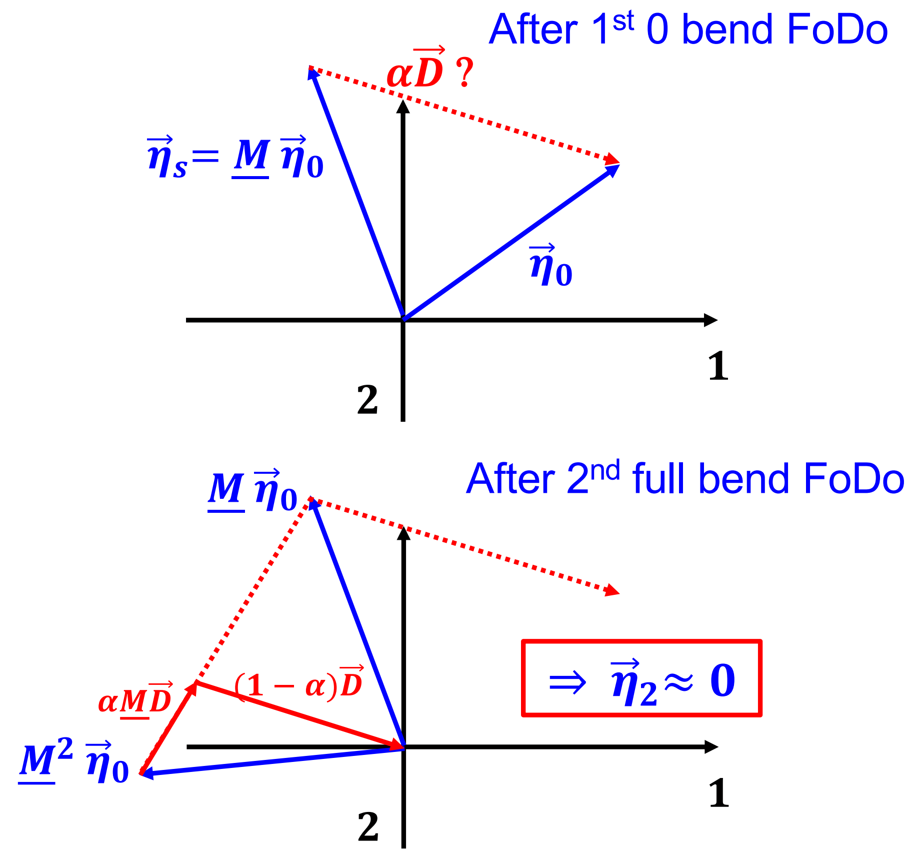
For every FoDo phase advance there is an α to make η 0.

Dispersion suppression by missing bends

After an arc of periodic FoDo cells one would often like to suppress the dispersion to 0 while not changing the betas much from the periodic cells. This can be done by having one FoDo with bends reduced by a factor α followed by one reduced by (1- α).



For any other FoDo phase advance: is there an α ?



For every FoDo phase advance there is an α to make η 0.

Closed orbit in periodic accelerators

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \Delta f$$

The extra force can for example come from an erroneous dipole field or from a correction coil: $\Delta f = \frac{q}{p}\Delta B_y = \Delta\kappa$

Variation of constants: $\vec{z} = \underline{M}\vec{z}_0 + \Delta\vec{z}$ with $\Delta\vec{z} = \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix}0\\ \Delta\kappa(\hat{s})\end{pmatrix}d\hat{s}$

For the periodic or closed orbit: $\vec{z}_{co} = \underline{M}_0\vec{z}_{co} + \underline{M}_0\int_0^L \underline{M}^{-1}(\hat{s})\begin{pmatrix}0\\ \Delta\kappa(\hat{s})\end{pmatrix}d\hat{s}$

$$\vec{z}_{co} = [\underline{M}_0^{-1} - \underline{1}]^{-1}\int_0^L \underline{M}^{-1}(\hat{s})\begin{pmatrix}0\\ \Delta\kappa(\hat{s})\end{pmatrix}d\hat{s}$$

$$= \frac{(\cos\mu - 1)\underline{1} + \sin\mu\underline{\beta}}{(\cos\mu-1)^2 + \sin^2\mu}\int_0^L \begin{pmatrix}-\sqrt{\beta\hat{\beta}}\sin\hat{\psi}\\ \sqrt{\frac{\hat{\beta}}{\beta}}[\cos\hat{\psi} + \alpha\sin\hat{\psi}]\end{pmatrix}\Delta\kappa(\hat{s})\,d\hat{s}$$

Closed orbit in periodic accelerators

$$\vec{z}_{co}(L) = \frac{\cos\mu\underline{1} + \sin\mu\underline{\beta} - \underline{1}}{2 - 2\cos\mu}\int_0^L \begin{pmatrix} \sqrt{\beta\hat{\beta}}\sin(-\hat{\psi}) \\ \sqrt{\frac{\hat{\beta}}{\beta}}[\cos(-\hat{\psi}) - \alpha\sin(-\hat{\psi})] \end{pmatrix} \Delta\kappa(\hat{s})\, d\hat{s}$$

$$x_{co}(L) = \frac{1}{4\sin^2\frac{\mu}{2}}\int_0^L \sqrt{\beta\hat{\beta}}[\sin(\mu - \hat{\psi}) + \sin\hat{\psi}]\Delta\kappa(\hat{s})\, d\hat{s}$$

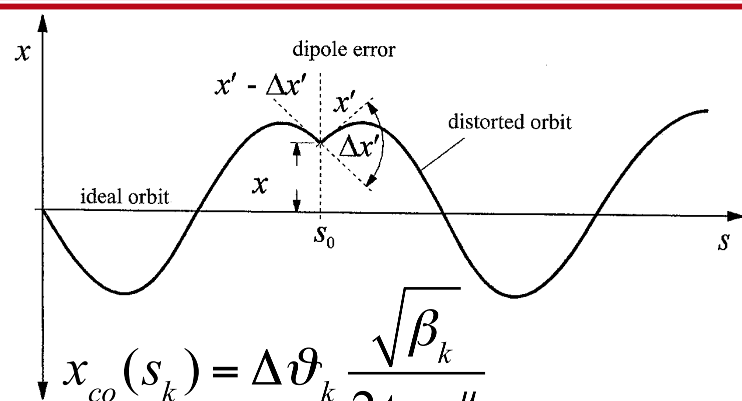
$$\mathbf{\sin\left(\left(\psi - \frac{\mu}{2}\right) + \frac{\mu}{2}\right) - \sin\left(\left(\psi - \frac{\mu}{2}\right) - \frac{\mu}{2}\right)}$$

$$= \frac{1}{2\sin\frac{\mu}{2}}\int_0^L \sqrt{\beta\hat{\beta}}\cos(\hat{\psi} - \tfrac{\mu}{2})\Delta\kappa(\hat{s})\, d\hat{s}$$

$$x_{co}(s) = \frac{1}{2\sin\frac{\mu}{2}}[\int_s^L \sqrt{\beta\hat{\beta}}\cos(\hat{\psi} - \psi - \tfrac{\mu}{2})\Delta\hat{\kappa}\, d\hat{s} + \int_0^s \sqrt{\beta\hat{\beta}}\cos(\hat{\psi} - \psi + \tfrac{\mu}{2})\Delta\hat{\kappa}\, d\hat{s}$$

$$= \frac{\sqrt{\beta}}{2\sin\frac{\mu}{2}}\int_0^L \sqrt{\hat{\beta}}\cos(|\hat{\psi} - \psi| - \tfrac{\mu}{2})\Delta\hat{\kappa}\, d\hat{s}$$

Closed orbit for one kick



$$x_{co}(s_k) = \Delta\vartheta_k \frac{\sqrt{\beta_k}}{2\tan\frac{\mu}{2}}$$

$$x_{co}(s) = \Delta\vartheta_k \frac{\sqrt{\beta\beta_k}}{2\sin\frac{\mu}{2}} \cos(|\psi - \psi_k| - \tfrac{\mu}{2})$$

Free betatron oscillation

$$x'_{co}(s_k) - x'_{co}(s_k + L) = \Delta\vartheta_k \frac{-\sin(-\frac{\mu}{2}) + \sin(\frac{\mu}{2})}{2\sin\frac{\mu}{2}} = \Delta\vartheta_k$$

$$x_{max} = \sqrt{2J\beta}$$

$$x_{co}(s) = \sqrt{2J\beta}\sin(\psi + \varphi_0)\,, \quad J = \frac{\Delta\vartheta_k^2 \beta_k}{8\sin^2\frac{\mu}{2}}$$

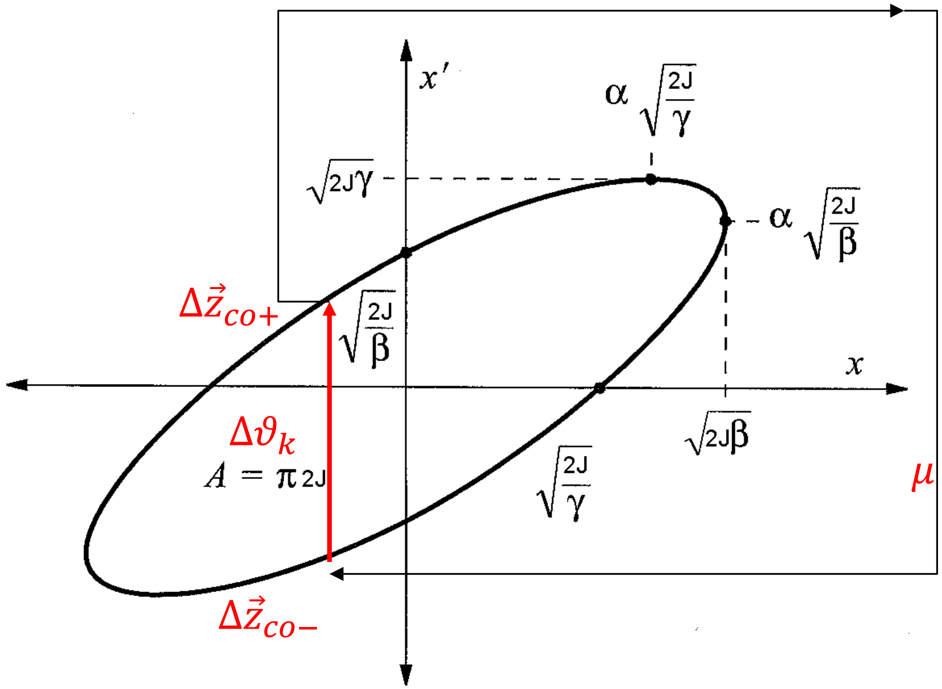
The oscillation amplitude J diverges when the tune ν is close to an integer.

$$s < s_k : \varphi_0 = \tfrac{\pi}{2} - \psi_k + \tfrac{\mu}{2} \quad , \quad s > s_k : \varphi_0 = \tfrac{\pi}{2} - \psi_k - \tfrac{\mu}{2}$$

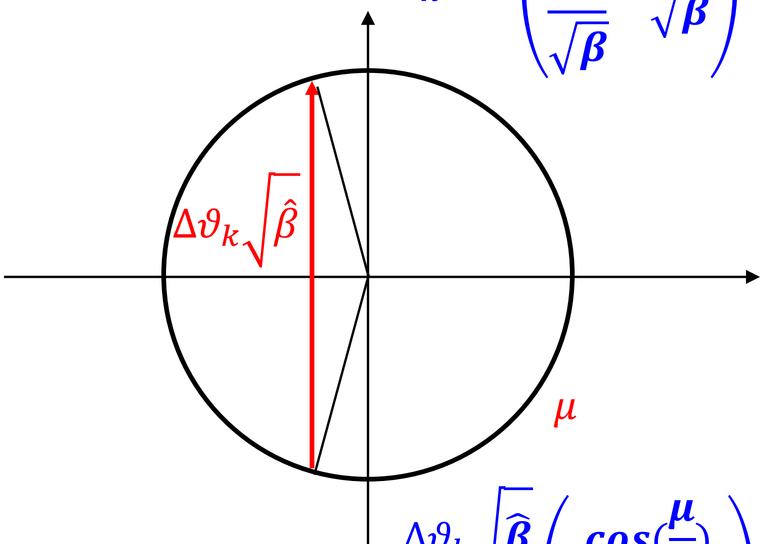
Phase jump by μ

Closed orbit for one kick

Free betatron oscillation



$$\begin{pmatrix} x_n \\ a_n \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$



$$\begin{pmatrix} x_{n\pm} \\ a_{n\pm} \end{pmatrix} = \frac{\Delta\vartheta_k\sqrt{\hat{\beta}}}{\sin(\frac{\mu}{2})} \begin{pmatrix} \cos(\frac{\mu}{2}) \\ \pm\sin(\frac{\mu}{2}) \end{pmatrix}$$

$$x_n(s) = \frac{\Delta\vartheta_k\sqrt{\beta\hat{\beta}}}{2\sin(\frac{\mu}{2})}\cos\left(\frac{\mu}{2} - |\psi - \psi_k|\right)$$

$$\begin{pmatrix} x_{n\pm} \\ a_{n\pm} \end{pmatrix} = \frac{\Delta\vartheta_k\sqrt{\hat{\beta}}}{\sin(\frac{\mu}{2})} \begin{pmatrix} \frac{\mu}{2} - |\psi - \psi_k| \\ \dots \end{pmatrix}$$

Closed orbit correction in periodic accelerators

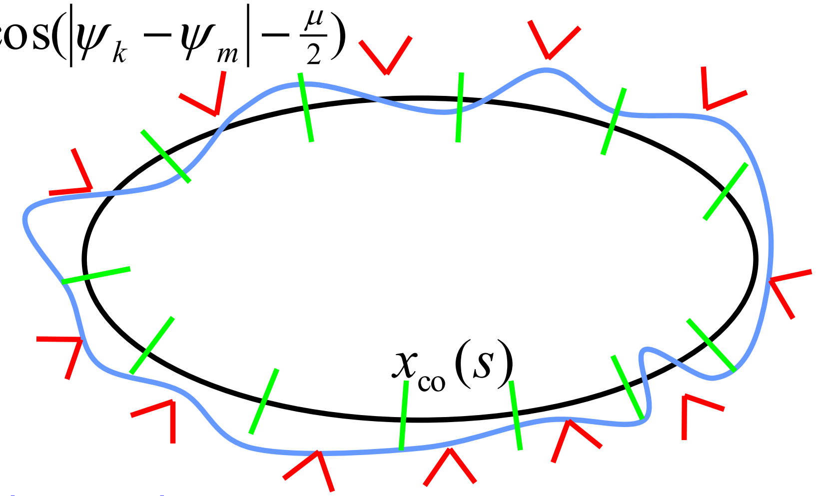
When the closed orbit $x_{\rm co}^{\rm old}(s_m)$ is measured at beam position monitors (BPMs, index m) and is influenced by corrector magnets (index k), then the monitor readings before and after changing the kick angles created in the correctors by $\Delta\vartheta_k$ are related by

$$x_{\rm co}^{\rm new}(s_m) = x_{\rm co}^{\rm old}(s_m) + \sum_k \Delta\vartheta_k \frac{\sqrt{\beta_m\beta_k}}{2\sin\frac{\mu}{2}}\cos(|\psi_k - \psi_m| - \tfrac{\mu}{2})$$

$$= x_{\rm co}^{\rm old}(s_m) + \sum_k O_{mk}\Delta\vartheta_k$$

$$\vec{x}_{\rm co}^{\rm new} = \vec{x}_{\rm co}^{\rm old} + \underline{O}\Delta\vec{\vartheta}$$

$$\Delta\vec{\vartheta} = -\underline{O}^{-1}\vec{x}_{\rm co}^{\rm old} \quad \Rightarrow \quad \vec{x}_{\rm co}^{\rm new} = 0$$

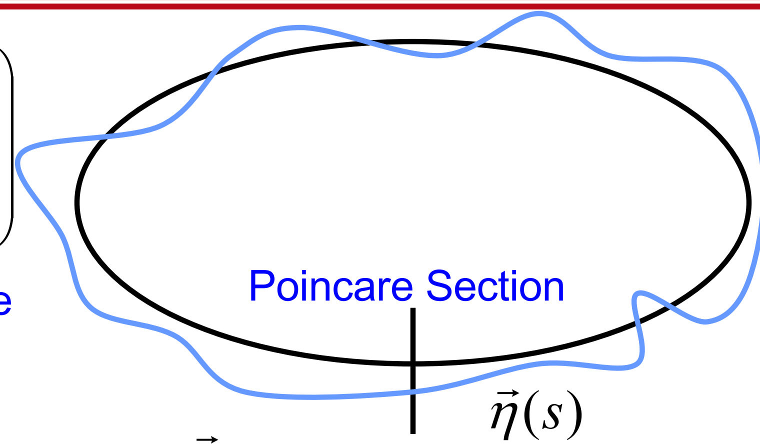


It is often better not to try to correct the closed orbit at the the BPMs to zero in this way since

1. computation of the inverse can be numerically unstable, so that small errors in the old closed orbit measurement lead to a large error in the corrector coil settings.
2. A zero orbit at all BPMs can be a bad orbit inbetween BPMs

Periodic dispersion

$$\begin{pmatrix} \underline{M}_{0x}\vec{z}_0 + \vec{D}(L)\delta \\ M_{56}\delta \\ \delta \end{pmatrix} = \begin{pmatrix} \underline{M}_{0x} & \vec{0} & \vec{D}(L) \\ \vec{T}^T & 1 & M_{56} \\ \vec{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{z}_0 \\ 0 \\ \delta \end{pmatrix}$$



The periodic orbit for particles with relative energy deviation δ is

$$\vec{\eta}(L) = \underline{M}_0\vec{\eta}(0) + \vec{D}(L) \quad \text{with} \quad \vec{\eta}(L) = \vec{\eta}(0)$$

$$\vec{\eta}(0) = \underline{M}_0\vec{\eta}(0) + \vec{D}(L)$$

$$\Downarrow$$

$$\vec{\eta}(0) = [\underline{1} - \underline{M}_0(0)]^{-1}\vec{D}(L)$$

Particles with energy deviation δ oscillates around this periodic orbit.

$$\vec{z} = \vec{z}_\beta + \delta\vec{\eta}$$

$$\underline{\vec{z}_\beta(L) + \delta}\vec{\eta}(L) = \vec{z}(L) = \underline{M}_0\vec{z}(0) + \vec{D}(L)\delta = \underline{M}_0[\vec{z}_\beta(0) + \delta\vec{\eta}(0)] + \vec{D}(L)\delta$$

$$= \underline{M}_0\underline{\vec{z}_\beta(0) + \delta}\vec{\eta}(L)$$

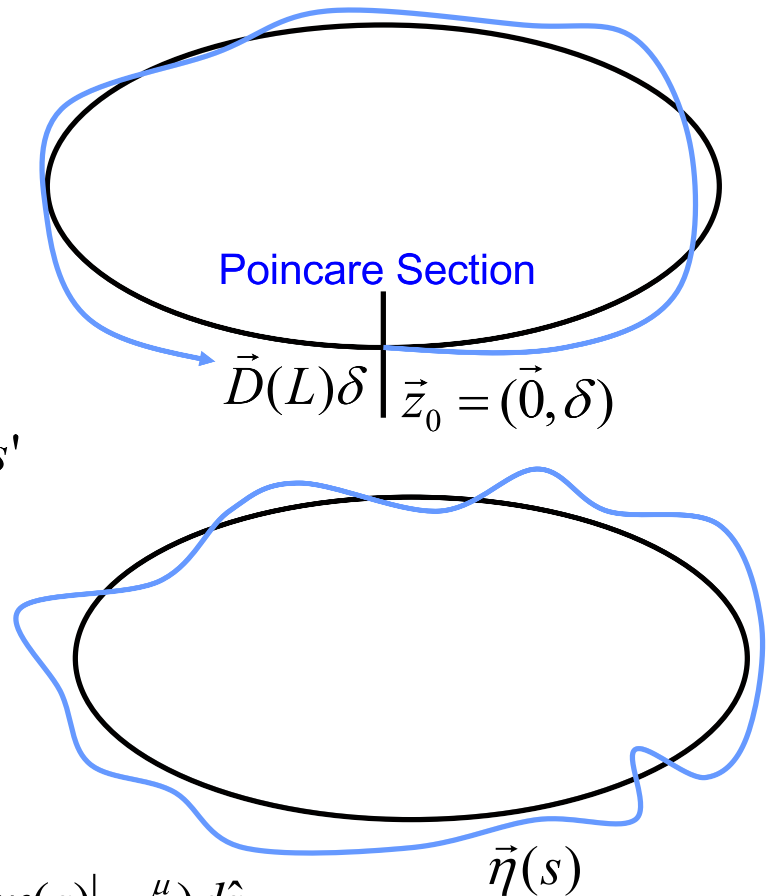
Periodic dispersion integral

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \kappa\delta$$

$$\vec{z} = \underline{M}\vec{z}_0 + \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$$

$$\Rightarrow \quad \vec{D}(L) = \int_0^L \underline{M}(L-\hat{s})\begin{pmatrix} 0 \\ \kappa(\hat{s}) \end{pmatrix} ds'$$



$$\Delta\kappa = \delta\kappa$$

$$\eta(s) = \frac{\sqrt{\beta(s)}}{2\sin\frac{\mu}{2}} \oint \kappa(\hat{s})\sqrt{\beta(\hat{s})}\cos\left(\left|\psi(\hat{s}) - \psi(s)\right| - \tfrac{\mu}{2}\right)d\hat{s}$$

Qadrupole errors in one-pass accelerators

$$\vec{z}' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z},s)$$

$$\vec{z}(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z},\hat{s})\,d\hat{s} \approx \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_H,\hat{s})\,d\hat{s}$$

$$x'' = -(\kappa^2 + k)x - \Delta k(s)x \quad \Rightarrow \quad \begin{pmatrix} x' \\ a' \end{pmatrix} = \begin{pmatrix} a \\ -(\kappa^2 + k)x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \Delta k(s) & 0 \end{pmatrix}\begin{pmatrix} x \\ a \end{pmatrix}$$

$$\vec{z}(s) = \left\{ \underline{M}(s) - \int_0^s \underline{M}(s,\hat{s})\begin{pmatrix} 0 & 0 \\ \Delta k(\hat{s}) & 0 \end{pmatrix}\underline{M}(\hat{s},0)\,d\hat{s} \right\}\vec{z}_0$$

One quadrupole error:

$$\Delta\underline{M}(s,\hat{s}) = -\underline{M}(s,\hat{s})\begin{pmatrix} 0 & 0 \\ \Delta kl(\hat{s}) & 0 \end{pmatrix}$$

Qadrupole errors and Twiss in one pass accelerators

$$\Delta\underline{M}(s,\hat{s}) = -\underline{M}(s,\hat{s})\begin{pmatrix} 0 & 0 \\ \Delta kl(\hat{s}) & 0 \end{pmatrix} \qquad \underline{M}(s) = \begin{pmatrix} \dots & \sqrt{\beta_0\beta}\sin\tilde{\psi} \\ \dots & \sqrt{\frac{\beta_0}{\beta}}[\cos\tilde{\psi} - \alpha\sin\tilde{\psi}] \end{pmatrix}$$

$$\Delta\underline{M}(s,\hat{s}) = -\Delta kl(\hat{s})\begin{pmatrix} \sqrt{\hat{\beta}\beta}\sin\psi & 0 \\ \sqrt{\frac{\hat{\beta}}{\beta}}[\cos\psi - \alpha\sin\psi] & 0 \end{pmatrix} , \qquad \psi = \psi(s) - \psi(\hat{s})$$

$$= \begin{pmatrix} \frac{\frac{1}{2}\Delta\beta[\cos\psi + \hat{\alpha}\sin\psi] + \Delta\psi\beta[\hat{\alpha}\cos\psi - \sin\psi]}{\sqrt{\hat{\beta}\beta}} & \sqrt{\hat{\beta}}\left(\frac{\frac{\Delta\beta}{2}\sin\psi + \Delta\psi\beta\cos\psi}{\sqrt{\beta}}\right) \\ \dots & \dots \end{pmatrix}$$

$$\frac{1}{2}\Delta\beta\cos\psi + \frac{1}{2}\Delta\beta\frac{\sin^2\psi}{\cos\psi} = \frac{1}{2}\Delta\beta\frac{1}{\cos\psi} = -\Delta kl(\hat{s})\beta\hat{\beta}\sin\psi \qquad \Delta\psi = -\frac{\Delta\beta}{2\beta}\tan\psi$$

$$\boxed{\Delta\beta = -\Delta kl(\hat{s})\beta\hat{\beta}\sin 2\psi} \qquad \boxed{\Delta\psi = \Delta kl(\hat{s})\hat{\beta}\tfrac{1}{2}(1-\cos 2\psi)}$$

Twiss changes in one-pass accelerators

$$\Delta\psi = \Delta k l_j \beta_j \sin^2(\psi - \psi_j)$$

→ More focusing always increases the tune

$$\frac{\Delta\beta}{\beta} = -\Delta k l_j \beta_j \sin(2[\psi - \psi_j])$$

→ Beta beat oscillates twice as fast as orbit.

Notice the self consistency: $\Delta\psi = \int_0^s \left(\frac{1}{\hat\beta + \Delta\hat\beta} - \frac{1}{\hat\beta}\right) d\hat s = -\int_0^s \frac{\Delta\hat\beta}{\hat\beta^2} d\hat s = -\int_0^s \frac{\Delta\hat\beta}{\hat\beta} d\hat\psi$

$$\Delta\alpha = -\frac{\Delta\beta'}{2} = \Delta k l_j \beta_j [\cos(2[\psi - \psi_j]) - \alpha \sin(2[\psi - \psi_j])]$$

$$\begin{pmatrix} \frac{\Delta\beta}{\beta} \\ \frac{\beta\Delta\alpha - \alpha\Delta\beta}{\beta} \end{pmatrix} = \Delta k l_j \, \beta_j \begin{pmatrix} \sin(2[\psi - \psi_j]) \\ \cos(2[\psi - \psi_j]) \end{pmatrix}$$

$$\left(\Delta k l_j \beta_j\right)^2 = [\left(\frac{\Delta\beta}{\beta}\right)^2 + \left(\Delta\alpha - \alpha\frac{\Delta\beta}{\beta}\right)^2]$$

Twiss correction in one-pass accelerators

$$\frac{\Delta\beta}{\beta} = -\sum_j \Delta k l_j \beta_j \sin(2[\psi - \psi_j]) \qquad \Delta\psi = \sum_j \Delta k l_j \beta_j \tfrac{1}{2}[1 - \cos(2[\psi - \psi_j])]$$

When beta functions and betatron phases have been measured at many places, quadrupoles can be changed with these formulas to correct the Twiss errors.

Quadruple errors and the tune

$$\cos(\mu+\Delta\mu)=\tfrac{1}{2}\mathrm{Tr}[M_0(s_j)+\Delta M_0(s_j)]\approx\cos\mu-\Delta\mu\sin\mu$$

$$=\tfrac{1}{2}\mathrm{Tr}[\begin{pmatrix}1 & 0\\ -\Delta kl_j & 1\end{pmatrix}\begin{pmatrix}\cos\mu+\alpha_j\sin\mu & \beta_j\sin\mu\\ -\gamma_j\sin\mu & \cos\mu-\alpha_j\sin\mu\end{pmatrix}]$$

$$=\cos\mu-\tfrac{1}{2}\Delta kl_j\beta_j\sin\mu$$

$$\boxed{\Delta\mu=\tfrac{1}{2}\Delta kl_j\beta_j}$$

Oscillation frequencies can be measured relatively easily and accurately.

Measurement of beta function: Change k and measure tune.

Quadruple errors and periodic beta function

One pass accelerators:

Periodic accelerators:

$$\Delta x(s) = \Delta\vartheta\sqrt{\beta\hat{\beta}}\sin(\psi - \hat{\psi})$$

$$\Delta x_{co}(s) = \frac{\Delta\vartheta\sqrt{\beta\hat{\beta}}}{\sin(\frac{\mu}{2})}\cos(|\psi - \hat{\psi} - \frac{\mu}{2})$$

$$\frac{\Delta\beta}{\beta} = -\Delta k l \hat{\beta}\sin(2(\psi - \hat{\psi}))$$

$$\frac{\Delta\beta}{\beta} = -\frac{\Delta k l \hat{\beta}}{2\sin(\mu)}\cos(2|\psi - \hat{\psi}| - \mu)$$

Energy dependent Twiss parameters

Natural Chromaticity: $\xi_x = \frac{1}{2\pi}\partial\mu_x/\partial\delta\ ,\ \xi_y = \frac{1}{2\pi}\partial\mu_y/\partial\delta$

$$\Delta\mu_x = \frac{1}{2}\Delta k l\hat{\beta}_x \longrightarrow \partial\mu_x/\partial\delta = -\frac{1}{2}\int_0^L k(s)\beta_x(s)\,ds$$

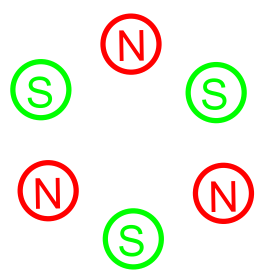
$$\Delta\mu_y = \frac{1}{2}\Delta k l\hat{\beta}_y \longrightarrow \partial\mu_y/\partial\delta = \frac{1}{2}\int_0^L k(s)\beta_y(s)\,ds$$

The periodic beta functions will similarly depend on energy.

Sextupoles (revisited)

$$\psi = \Psi_3 \,\mathrm{Im}\{(x-iy)^3\} = \Psi_3 \cdot (y^3 - 3x^2 y) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_3\, 3\begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

C_3 Symmetry



i) Sextupole fields hardly influence the particles close to the center, where one can linearize in x and y.

ii) In linear approximation a by Δx shifted sextupole has a quadrupole field.

iii) When Δx depends on the energy, one can build an energy dependent quadrupole.

$$\vec{B} = -\vec{\nabla}\psi = \Psi_3\, 3\begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

$$x \mapsto \Delta x + x$$

$$\vec{B} \approx \Psi_3\, 3\begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix} + 6\Psi_3 \Delta x \begin{pmatrix} y \\ x \end{pmatrix} + O(\Delta x^2)$$

$$k_2 = \frac{q}{p} 3!\Psi_3 \Rightarrow k_1 = k_2 \Delta x$$

Chromaticity and its correction

Chromaticity ξ = energy dependence of the tune

$$\upsilon(\delta) = \upsilon + \tfrac{\partial \upsilon}{\partial \delta}\delta + \dots$$

$$\xi = \tfrac{\partial \upsilon}{\partial \delta} \quad \text{with} \quad \upsilon = \tfrac{\mu}{2\pi}$$

Natural chromaticity ξ_0 = energy dependence of the tune due to quadrupoles only

$$\xi_{x0} = -\frac{1}{4\pi}\oint \beta_x(\hat{s})k_1(\hat{s})d\hat{s} \qquad \xi_{y0} = \frac{1}{4\pi}\oint \beta_y(\hat{s})k_1(\hat{s})d\hat{s}$$

Particles with energy difference oscillate around the periodic dispersion leading to a quadrupole effect in sextupoles that also shifts the tune:

$$\xi_x = \frac{1}{4\pi}\oint \beta_x(-k_1 + \eta_x k_2)d\hat{s} \qquad \xi_y = \frac{1}{4\pi}\oint \beta_y(k_1 - \eta_x k_2)d\hat{s}$$

Typically the the chormaticity ξ is chosen to be slightly positive, between 0 and 3.

Chromatic beta beat minimization

Chromatic beta beat is mostly created by the strongest quadrupoles, it can be influenced by sextupoles, under the provision that these on average still correct the chromaticity.

$$\frac{d\beta}{d\delta} = \beta(k_1 l - k_2 l \cdot \eta)\hat{\beta}\sin(2|\psi - \hat{\psi}| - \mu)$$

Periodic accelerators:

$$\frac{d\beta}{d\delta} = \frac{\beta}{2\sin\mu}(k_1 l - k_2 l \cdot \eta)\hat{\beta}\cos(2|\psi - \hat{\psi}| - \mu)$$

Sextupoles are used to compensate the chromaticity.

Several sextupoles are used to have their average compensate the chromaticity but have their regional variation compensate the chromatic beta beat on average and at critical sections, e.g. interaction points.

Single resonance model

$$x'' = -K\,x + \Delta f_x(x,y,s)$$

$$\tfrac{d}{d\vartheta} J = \sum_{n,m=-\infty}^{\infty} m H_{nm}(J)\sin(n\vartheta + m\varphi + \Psi_{nm}(J))$$

$$\tfrac{d}{d\vartheta}\varphi = \upsilon + \partial_J \sum_{n,m=-\infty}^{\infty} H_{nm}(J)\cos(n\vartheta + m\varphi + \Psi_{nm}(J))$$

Strong deviation from: $J = J_0\ ,\quad \varphi = \upsilon\vartheta + \varphi_0$

Occur when there is coherence between the perturbation and the phase space rotation: $n + m\tfrac{d}{ds}\varphi \approx 0$

Resonance condition: tune is rational $\boxed{n + m\ \upsilon = 0}$

On resonance the integral would increases indefinitely !

Neglecting all but the most important term

$$H(\varphi, J, \vartheta) \approx \upsilon J + H_{00}(J) + H_{nm}(J)\cos(n\vartheta + m\varphi + \Psi_{nm}(J))$$

Sum and difference resonances

$n + m_x \upsilon_x + m_y \upsilon_y \approx 0$ means that oscillations in y can drive oscillations in x in

$$x'' = -K\,x + \Delta f_x(x,y,s)$$

The resonance term in the Hamiltonian then changes only slowly:

$$H(\vec{\varphi},\vec{J},\vartheta) = \vec{\upsilon}\cdot\vec{J} + H_{00}(\vec{J}) + H_{n\vec{m}}(\vec{J})\cos(n\vartheta + \vec{m}\cdot\vec{\varphi} + \Psi_{n\vec{m}}(\vec{J}))$$

$$\tfrac{d}{d\vartheta}\vec{\varphi} = \vec{\partial}_J H \quad , \quad \tfrac{d}{d\vartheta}\vec{J} = -\vec{\partial}_\varphi H$$

$$J = \vec{m}\cdot\vec{J}$$

$$J_\perp = m_x J_y - m_y J_x = \vec{m}\times\vec{J} \quad \Rightarrow \quad \tfrac{d}{d\vartheta} J_\perp = 0$$

Difference resonances lead to stable motion since:

$$n + |m_x|\upsilon_x - |m_y|\upsilon_y \approx 0 \Rightarrow \quad |m_y|\,J_x + |m_x|\,J_y = \text{const}$$

Sum resonances lead to unstable motion since:

$$n + |m_x|\upsilon_x + |m_y|\upsilon_y \approx 0 \Rightarrow \quad |m_y|\,J_x - |m_x|\,J_y = \text{const}$$

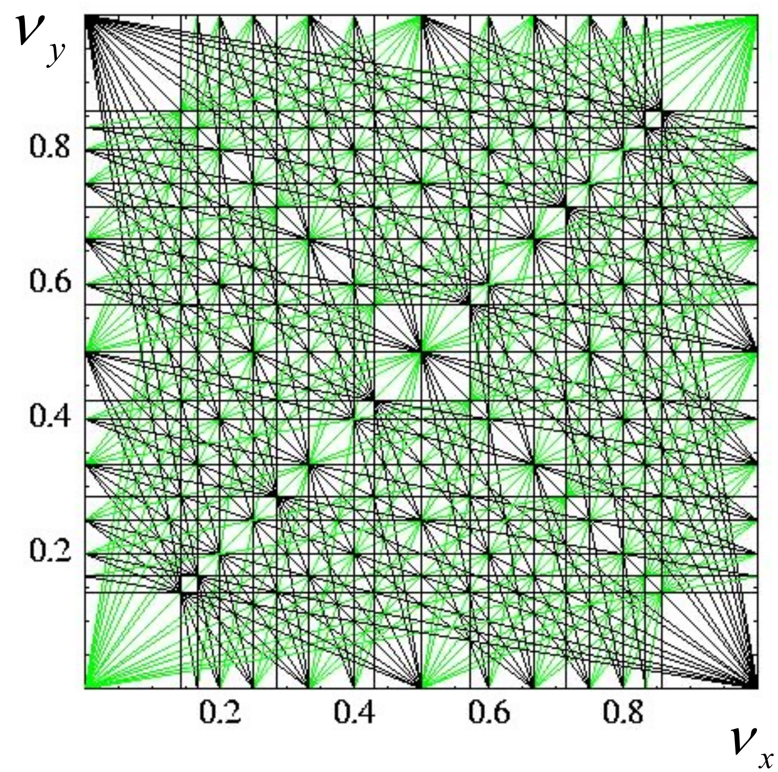
Common reasons for working points

(1) Avoid resonances $n+m_x\upsilon_x+m_y\upsilon_y\approx 0$

(2) +/- Colliders: be above a half integer to squeze the beam size $\frac{\Delta\beta}{\beta}=-\frac{\Delta k\beta}{2\tan\mu}$

(3) Polarized beams: close to integer

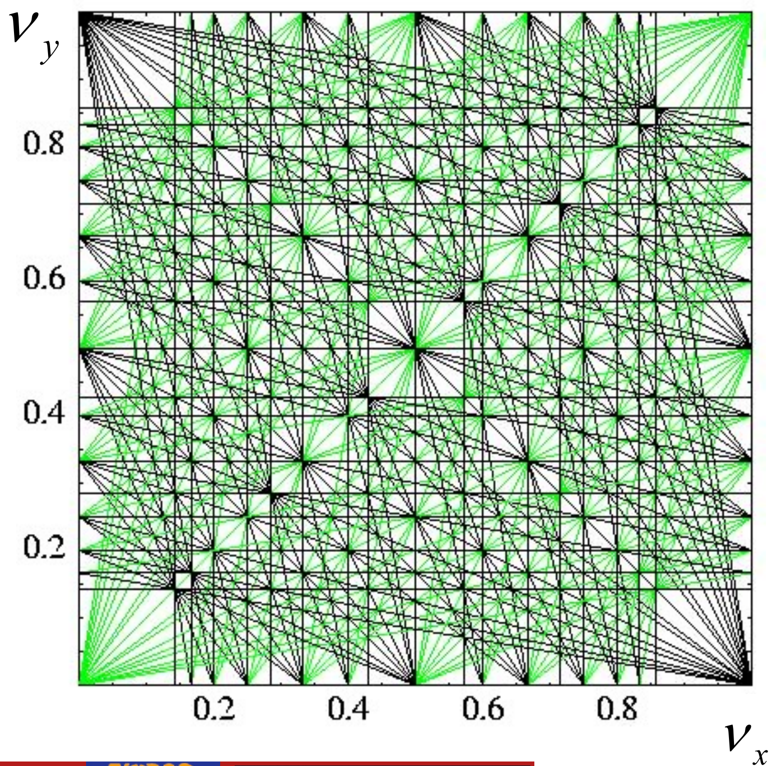
(4) Where the loss rate is smallest



Resonance diagram and choosing the tune

$n + m_x \upsilon_x + m_y \upsilon_y \approx 0$ means that oscillations in y can drive oscillations in x in

$$x'' = -K\, x + \Delta f_x(x, y, s)$$



All these resonances have to be avoided by their respective resonance width.

The position of an accelerator in the tune plane s called its Working Point.

Perturbations

$$\begin{pmatrix} x' \\ a' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix}\begin{pmatrix} x \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta f \end{pmatrix}$$

$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi+\phi_0) \\ \cos(\psi+\phi_0) \end{pmatrix} = \sqrt{2J}\,\underline{\beta}\,\vec{S}$$

This would be a solution with constant J and ϕ when $\Delta f=0$.

Variation of constants:

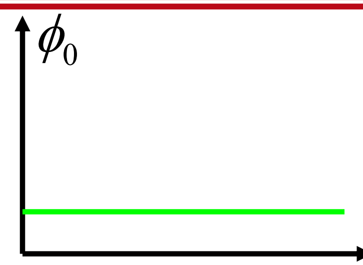
$$\frac{J'}{\sqrt{2J}}\underline{\beta}\,\vec{S} + \sqrt{2J}\,\phi_0'\begin{pmatrix} 0 & \sqrt{\beta} \\ -\frac{1}{\sqrt{\beta}} & -\frac{\alpha}{\sqrt{\beta}} \end{pmatrix}\vec{S} = \begin{pmatrix} 0 \\ \Delta f \end{pmatrix}$$

$$\frac{J'}{\sqrt{2J}}\vec{S} + \sqrt{2J}\,\phi_0'\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\vec{S} = \underline{\beta}^{-1}\begin{pmatrix} 0 \\ \Delta f \end{pmatrix} \quad \text{with} \quad \underline{\beta}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

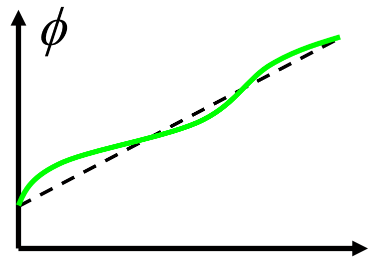
$$\frac{J'}{\sqrt{2J}} = \cos(\psi+\phi_0)\sqrt{\beta}\,\Delta f \quad , \quad \sqrt{2J}\,\phi_0' = -\sin(\psi+\phi_0)\sqrt{\beta}\,\Delta f$$

Simplification of linear motion

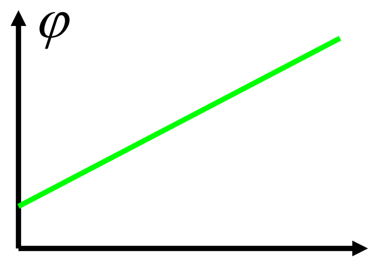
$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi + \phi_0) \\ \cos(\psi + \phi_0) \end{pmatrix} \Rightarrow \begin{matrix} J' = 0 \\ \phi_0' = 0 \end{matrix}$$



$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix} \Rightarrow \begin{matrix} J' = 0 \\ \phi' = \frac{1}{\beta} \end{matrix}$$



$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi - \mu\frac{s}{L} + \varphi) \\ \cos(\psi - \mu\frac{s}{L} + \varphi) \end{pmatrix} \Rightarrow \begin{matrix} J' = 0 \\ \varphi' = \mu\frac{1}{L} \end{matrix}$$



$$\tilde{\psi} = \psi - \mu\frac{s}{L} \Rightarrow \tilde{\psi}(s+L) = \tilde{\psi}(s)$$

Corresponds to Floquet's Theorem

Quasi-periodic perturbation

$$J' = \cos(\psi + \phi_0)\sqrt{2J\beta}\Delta f \quad , \quad \phi_0' = -\sin(\psi + \phi_0)\sqrt{\tfrac{\beta}{2J}}\Delta f$$

$$\tilde{\psi} = \psi - \mu\tfrac{s}{L} \; , \; \varphi = \mu\tfrac{s}{L} + \phi_0$$

$$J' = \cos(\tilde{\psi} + \varphi)\sqrt{2J\beta}\Delta f \quad , \quad \varphi' = \mu\tfrac{1}{L} - \sin(\tilde{\psi} + \varphi)\sqrt{\tfrac{\beta}{2J}}\Delta f$$

New independent variable $\vartheta = 2\pi\frac{s}{L}$

$$\tfrac{d}{d\vartheta}J = \cos(\tilde{\psi} + \varphi)\sqrt{2J\beta}\Delta f\,\tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi = \upsilon - \sin(\tilde{\psi} + \varphi)\sqrt{\tfrac{\beta}{2J}}\Delta f\,\tfrac{L}{2\pi}$$

$$\Delta f(x) = \Delta f(\sqrt{2J\beta}\sin(\tilde{\psi} + \varphi))$$

The perturbations are 2π periodic in ϑ and in φ

φ is approximately $\varphi \approx \upsilon \cdot \vartheta$

For irrational ν, the perturbations are quasi-periodic.

Tune shift with amplitude

$$\tfrac{d}{d\vartheta}J = \cos(\tilde{\psi}+\varphi)\sqrt{2J\beta}\,\Delta f\,\tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi = \upsilon - \sin(\tilde{\psi}+\varphi)\sqrt{\tfrac{\beta}{2J}}\,\Delta f\,\tfrac{L}{2\pi}$$

$$\tfrac{d}{d\vartheta}\varphi = \partial_J H \quad , \quad \tfrac{d}{d\vartheta}J = -\partial_\phi H \quad , \quad H(\varphi,J,\vartheta) = \upsilon\; J - \tfrac{L}{2\pi}\int_0^x \Delta f(\hat{x},s)\,d\hat{x}$$

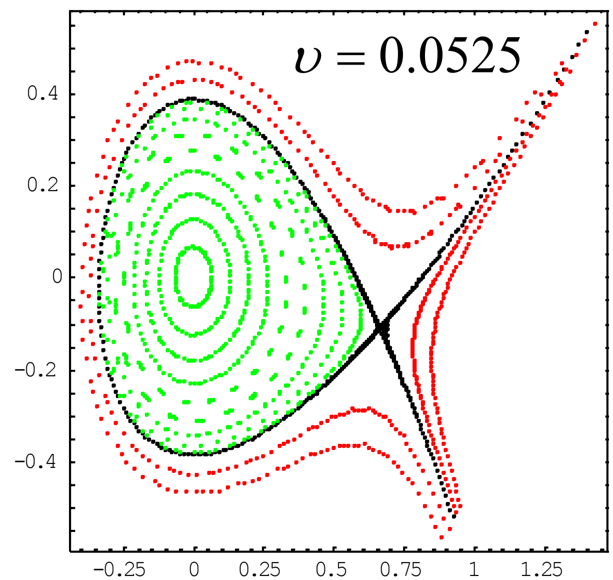
The motion remains Hamiltonian in the perturbed coordinates !

If there is a part in $\partial_J H$ that does not depend on φ, s $\Rightarrow$ Tune shift

The effect of other terms tends to average out.

$$\varphi(\vartheta) - \varphi_0 \approx \vartheta \cdot \partial_J \langle H \rangle_{\varphi,\vartheta}(J)$$

$$\upsilon(J) = \upsilon + \partial_J \langle \Delta H \rangle_{\varphi,\vartheta}(J)$$



Shift examples

$$H(\varphi,J)=\upsilon\cdot J-\tfrac{L}{2\pi}\int_0^x \Delta f(\hat{x},s)\,d\hat{x} \quad , \quad \Delta\upsilon(J)=\partial_J\langle\Delta H\rangle_{\varphi,\vartheta}$$

Quadrupole: $\Delta f=-\Delta k\,x$

$$\Delta H=\tfrac{L}{2\pi}\Delta k\,\tfrac{1}{2}x^2=\tfrac{L}{2\pi}\Delta k\,J\beta\sin^2(\tilde{\psi}+\varphi)$$

$$\langle\Delta H\rangle_{\varphi,\vartheta}=\tfrac{1}{2\pi}\int_0^{2\pi}\Delta k\beta\,d\vartheta\,L\frac{J}{4\pi}=\int_0^L\Delta k\beta\,ds\,\frac{J}{4\pi}\Rightarrow \Delta\upsilon=\tfrac{1}{4\pi}\oint\Delta k\beta\,ds$$

Sextupole: $\Delta f=-k_2\tfrac{1}{2}x^2$

$$\Delta H=\tfrac{L}{2\pi}k_2\tfrac{1}{3!}x^3=\tfrac{L}{2\pi}k_2\tfrac{1}{3!}\sqrt{2J\beta}^{\,3}\sin^3(\tilde{\psi}+\varphi)$$

$$\langle\Delta H\rangle_{\varphi,\vartheta}=0 \quad\Rightarrow\quad \Delta\upsilon=0$$

Octupole: $\Delta f=-k_3\tfrac{1}{3!}x^3$

$$\Delta H=\tfrac{L}{2\pi}k_3\tfrac{1}{4!}x^4=\tfrac{L}{2\pi}k_3\tfrac{1}{3!}(J\beta)^2\sin^4(\tilde{\psi}+\varphi)$$

$$\langle\Delta H\rangle_{\varphi,\vartheta}=\tfrac{J^2}{3!2\pi}\oint k_3\beta^2 ds\left\langle\tfrac{1}{2^4}(e^{i\varphi}-e^{-i\varphi})^4\right\rangle_{\varphi}\Rightarrow\Delta\upsilon=J\tfrac{1}{16\pi}\oint k_3\beta^2 ds$$

Nonlinear resonances

$$\tfrac{d}{d\vartheta}J = \cos(\tilde{\psi}+\varphi)\sqrt{2J\beta}\Delta f\,\tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi = \upsilon - \sin(\tilde{\psi}+\varphi)\sqrt{\tfrac{\beta}{2J}}\Delta f\,\tfrac{L}{2\pi}$$

$$\tfrac{d}{d\vartheta}\varphi = \partial_J H \quad , \quad \tfrac{d}{d\vartheta}J = -\partial_\phi H \quad , \quad H(\varphi, J, \vartheta) = \upsilon \cdot J - \tfrac{L}{2\pi}\int_0^x \Delta f(\hat{x}, s)\,d\hat{x}$$

The effect of the perturbation is especially strong when

$\cos(\tilde{\psi}+\varphi)\sqrt{\beta}\Delta f$ or $\sin(\tilde{\psi}+\varphi)\sqrt{\beta}\Delta f$

has contributions that hardly change, i.e. the change of

$\sqrt{\beta(\vartheta)}\Delta f(x(\vartheta),\vartheta)$ is in resonance with the rotation angle $\varphi(\vartheta)$.

Periodicity allows Fourier expansion:

$$H(\varphi, J, \vartheta) = \sum_{n,m=-\infty}^{\infty} \widehat{H}_{nm}(J)\,e^{i[n\vartheta + m\varphi]} = \sum_{n,m=-\infty}^{\infty} H_{nm}(J)\cos(n\vartheta + m\varphi + \Psi_{nm}(J))$$

$H_{00}(J) = \langle H(\varphi, J, s)\rangle_{\varphi,s} \Rightarrow$ Tune shift

Choosing: $\Psi_{00}(J) = 0$

Nonlinear motion

Sextupoles cause nonlinear dynamics, which can be chaotic and unstable.

$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \underline{M}_0 \left[\begin{pmatrix} x_n \\ x'_n \end{pmatrix} - \frac{k_2 l_s}{2} \begin{pmatrix} 0 \\ x_n{}^2 \end{pmatrix} \right] \qquad \begin{pmatrix} x_n \\ x'_n \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \hat{x}_n \\ \hat{x}'_n \end{pmatrix}$$

$$\begin{pmatrix} \hat{x}_{n+1} \\ \hat{x}'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \left[\begin{pmatrix} \hat{x}_n \\ \hat{x}'_n \end{pmatrix} - \frac{k_2 l_s}{2} \sqrt{\beta} \begin{pmatrix} 0 \\ \beta\, \hat{x}_n^2 \end{pmatrix} \right]$$

$$\begin{pmatrix} \hat{x}_f \\ \hat{x}'_f \end{pmatrix} = \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \begin{pmatrix} 1-\cos\mu & \sin\mu \\ -\sin\mu & 1-\cos\mu \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \hat{x}_f^2 \end{pmatrix} = \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \frac{1}{2\sin\frac{\mu}{2}} \begin{pmatrix} -\cos\frac{\mu}{2} \\ \sin\frac{\mu}{2} \end{pmatrix} \hat{x}_f^2$$

$$\left.\begin{aligned} \hat{x}_f &= -\tfrac{4}{k_2 l_s} \beta^{-\frac{3}{2}} \tan\tfrac{\mu}{2} \\ \hat{x}'_f &= \tfrac{4}{k_2 l_s} \beta^{-\frac{3}{2}} \tan^2\tfrac{\mu}{2} \end{aligned}\right\} \hat{x} = \hat{x}_f + \Delta\hat{x} \qquad J_f = \tfrac{1}{2}(\hat{x}_f^2 + \hat{x}'^2_f) = \tfrac{1}{2\beta^3}\left(\tfrac{4}{k_2 l_s} \tfrac{\tan\frac{\mu}{2}}{\cos\frac{\mu}{2}}\right)^2$$

$$\begin{pmatrix} \Delta\hat{x}_{n+1} \\ \Delta\hat{x}'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \left[\begin{pmatrix} \Delta\hat{x}_n \\ \Delta\hat{x}'_n \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{k_2 l_s}{2} \beta^{\frac{3}{2}} \Delta\hat{x}_n^2 - 4\tan\frac{\mu}{2} \Delta\hat{x}_n \end{pmatrix} \right]$$

Dynamic aperture (e.g., close to integer tunes)

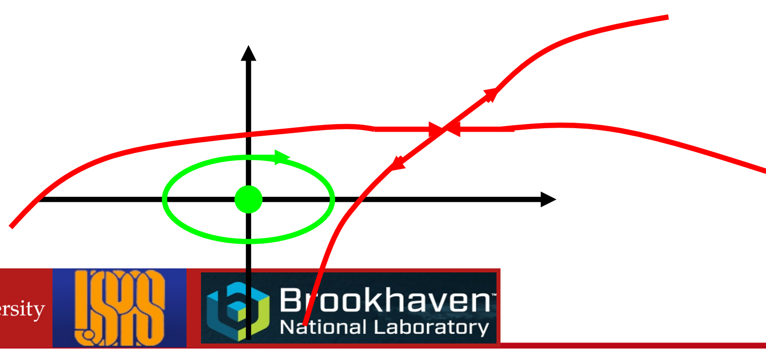
$$\begin{pmatrix}\Delta\hat{x}_{n+1}\\ \Delta\hat{x}'_{n+1}\end{pmatrix}=\begin{pmatrix}\cos\mu & \sin\mu\\ -\sin\mu & \cos\mu\end{pmatrix}\left[\begin{pmatrix}\Delta\hat{x}_n\\ \Delta\hat{x}'_n\end{pmatrix}-\begin{pmatrix}0\\ \frac{k_2 l_s}{2}\beta^{\frac{3}{2}}\Delta\hat{x}_n^2-4\tan\frac{\mu}{2}\Delta\hat{x}_n\end{pmatrix}\right]$$

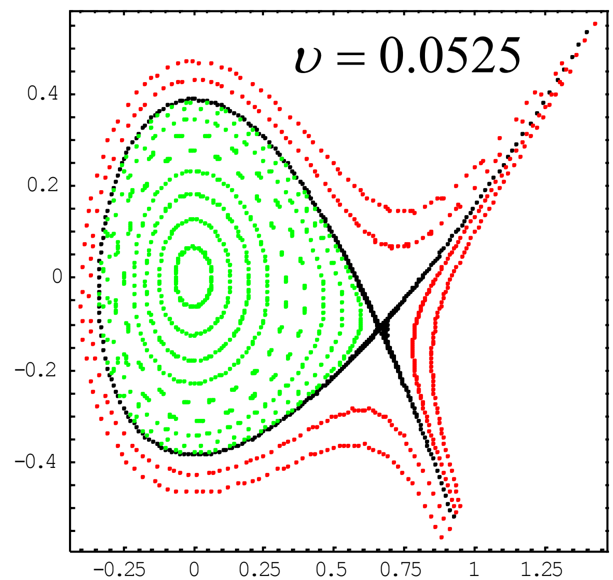
$$\begin{pmatrix}\Delta\hat{x}_{n+1}\\ \Delta\hat{x}'_{n+1}\end{pmatrix}=\begin{pmatrix}\cos\mu+4\sin\mu\tan\frac{\mu}{2} & \sin\mu\\ -\sin\mu+4\cos\mu\tan\frac{\mu}{2} & \cos\mu\end{pmatrix}\left[\begin{pmatrix}\Delta\hat{x}_n\\ \Delta\hat{x}'_n\end{pmatrix}-\begin{pmatrix}0\\ \frac{k_2 l_s}{2}\beta^{\frac{3}{2}}\Delta\hat{x}_n^2\end{pmatrix}\right]$$

Example of one sextupole:

$$\tfrac{1}{2}Tr[\underline{M}]=1-2\sin^2\tfrac{\mu}{2}+4\sin^2\tfrac{\mu}{2}=1+2\sin^2\tfrac{\mu}{2}\geq 1$$

The additional fixed point is unstable !



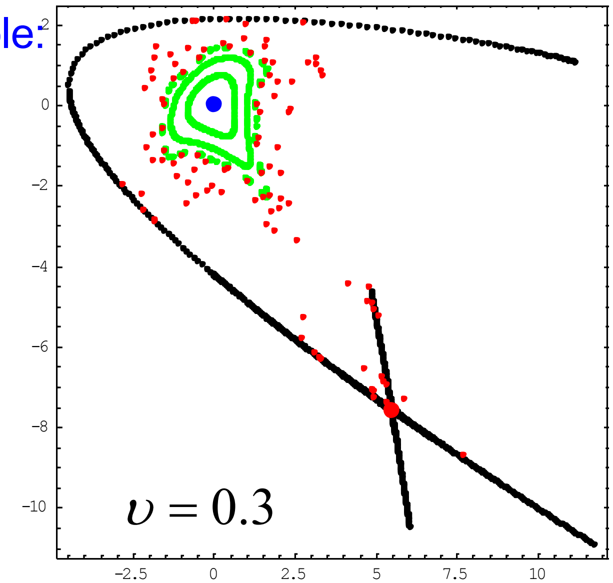


Sextupole Aperture

If the chormaticity is corrected by a single sextupole:

$$\xi_x = \xi_{0x} + \frac{1}{4\pi}\beta_x\eta_x k_2 l \approx 0$$

$$J_f = \frac{1}{2\beta^3}\left(\frac{4}{k_2 l_s}\frac{\tan\frac{\mu}{2}}{\cos\frac{\mu}{2}}\right)^2 \approx \frac{1}{2\beta}\left(\frac{\eta}{\xi_0\pi}\frac{\sin\frac{\mu}{2}}{\cos^2\frac{\mu}{2}}\right)^2$$



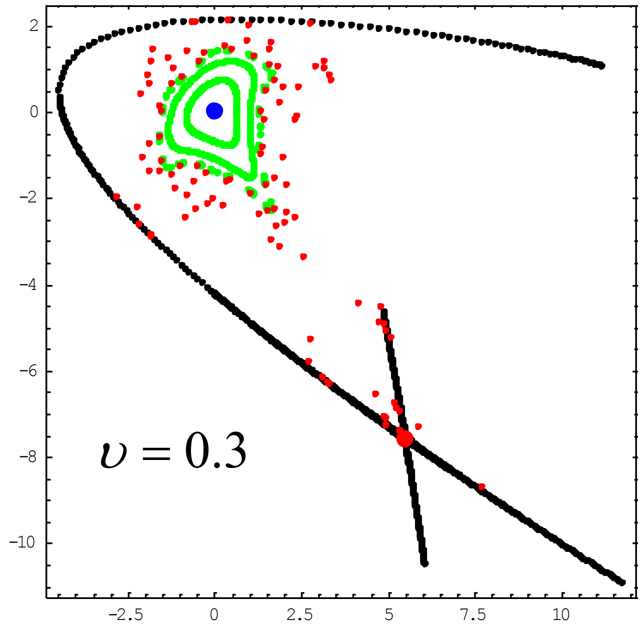
Often the dynamic aperture is much smaller than the fixed point indicates !

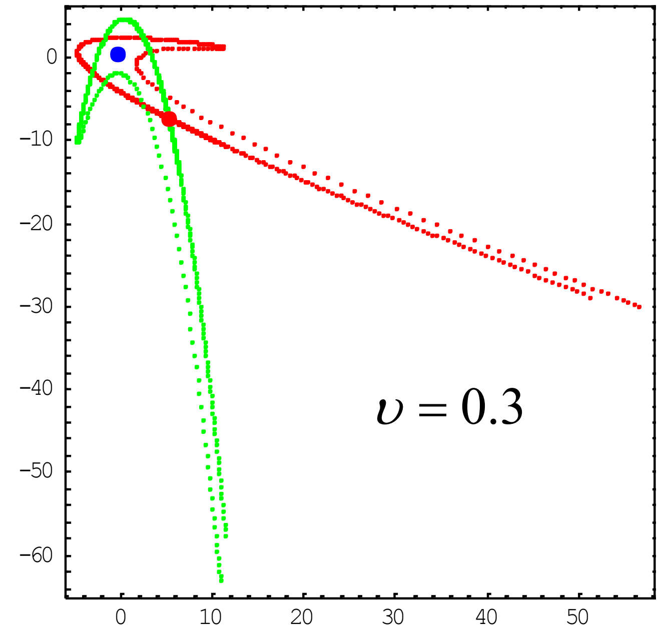
When many sextupoles are used: $$\xi_{0x} + \frac{N}{4\pi}\beta_x\eta_x k_2 l \approx 0$$

The sum of all k_2^2 is then reduced to about $\sum (k_2 l\beta)^2 \approx N(k_2 l\beta)^2 \approx \frac{1}{N}\left(\frac{4\pi}{\eta}\xi_0\right)^2$

The dynamic aperture is therefore greatly increased when distributed sextupoles are used.

Sextupole extraction



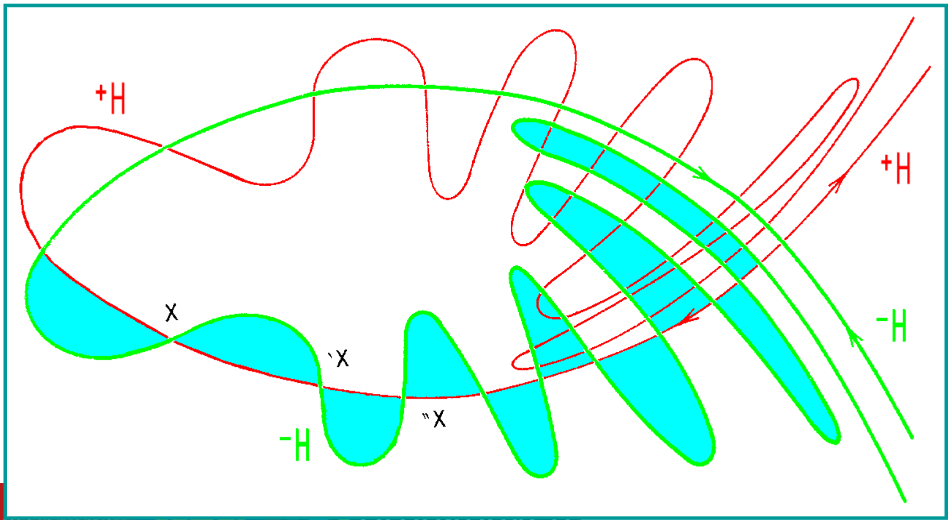


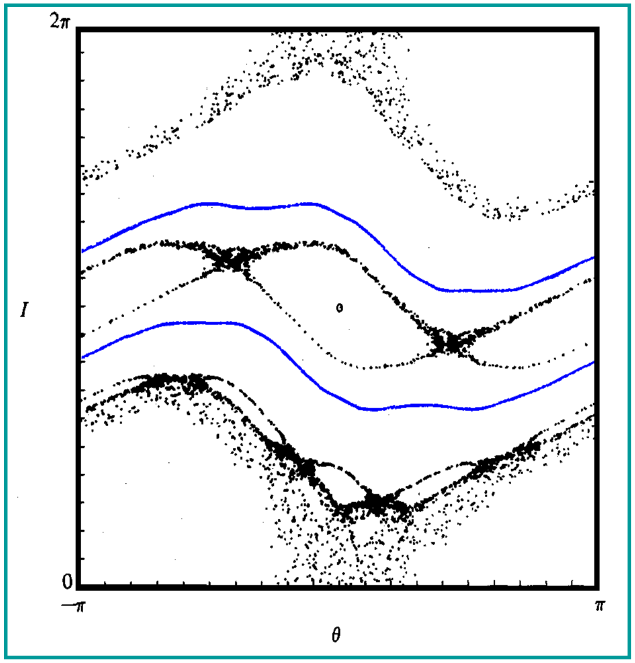
Due to the narrow region of unstable trajectories, sextupoles are used for slow particle extraction at a tune of 1/3.

The intersection of stable and unstable manifolds is a certain indication of chaos.

Homoclinic points

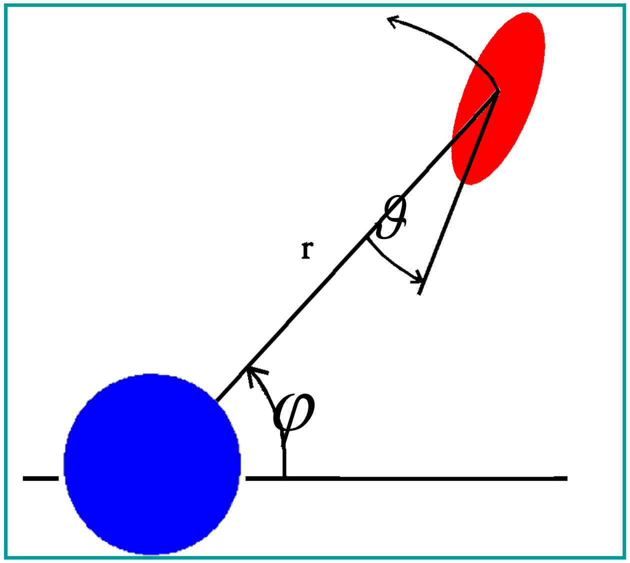
- At instable fixed points, there is a stable and an instabile invariant curve.
- Intersections of these curves (homoclinic points) lead to chaos.

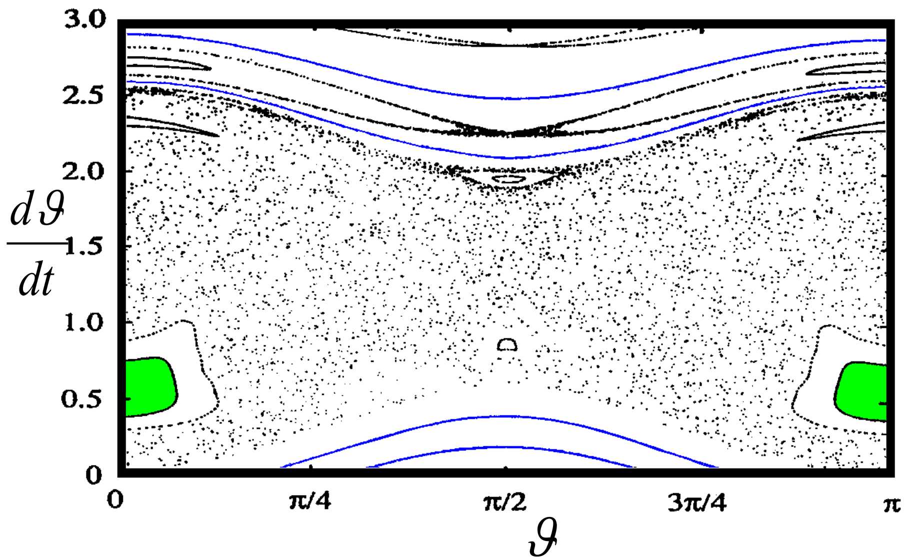




Hyperion: rotation around the vertical

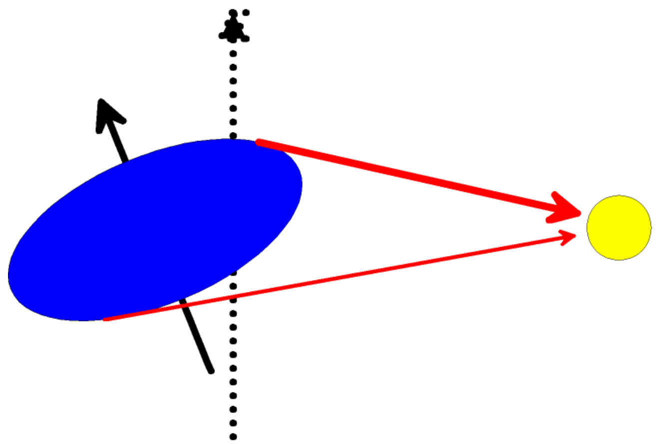
$$\frac{d^2(\vartheta+\varphi(t))}{dt^2} = -\alpha\left(\frac{a}{r(t)}\right)^3 \sin 2\vartheta$$

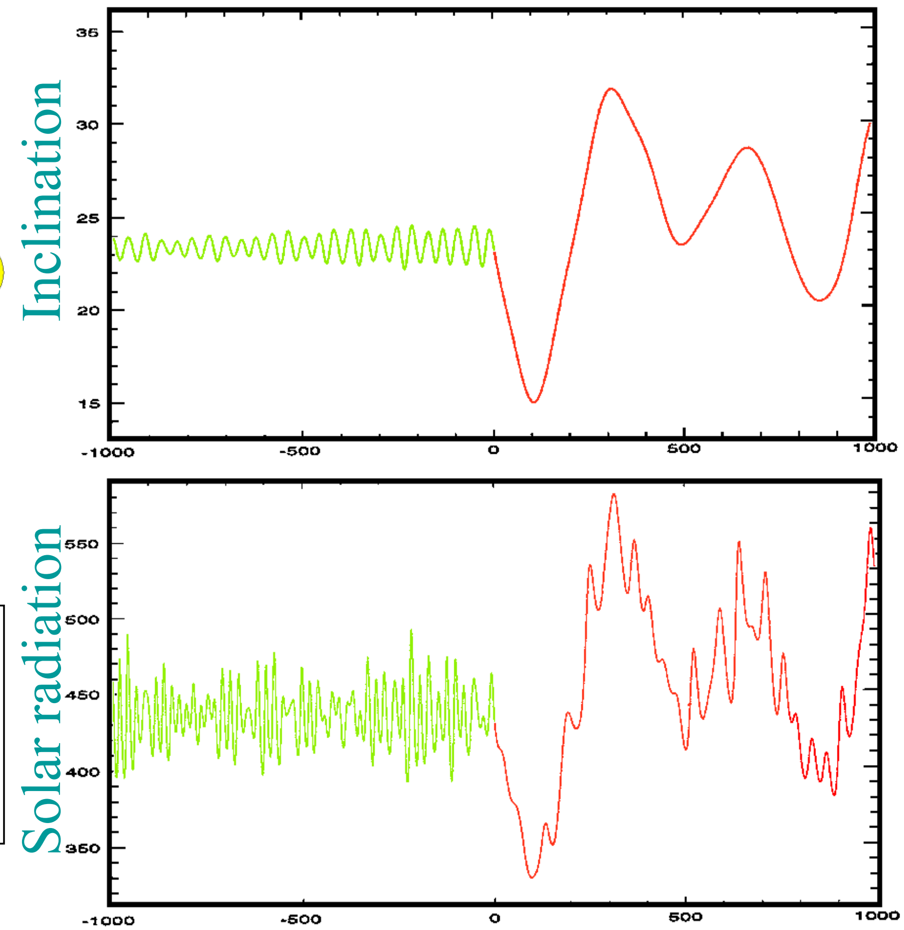




- On the path from Rotation to Libration around the Spin-Orbit-Coupling is a strong chaotic region.

Tilt of the earth





-1Ma with and without moon 1Ma

- Tidal forces from moon and sun cause a stabilization of the rotation axis.

Single resonance model

$$\frac{d}{d\vartheta} J = \sum_{n,m=-\infty}^{\infty} m H_{nm}(J) \sin(n\vartheta + m\varphi + \Psi_{nm}(J))$$

$$\frac{d}{d\vartheta} \varphi = \upsilon + \partial_J \sum_{n,m=-\infty}^{\infty} H_{nm}(J) \cos(n\vartheta + m\varphi + \Psi_{nm}(J))$$

Strong deviation from: $J = J_0 \,, \quad \varphi = \upsilon\vartheta + \varphi_0$

Occur when there is coherence between the perturbation and the phase space rotation: $n + m \frac{d}{ds}\varphi \approx 0$

Resonance condition: tune is rational $\boxed{n + m\,\upsilon = 0}$

On resonance the integral would increases indefinitely !

Neglecting all but the most important term

$$H(\varphi, J, \vartheta) \approx \upsilon J + H_{00}(J) + H_{nm}(J) \cos(n\vartheta + m\varphi + \Psi_{nm}(J))$$

Fixed points

$$\frac{d}{d\vartheta} J = mH_{nm}(J)\sin(n\vartheta + m\varphi + \Psi_{nm}(J))$$

$$\frac{d}{d\vartheta} \varphi = \upsilon + \Delta\upsilon(J) + \partial_J [H_{nm}(J)\cos(n\vartheta + m\varphi + \Psi_{nm}(J))]$$

$$\Phi = \tfrac{1}{m}[n\vartheta + m\varphi + \Psi_{nm}(J)] \ , \quad \delta = \upsilon + \tfrac{n}{m}$$

$$\frac{d}{d\vartheta} J = mH_{nm}(J)\sin(m\Phi) \ , \quad \frac{d}{d\vartheta}\Phi = \delta + \Delta\upsilon(J) + H'_{nm}(J)\cos(m\Phi)$$

$$H(\Phi, J, \vartheta) \approx \delta J + H_{00}(J) + H_{nm}(J)\cos(m\Phi)$$

Fixed points: $\frac{d}{d\vartheta} J = mH_{nm}(J_f)\sin(m\Phi_f) = 0 \quad \Rightarrow \quad \Phi_f = \frac{k}{m}\pi$

If $\delta + \Delta\upsilon(J_f) \pm H'_{nm}(J_f) = 0$ has a solution.

$$\frac{d}{d\vartheta}\Delta J = \pm m^2 H_{nm}(J_f)\Delta\Phi \ , \quad \frac{d}{d\vartheta}\Delta\Phi = [\Delta\upsilon'(J_f) \pm H''_{nm}(J_f)]\Delta J$$

Stable fixed point for: $H_{nm}(J_f)[H''_{nm}(J_f) \pm \Delta\upsilon'(J_f)] < 0$

Third integer resonances

Sextupole: $\Delta f = -k_2 \frac{1}{2} x^2$

$$\Delta H = \frac{L}{2\pi} k_2 \frac{1}{3!} x^3 = \frac{L}{2\pi} k_2 \frac{1}{3!} \sqrt{2J\beta}^3 \sin^3(\tilde{\psi} + \varphi)$$

$$= \frac{L}{2\pi} k_2 \frac{1}{3!4} \sqrt{2J\beta}^3 [\sin(3[\tilde{\psi} + \varphi]) + 3\sin(\tilde{\psi} + \varphi)]$$

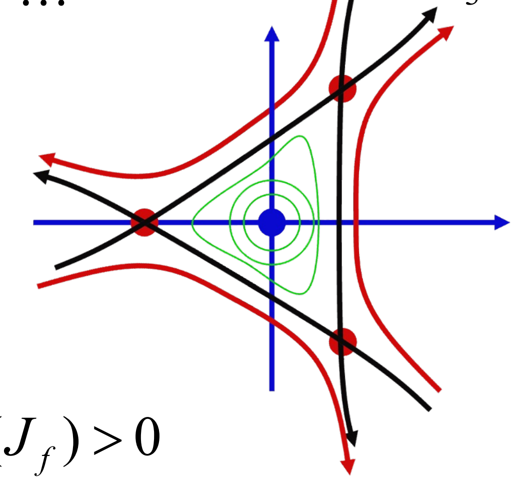
Simplification: one sextupole $k_2(\vartheta) = k_2 \delta(\vartheta) = k_2 \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \cos(n\vartheta)$

$$\Delta H = \frac{L}{2\pi} k_2 \frac{1}{3!4} \sqrt{2J\beta}^3 \frac{1}{2\pi} \cos(-n\vartheta + 3\varphi + 3\tilde{\psi} - \frac{\pi}{2}) + \dots$$

for $\upsilon \approx \frac{n}{3}$

$$\Delta H \approx A_2 \sqrt{J}^3 \cos(3\Phi)$$

$$\left.\begin{aligned} &\Phi_f = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \dots \\ &\delta \pm A_2 \tfrac{3}{2} \sqrt{J} = 0 \end{aligned}\right\} \begin{aligned} &\Phi_f = \tfrac{1}{3}\pi, \pi, \tfrac{5}{3}\pi \\ &\text{for } \delta > 0 \end{aligned}$$



All these fixed points are instable since $H_{nm}(J_f) H''_{nm}(J_f) > 0$

Fourth integer resonances

Octupole: $\Delta f = -k_3 \frac{1}{3!} x^3$, $\Delta H = \frac{L}{2\pi} k_3 \frac{1}{4!} x^4 = \frac{L}{2\pi} k_3 \frac{1}{3!} J^2 \beta^2 \sin^4(\tilde{\psi} + \varphi)$

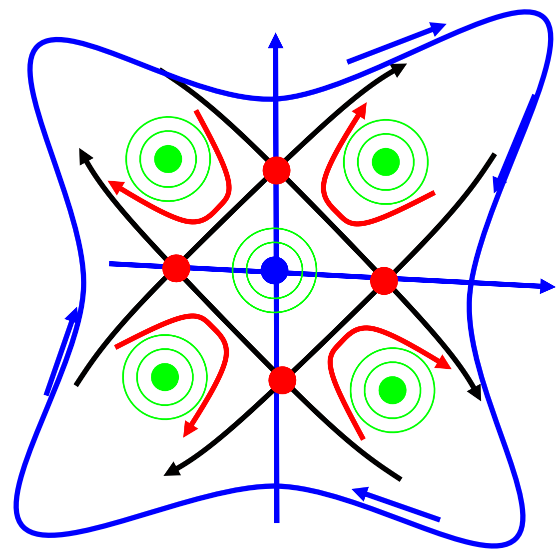
$$= \frac{L}{2\pi} k_3 \frac{1}{3!8} J^2 \beta^2 [\cos(4[\tilde{\psi} + \varphi]) - 4\cos(2[\tilde{\psi} + \varphi]) + 3]$$

Simplification: one octupole $k_3(\vartheta) = k_3 \delta(\vartheta) = k_3 \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \cos(n\vartheta)$

$\Delta H \approx A_3 J^2 [3 + \cos(4\Phi)]$ for $\upsilon \approx \frac{n}{4}$

$\Phi_f = 0, \frac{1}{4}\pi, \frac{2}{4}\pi, \ldots$ Either 8 fixed points: $\delta < 0$

$\delta + A_3 2J(3 \pm 1) = 0$ or none for: $\delta > 0$

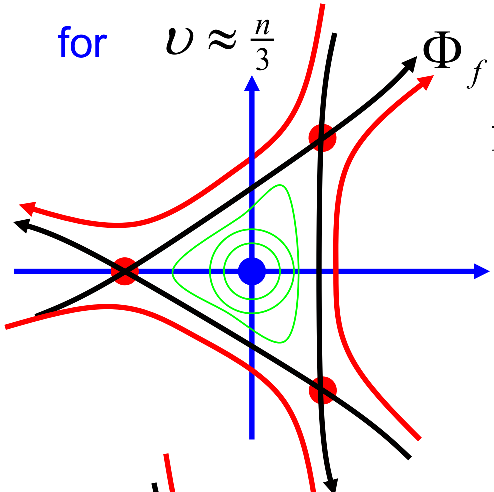


$$H_{nm}(J_f)[H''_{nm}(J_f) \pm \Delta\upsilon'(J_f)] < 0$$

Stability for $(2A_3 J)^2 [1 \pm 3] < 0$,
i.e. for the 4 outer fixed points.

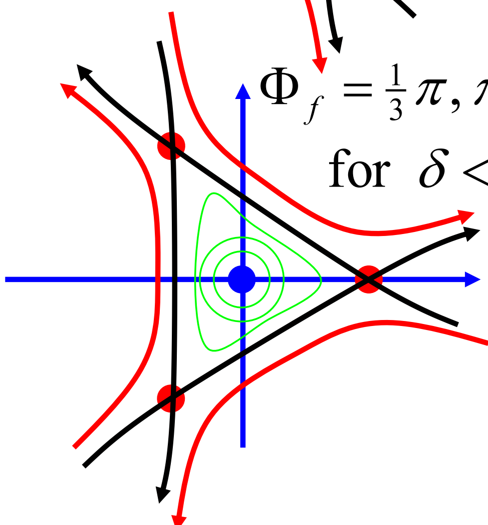
Particle motion in the single resonance model

for $\upsilon \approx \frac{n}{3}$



$\Phi_f = \frac{1}{3}\pi, \pi, \frac{5}{3}\pi$

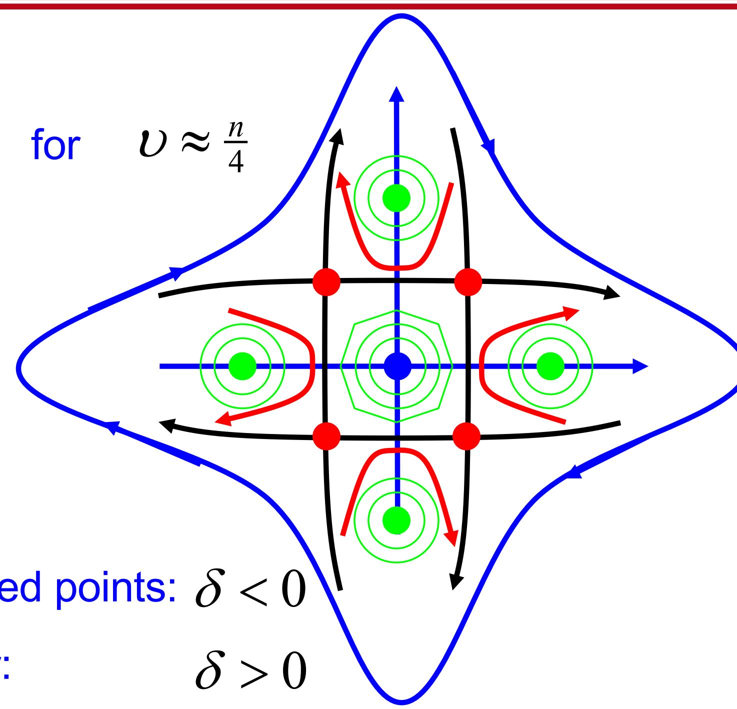
for $\delta > 0$



$\Phi_f = \frac{1}{3}\pi, \pi, \frac{5}{3}\pi$

for $\delta < 0$

for $\upsilon \approx \frac{n}{4}$



Either 8 fixed points: $\delta < 0$

or none for: $\delta > 0$

How can the motion inside the fixed points be simplified for a real accelerator ?

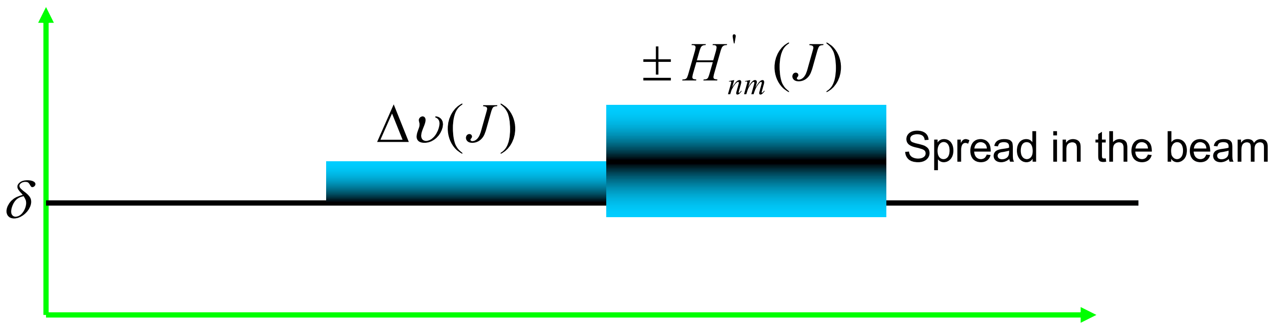
→ Normal From Theory

Resonance width (strength)

Fixed points: $\frac{d}{d\vartheta}J = mH_{nm}(J_f)\sin(m\Phi_f) = 0 \quad \Rightarrow \quad \Phi_f = \frac{k}{m}\pi$

If $\delta + \Delta\upsilon(J_f) \pm H'_{nm}(J_f) = 0$ has a solution.

δ has to avoid the region $\delta + \Delta\upsilon(J) \pm H'_{nm}(J) = 0$ for all particles.



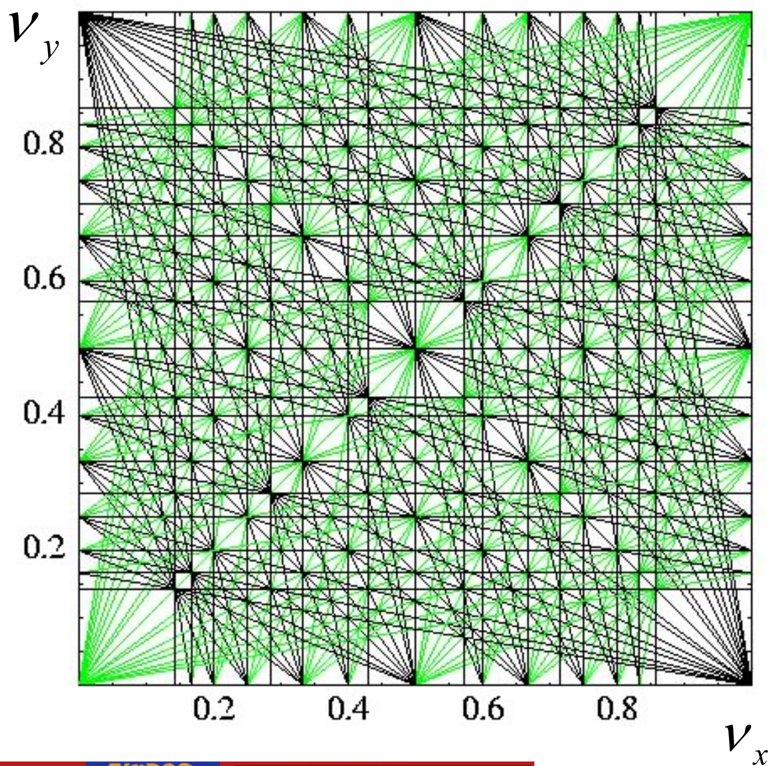
Assuming that the tune shift and perturbation are monotonous in J:

This tune region has the width $\Delta_{nm} = 2\,|\,H'_{nm}(J_{\max})|$ for strong resonances.

Δ_{nm} Is called Resonance Width, Resonance Strength, or Stop-Band Width

Resonance width around resonance lines

$n + m_x \upsilon_x + m_y \upsilon_y \approx 0$ means that oscillations in y can drive oscillations in x in



All these resonances have to be avoided by their respective resonance width.

The position of an accelerator in the tune plane s called its Working Point.

Coupling resonances

$$\tfrac{d}{d\vartheta}J_x = \cos(\tilde{\psi}_x+\varphi_x)\sqrt{2J_x\beta_x}\Delta f_x\tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi_x = \upsilon_x - \sin(\tilde{\psi}_x+\varphi_x)\sqrt{\tfrac{\beta_x}{2J_x}}\Delta f_x\tfrac{L}{2\pi}$$

$$\tfrac{d}{d\vartheta}J_y = \cos(\tilde{\psi}_y+\varphi_y)\sqrt{2J_y\beta_y}\Delta f_y\tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi_y = \upsilon_y - \sin(\tilde{\psi}_y+\varphi_y)\sqrt{\tfrac{\beta_y}{2J_y}}\Delta f_y\tfrac{L}{2\pi}$$

$$\tfrac{d}{d\vartheta}\vec{\varphi} = \vec{\partial}_J H \quad , \quad \tfrac{d}{d\vartheta}\vec{J} = -\vec{\partial}_\varphi H \quad , \quad H(\vec{\varphi},\vec{J},\vartheta) = \vec{\upsilon}\cdot\vec{J} - \tfrac{L}{2\pi}\int_0^{\vec{x}}\Delta\vec{f}(\hat{\vec{x}},s)d\hat{\vec{x}}$$

The integral form can be chosen since it is path independent. This is due to the Hamiltonian nature of the force: $\Delta f_{x,y}(x,y,s) = -\partial_{x,y}\Delta H(x,y,s)$

Single Resonance model for two dimensions means retaining only the amplitude dependent tune shift and one term in the two dimensional Fourier expansion:

$$H(\vec{\varphi},\vec{J},\vartheta) = \vec{\upsilon}\cdot\vec{J} + H_{00}(\vec{J}) + H_{n\vec{m}}(\vec{J})\cos(n\vartheta + m_x\varphi_x + m_y\varphi_y + \Psi_{n\vec{m}}(\vec{J}))$$

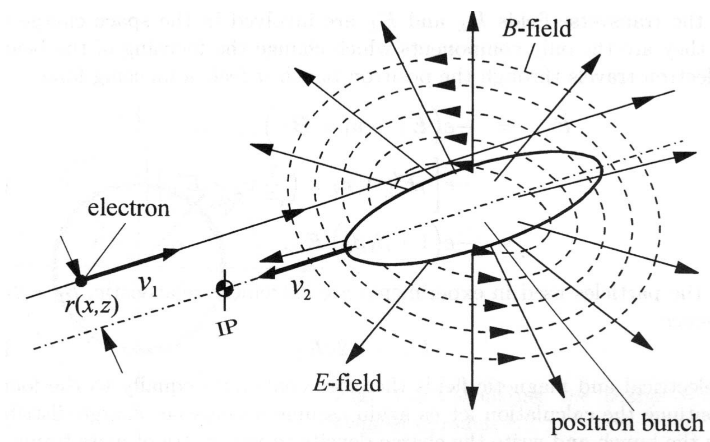
For $n + m_x\upsilon_x + m_y\upsilon_y \approx 0$

$$m_x\varphi_x + m_y\varphi_y = \vec{m}\cdot\vec{\varphi}$$

Beam-beam tune shift

The force that acts from one beam to the other during collisions is focusing or defocusing in both planes for small distances.

For large distances it is very nonlinear.



The effects of E and B forces add. Whereas they subtract for co-moving particles.

$$\Delta \nu_x^{(1)} = \frac{r_{cl}^{(1)} N_{cpb}^{(2)}}{2\pi} \frac{\beta_x^{(1)}}{\sigma_x^{(2)}(\sigma_x^{(2)} + \sigma_y^{(2)})}$$

Beam-beam force

$$\rho_{\text{lab}} = \frac{1}{2\pi\sigma_x\sigma_y} e^{-(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2})} \rho_{\text{lab}\,z}(z+v^{(2)}t) \qquad \rho_{\text{rest}} = \frac{1}{2\pi\sigma_x\sigma_y} e^{-(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2})} \rho_z(z)$$

$$\vec{j}_{\text{lab}} = \vec{\beta} c \rho_{\text{lab}\,z} \qquad \vec{j}_{\text{rest}} = 0$$

$$E_x(x,y,z) = \frac{Q\rho_z}{2\pi\varepsilon_0} \frac{1}{2\pi\sigma_x\sigma_y} \int \frac{x-x_0}{(x-x_0)^2+(y-y_0)^2} e^{-(\frac{x_0^2}{2\sigma_x^2}+\frac{y_0^2}{2\sigma_y^2})} dx_0 dy_0$$

$$\approx \frac{Q\rho_z}{2\pi\varepsilon_0} \frac{1}{2\pi\sigma_x\sigma_y} \int \frac{u}{u^2+v^2} e^{-(\frac{(x-u)^2}{2\sigma_x^2}+\frac{(y-v)^2}{2\sigma_y^2})} dudv$$

$$= \frac{Q\rho_z}{2\pi\varepsilon_0} \frac{1}{2\pi\sigma_x\sigma_y} (\sigma_x^2 \partial_x + x) \int \frac{1}{u^2+v^2} e^{-(\frac{(x-u)^2}{2\sigma_x^2}+\frac{(y-v)^2}{2\sigma_y^2})} dudv$$

$$= \frac{Q\rho_z}{2\pi\varepsilon_0} \frac{1}{2\pi\sigma_x\sigma_y} (\sigma_x^2 \partial_x + x) \int [\int_0^\infty e^{-t(u^2+v^2)} dt] e^{-(\frac{(x-u)^2}{2\sigma_x^2}+\frac{(y-v)^2}{2\sigma_y^2})} dudv$$

$$= \frac{Q\rho_z}{2\pi\varepsilon_0} \frac{1}{2\pi\sigma_x\sigma_y} (\sigma_x^2 \partial_x + x) \int e^{-(u^2(\frac{1}{2\sigma_x^2}+t)-\frac{2xu}{2\sigma_x^2}+v^2(\frac{1}{2\sigma_y^2}+t)-\frac{2yv}{2\sigma_y^2})} e^{-(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2})} dudvdt$$

Beam-beam force

$$E_x(x,y,z) \approx \frac{Q\rho_z}{4\pi\varepsilon_0}\frac{1}{\pi\sigma_x\sigma_y}(\sigma_x^2\partial_x + x)\int e^{-(u^2(\frac{1}{2\sigma_x^2}+t)-\frac{2xu}{2\sigma_x^2}+v^2(\frac{1}{2\sigma_y^2}+t)-\frac{2yv}{2\sigma_y^2})} e^{-(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2})} dudvdt$$

$$= \frac{Q\rho_z}{4\pi\varepsilon_0}\frac{1}{\pi\sigma_x\sigma_y}(\sigma_x^2\partial_x + x)\int e^{-(\tilde{u}^2(\frac{1}{2\sigma_x^2}+t)+\tilde{v}^2(\frac{1}{2\sigma_y^2}+t))} e^{-(\frac{x^2}{2\sigma_x^2}-\frac{x^2}{4\sigma_x^4(\frac{1}{2\sigma_x^2}+t)}+\frac{y^2}{2\sigma_y^2}-\frac{y^2}{4\sigma_y^4(\frac{1}{2\sigma_y^2}+t)})} dudvdt$$

$$= \frac{Q\rho_z}{4\pi\varepsilon_0}2(\sigma_x^2\partial_x + x)\int_0^\infty \frac{e^{-(\frac{x^2}{2\sigma_x^2+\frac{1}{t}}+\frac{y^2}{2\sigma_y^2+\frac{1}{t}})}}{\sqrt{2\sigma_x^2+\frac{1}{t}}\sqrt{2\sigma_y^2+\frac{1}{t}}}\frac{dt}{t}$$

$$= -\frac{Q\rho_z}{4\pi\varepsilon_0}\partial_x\int_0^\infty \frac{e^{-(\frac{x^2}{2\sigma_x^2+q}+\frac{y^2}{2\sigma_y^2+q})}}{\sqrt{2\sigma_x^2+q}\sqrt{2\sigma_y^2+q}}dq = -\partial_x U$$

$$\approx \frac{Q\rho_z}{2\pi\varepsilon_0}x\int_0^\infty \frac{1}{\sqrt{2\sigma_x^2+q}^3\sqrt{2\sigma_y^2+q}}dq = \frac{Q\rho_z}{2\pi\varepsilon_0}\frac{1}{\sigma_x(\sigma_x+\sigma_y)}x$$

Beam-beam force

$$\vec{E}(x,y,z) \approx \frac{Q\rho_z}{2\pi\varepsilon_0}\frac{1}{\sigma_x+\sigma_y}\begin{pmatrix}\frac{x}{\sigma_x}\\ \frac{y}{\sigma_y}\end{pmatrix} \qquad \vec{B}(x,y,z)=0$$

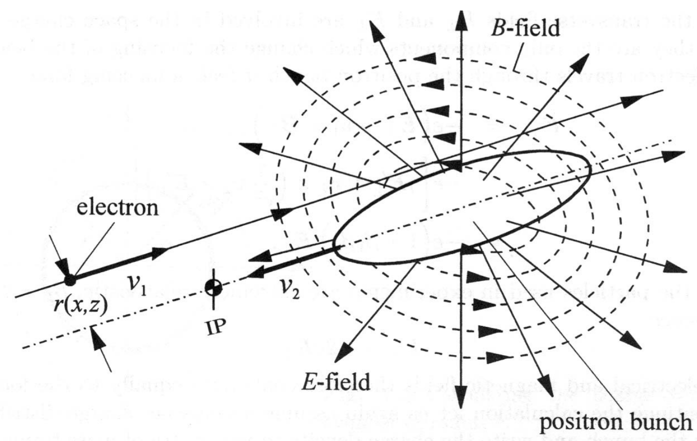
$$\begin{cases}\vec{E}_{\text{lab}}(x,y,z) \approx \frac{Q\rho_{\text{lab}\,z}}{2\pi\varepsilon_0}\frac{1}{\sigma_x+\sigma_y}\begin{pmatrix}\frac{x}{\sigma_x}\\ \frac{y}{\sigma_y}\end{pmatrix}\\ \vec{B}_{\text{lab}}(x,y,z)=\frac{1}{c}\vec{\beta}\times\vec{E}_{\text{lab}}(x,y,z)\end{cases}$$

$$\Delta\vec{p}(x,y)=\int\vec{F}(x,y,v^{(1)}t)dt$$

$$\approx \frac{Q}{2\pi\varepsilon_0}\frac{1}{\sigma_x+\sigma_y}\begin{pmatrix}\frac{x}{\sigma_x}\\ \frac{y}{\sigma_y}\end{pmatrix}\int\rho_{\text{lab}\,z}(v^{(1)}t+v^{(2)}t)dt(1+\beta^{(1)}\beta^{(2)})$$

$$=\frac{Q}{2\pi\varepsilon_0}\frac{1}{\sigma_x+\sigma_y}\begin{pmatrix}\frac{x}{\sigma_x}\\ \frac{y}{\sigma_y}\end{pmatrix}\frac{1+\beta^{(1)}\beta^{(2)}}{v^{(1)}+v^{(2)}}$$

$$\approx \frac{Q}{2\pi\varepsilon_0 c}\frac{1}{\sigma_x+\sigma_y}\begin{pmatrix}\frac{x}{\sigma_x}\\ \frac{y}{\sigma_y}\end{pmatrix}$$

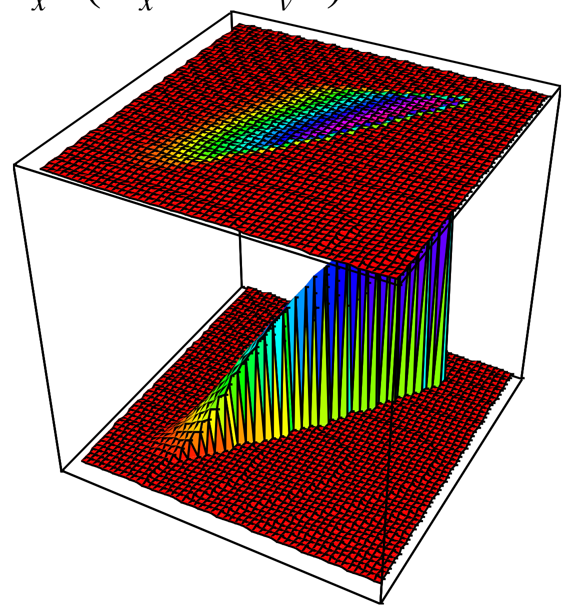


Beam-beam tune shift

$$\begin{pmatrix} \Delta x' \\ \Delta y' \end{pmatrix} = \frac{\Delta \vec{p}}{p} \approx \frac{q^{(1)}Q}{2\pi\varepsilon_0 p^{(1)}c} \frac{1}{\sigma_x + \sigma_y} \begin{pmatrix} \frac{x}{\sigma_x} \\ \frac{y}{\sigma_y} \end{pmatrix} = \begin{pmatrix} k_x x \\ k_y y \end{pmatrix}$$

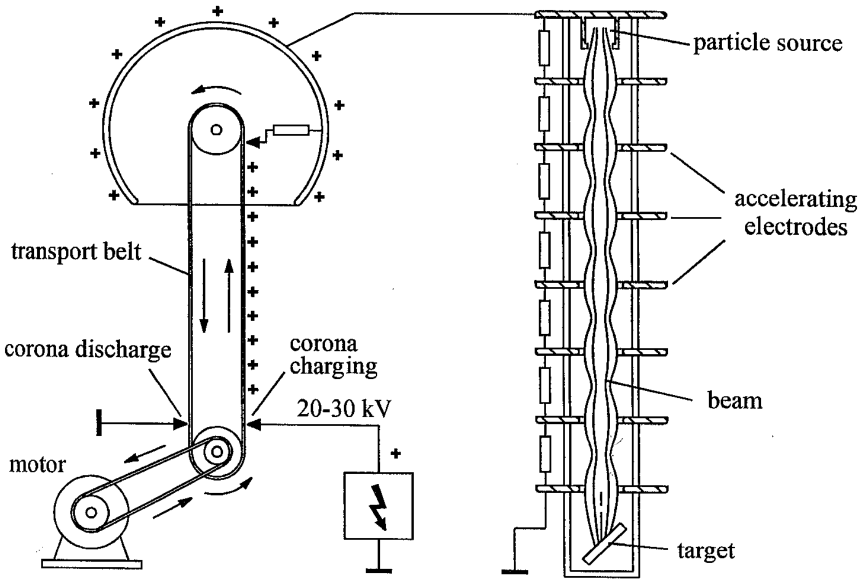
$$\Delta \nu_x^{(1)} \approx \frac{q^{(1)}Q}{8\pi^2\varepsilon_0 p^{(1)}c} \frac{\beta_x^{(1)}}{\sigma_x(\sigma_x + \sigma_y)} \approx \frac{r_{cl}^{(1)} N_{cpb}^{(2)}}{2\pi} \frac{\beta_x^{(1)}}{\sigma_x^{(2)}(\sigma_x^{(2)} + \sigma_v^{(2)})}$$

Tune spread over the beam,
amplitude dependent tune shift
and the tune shift cravat



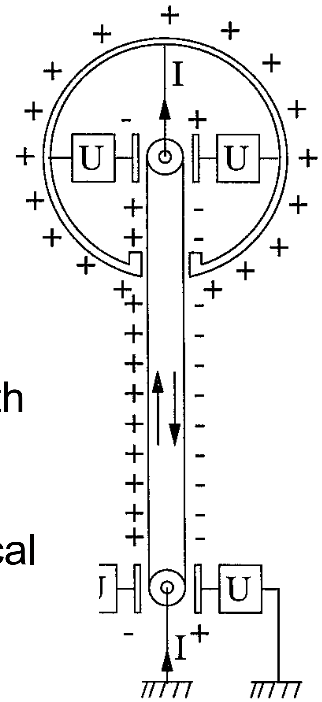
The Van de Graaff Accelerator

- 1930: van de Graaff builds the first 1.5MV high voltage generator



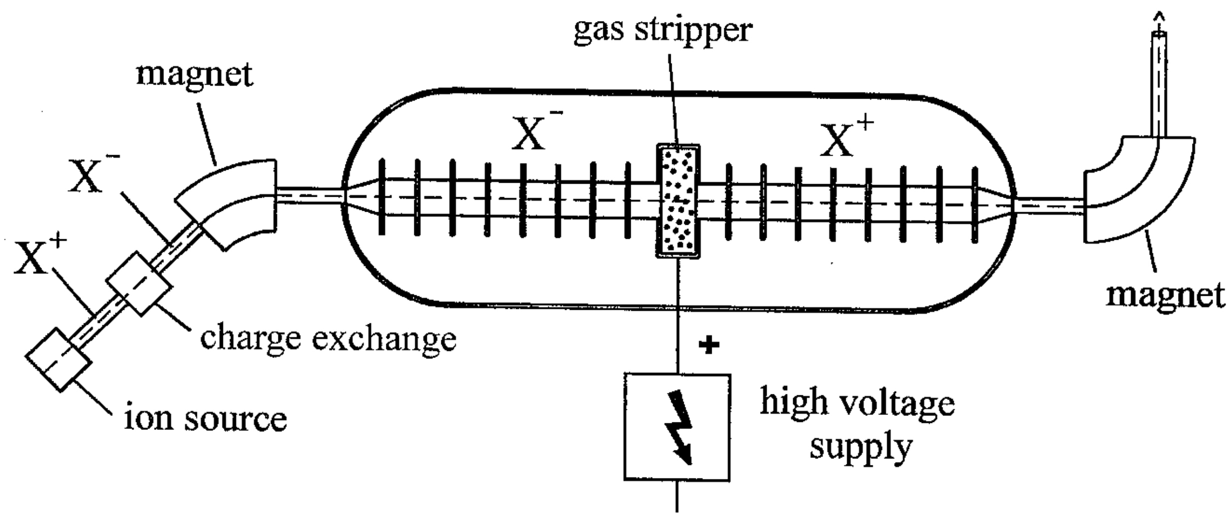
Van de Graaff

- Today Peletrons (with chains) or Laddertron (with stripes) that are charged by influence are commercially available.
- Used as injectors, for electron cooling, for medical and technical n-source via d + t ↦ n + α
- Up to 17.5 MV with insulating gas (1MPa SF_6)

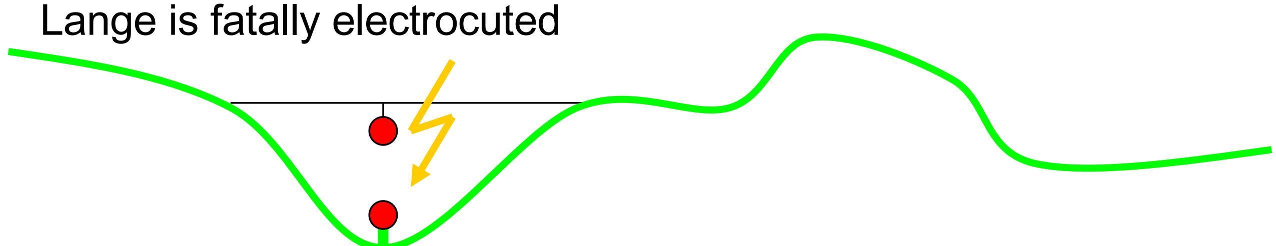


The Tandem (Van de Graaff) Accelerator

- Two Van de Graaffs, one + one –
- The Tandem Van de Graaff, highest energy 35MeV

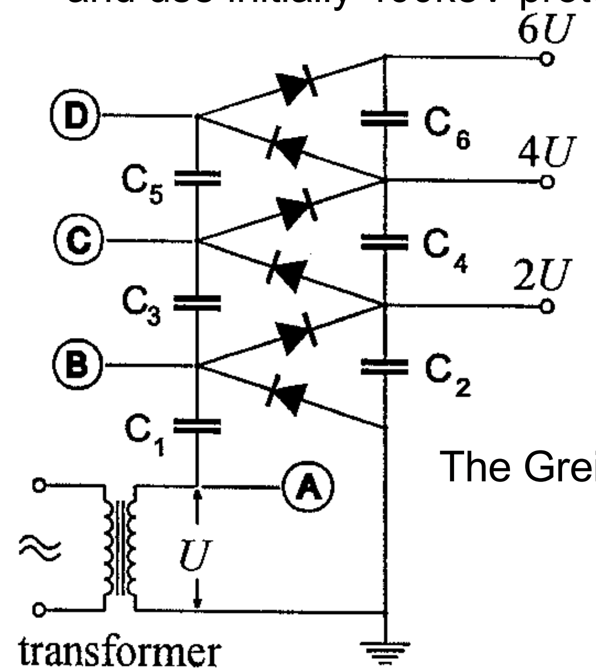


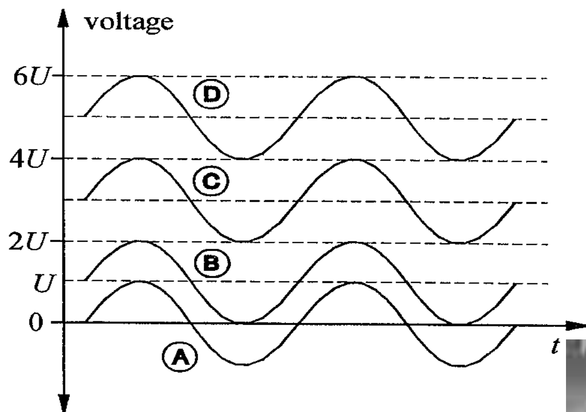
- 1932: Brasch and Lange use potential from lightening, in the Swiss Alps, Lange is fatally electrocuted



The Cockcroft-Walton Accelerator

1932: Cockcroft and Walton 1932: 700keV cascate generator (planed for 800keV) and use initially 400keV protons for ^{7}Li + p $\mapsto$ ^{4}He + ^{4}He and ^{7}Li + p $\mapsto$ ^{7}Be + n





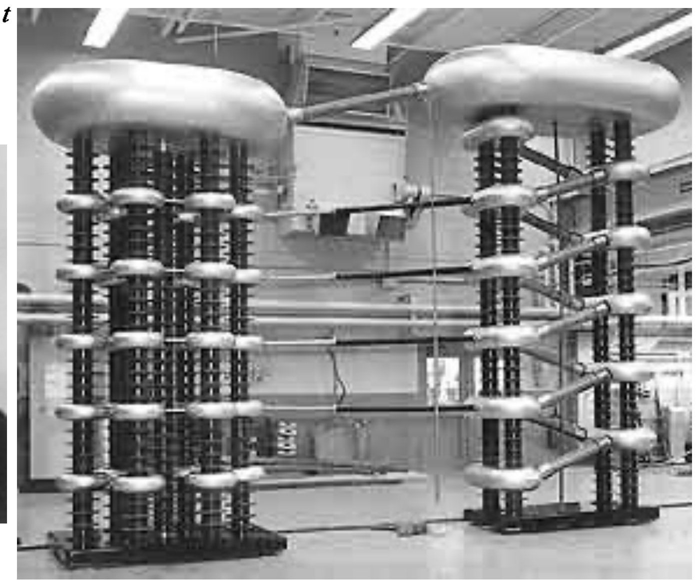
The Greinacker circuit

Up to 4MeV, 1A

NP 1951

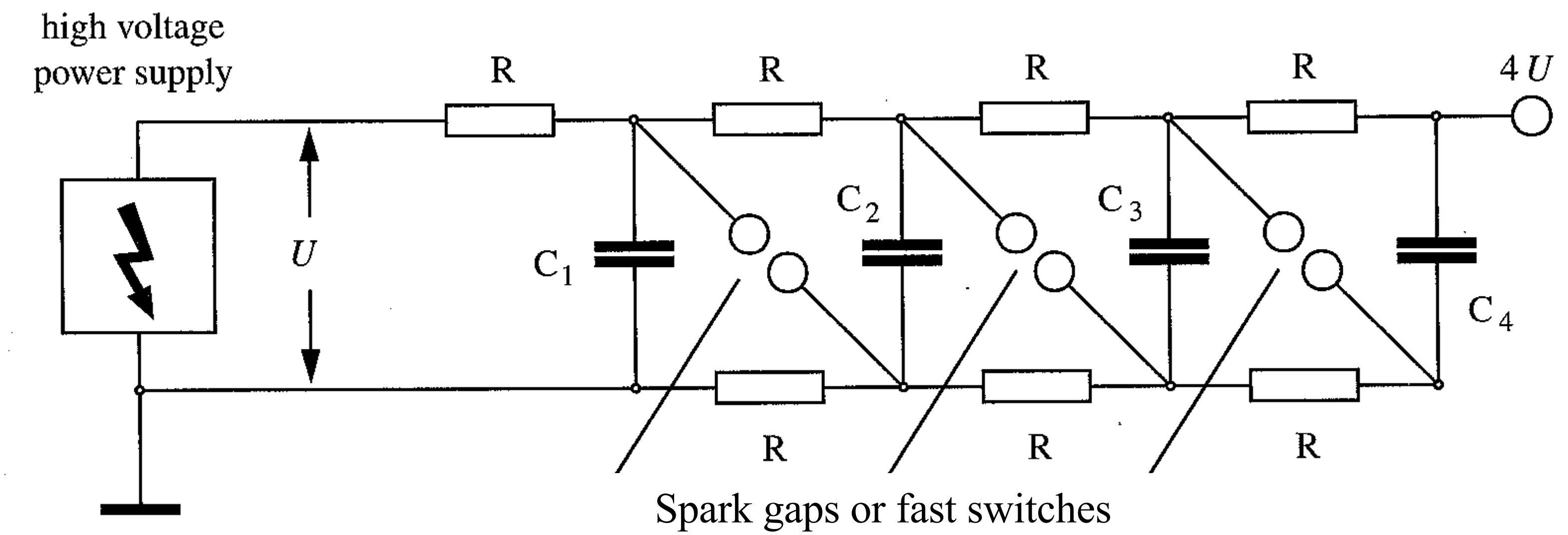
Sir John D Cockcrof

Ernest T S Walton



The Marx Generator

- 1932: Marx Generator achieves 6MV at General Electrics



After capacitors of around 2uF are filled to about 20kV, the spark gaps or switches close as fast as 40ns, allowing up to 500kA.

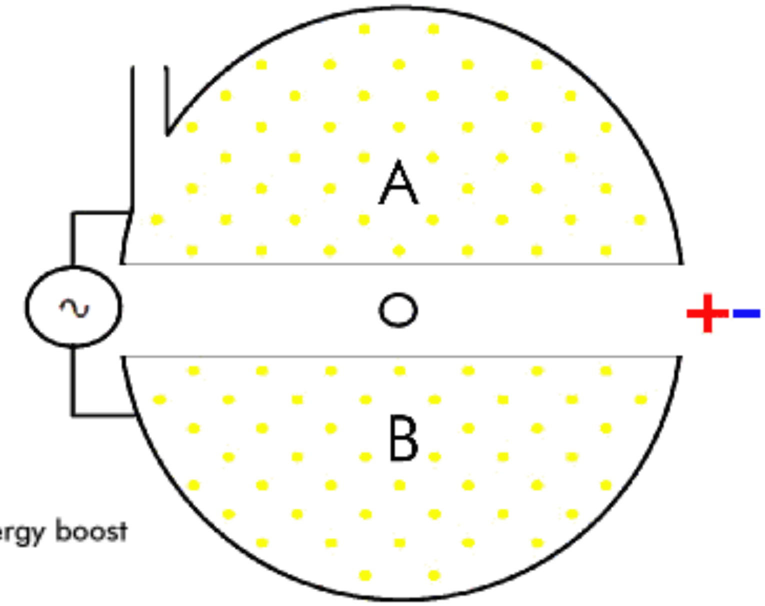
Today: The Z-machine (Physics Today July 2003) for z-pinch initial confinement fusion has 40TW for 100ns from 36 Marx generators

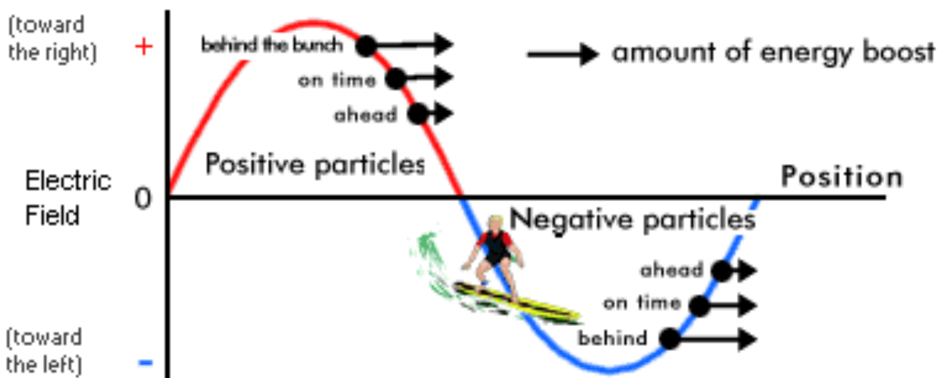
Three historic lines of accelerators

Resonant Accelerators

Direct Voltage Accelerators

Transformer Accelerator





Particles must have the correct phase relation to the accelerating voltage.

The Cyclotron frequency

$$F_r = m_0\gamma\omega_z v = qvB_z$$

$$\omega_z = \frac{q}{m_0\gamma} B_z = \text{const}$$

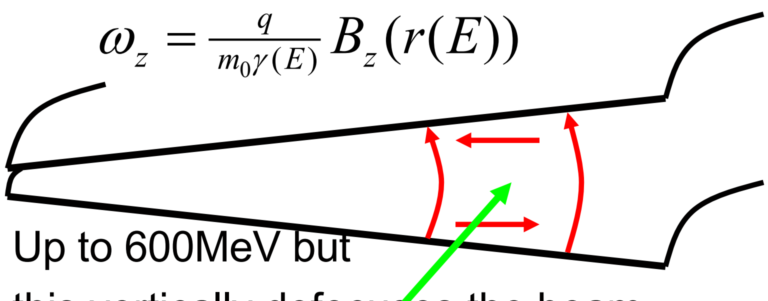
Condition: Non-relativistic particles.
Therefore, not for electrons.

- The synchrocyclotron:
  Acceleration of bunches with decreasing

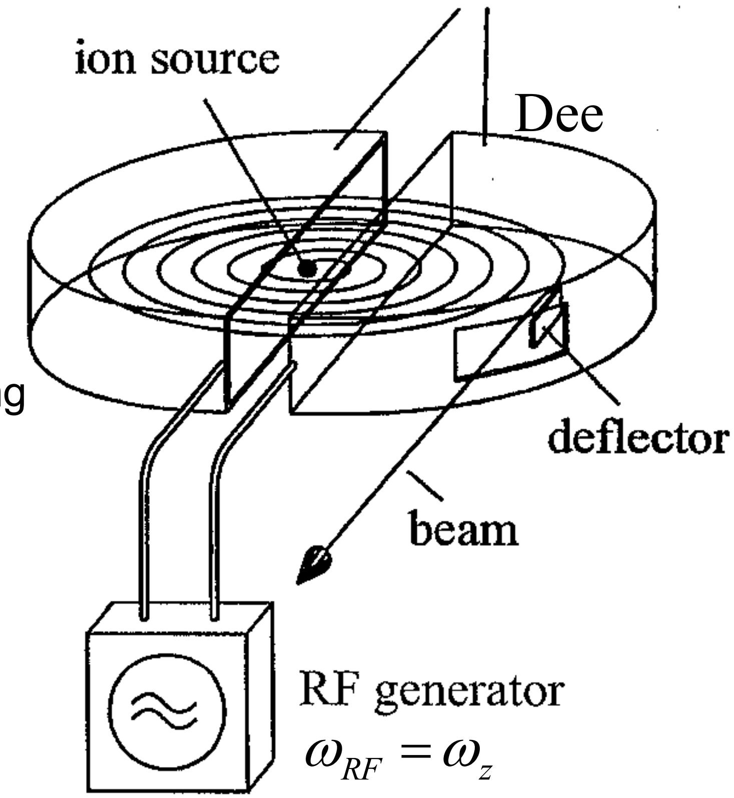
$$\omega_z(E) = \frac{q}{m_0\gamma(E)} B_z$$

- The isocyclotron with constant

$$\omega_z = \frac{q}{m_0\gamma(E)} B_z(r(E))$$



Up to 600MeV but
this vertically defocuses the beam



- 1938: Thomas proposes strong
  (transverse) focusing for a cyclotron

The Isocyclotron

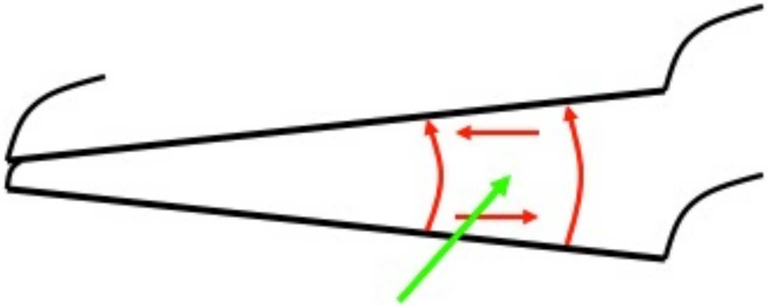
- The isocyclotron with constant

$$\omega_z = \frac{q}{m_0\gamma(E)} B_z(r(E))$$

Up to 600MeV but

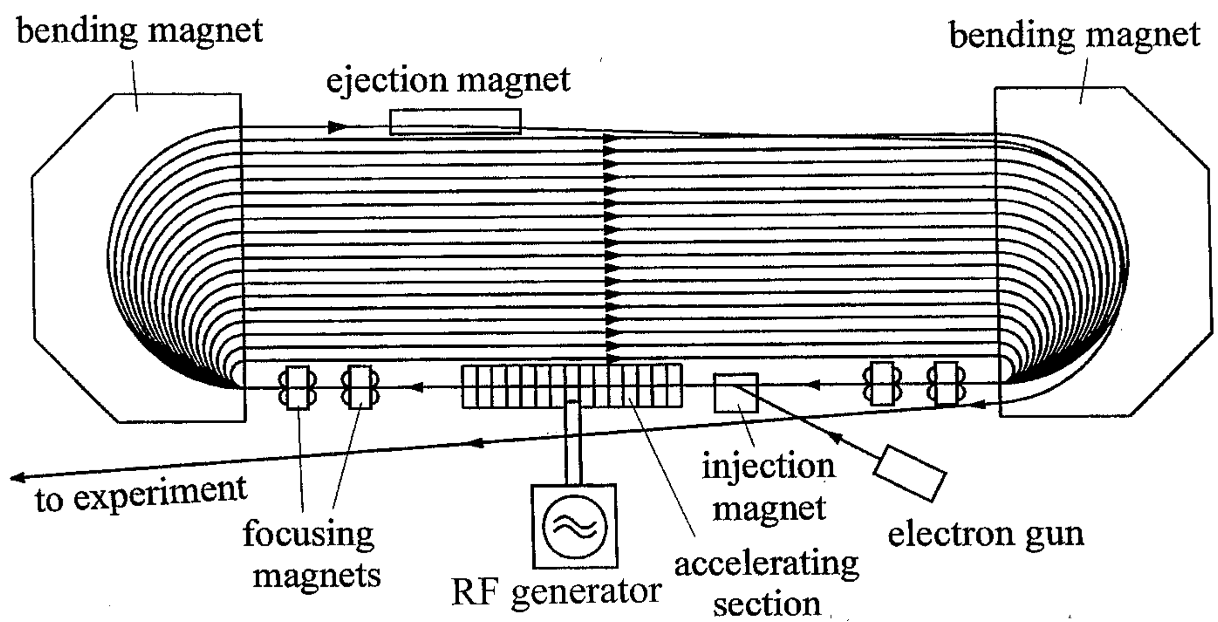
this vertically defocuses the beam.

Edge focusing is therefore used.



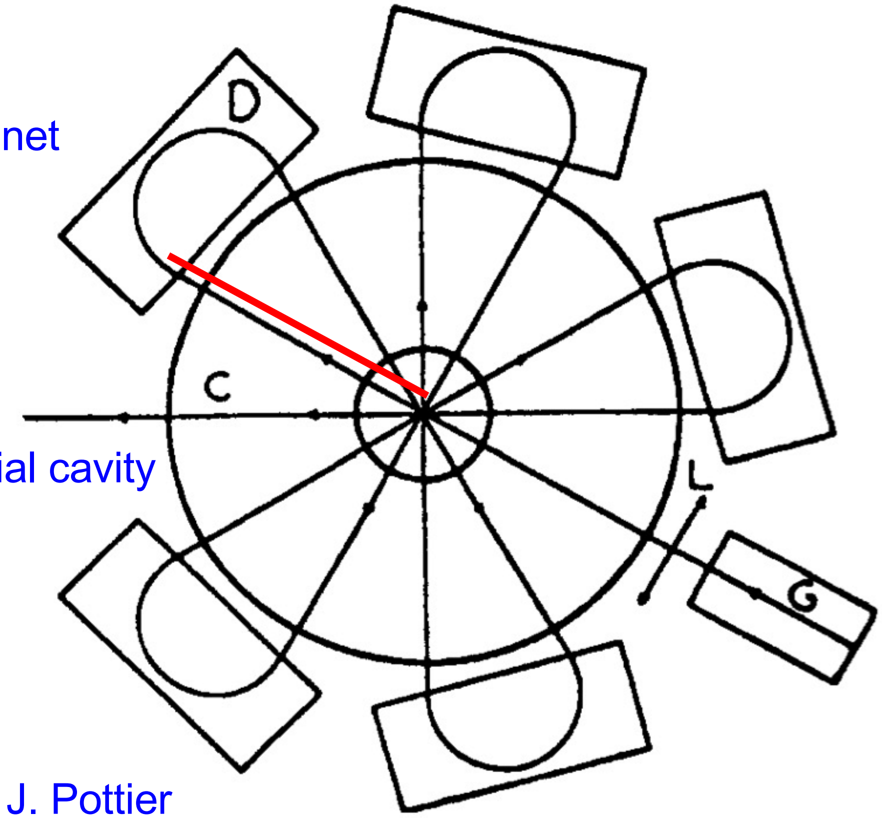
The Microtron

- Electrons are quickly relativistic and cannot be accelerated in a cyclotron.
- In a microtron the revolution frequency changes, but each electron misses an integer number of RF waves.



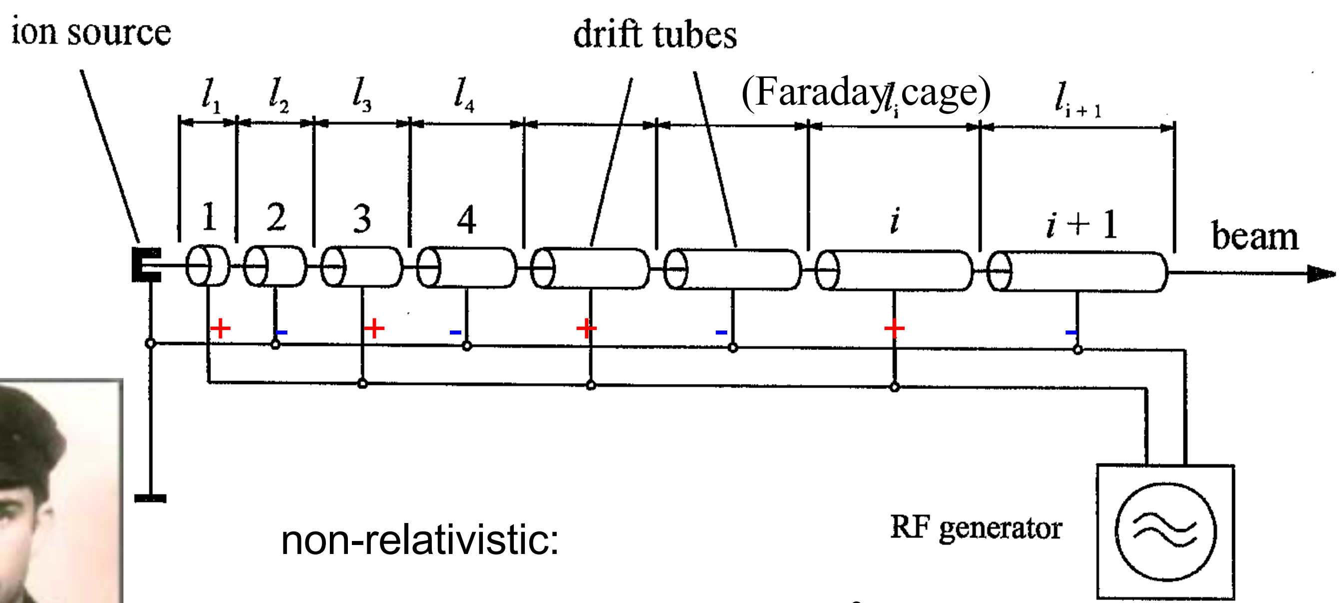
- Today: Used for medical applications with one magnet and 20MeV.
- Nuclear physics: MAMI designed for 820MeV as race track microtron.

The Rhodotron



- Invented in 1989 by J. Pottier (French, CEA)

The Wideroe linear accelerator



Wideroe

non-relativistic:

$$K_n = nqU_{\max} \sin \psi_0 = \tfrac{1}{2} m v_n^2$$

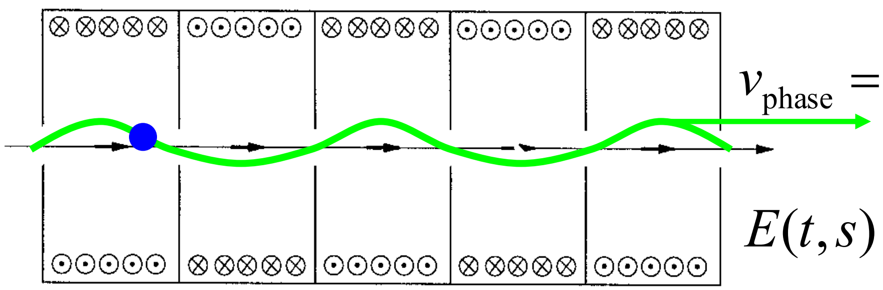
$$l_n = \tfrac{1}{2} v_n T_{RF} = \tfrac{1}{2} \beta_n \lambda_{RF} \propto \sqrt{n}$$

Called the π or the 1/2βλ mode

Accelerating Cavities

- 1933: J.W. Beams uses resonant cavities for acceleration

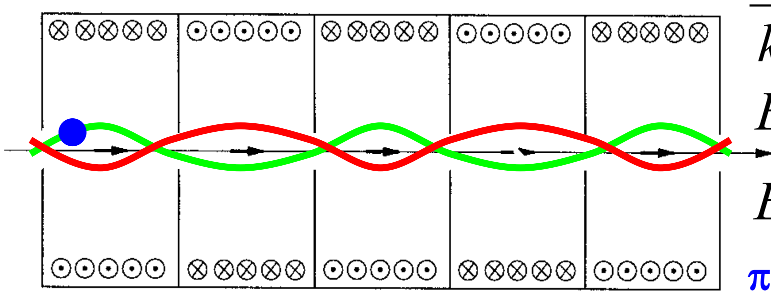
Traveling wave cavity:



$$v_{\text{phase}} = \frac{\omega}{k} = v_{\text{particle}}$$ Here v=c for electrons

$$E(t,s) \approx E_{\max} \sin(\omega t - ks)$$

Standing wave cavity:



$$\frac{\omega}{k} = v_{\text{particle}}$$

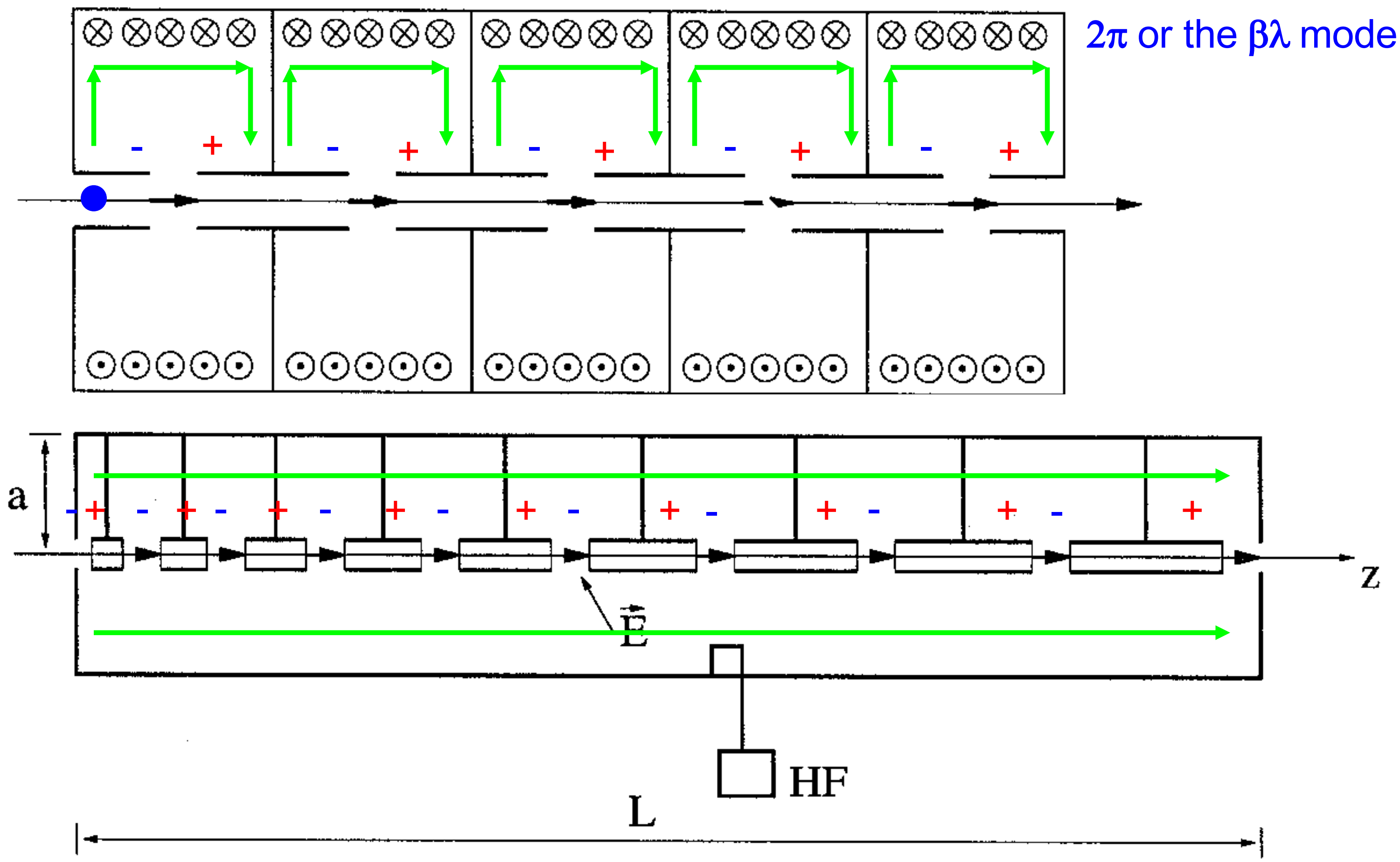
$$E(t,s) \approx E_{\max} \sin(\omega t) \sin(ks)$$

$$E(\tfrac{s}{v_{\text{particle}}}, s) \approx E_{\max} \sin^2(ks)$$

π or the 1/2βλ mode

Transit factor (for this example): $\langle E \rangle = \frac{1}{\lambda_{RF}} \int_0^{\lambda_{RF}} E(\tfrac{s}{v_{\text{particle}}}, s)\, ds \approx \tfrac{1}{2} E_{\max}$

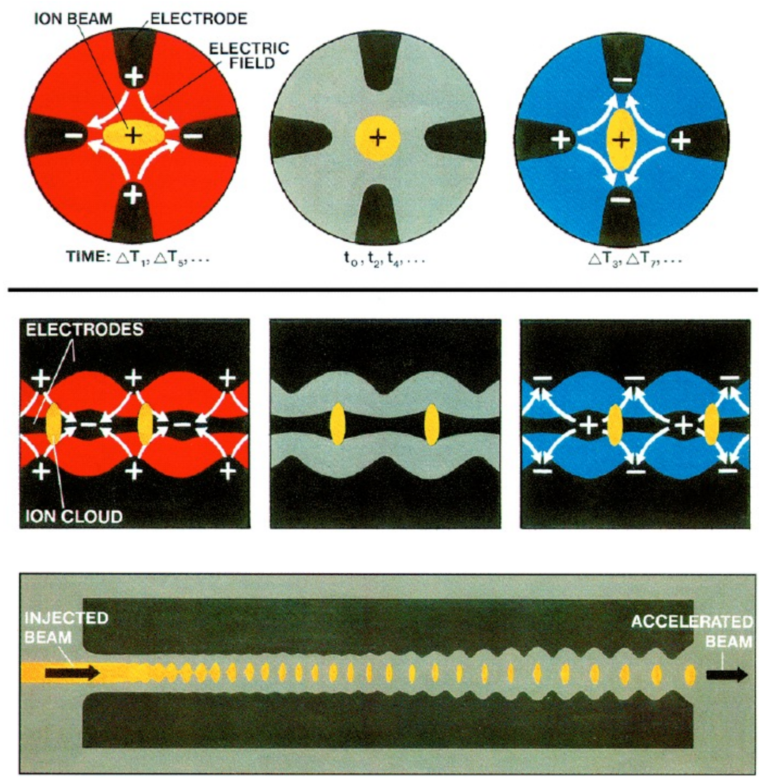
The Alvarez Linear Accelerator



Needs only one power input coupler and walls do not dissipate energy.

Radio Frequency Quadrupoles

- 1970: Kapchinskii and Teplyakov invent the RFQ



Three historic lines of accelerators

Direct Voltage Accelerators   Resonant Accelerators   Transformer Accelerator

- 1924: Wideroe invents the betatron
- 1940: Kerst and Serber build a betatron for 2.3MeV electrons and understand betatron (transverse) focusing (in 1942: 20MeV)

Betatron:

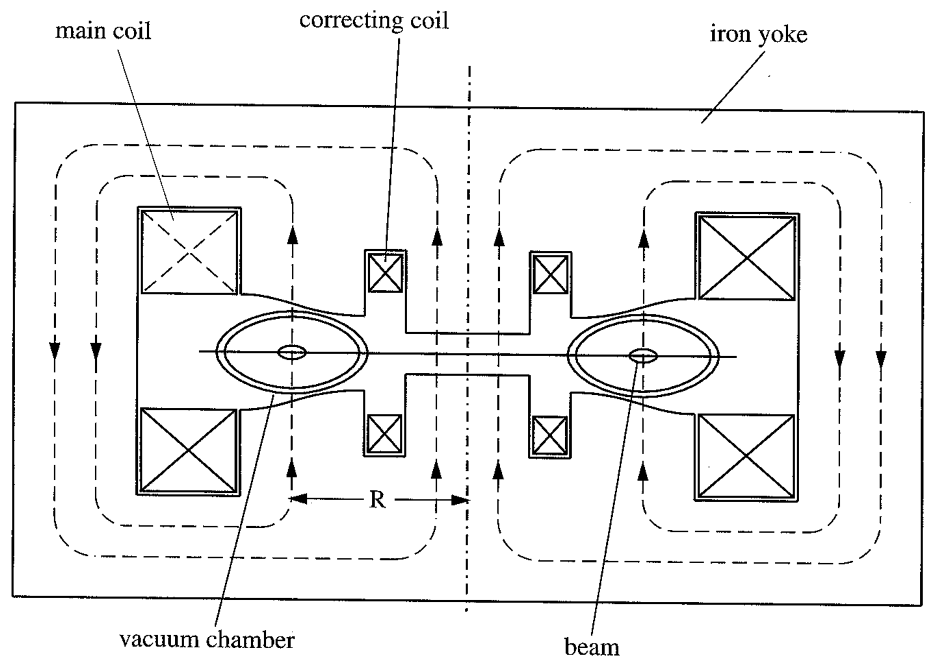
R=const, B=B(t)

Whereas for a cyclotron:

R(t), B=const

No acceleration section is needed since

$$\oint_{\partial A} \vec{E}\cdot d\vec{s} = -\iint_{A} \tfrac{d}{dt}\vec{B}\cdot d\vec{a}$$



The Synchrotron

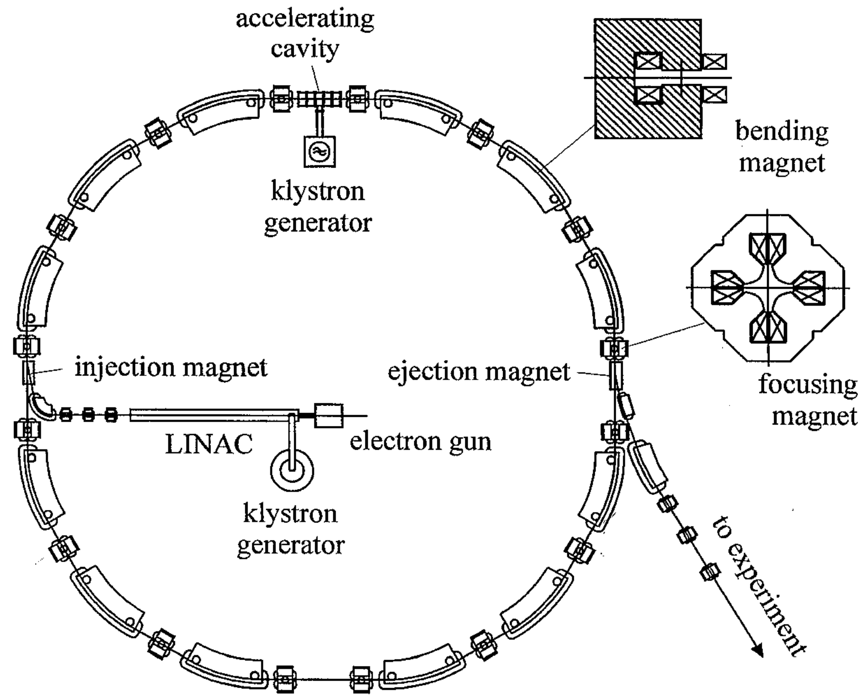
- 1945: Veksler (UDSSR) and McMillan (USA) invent the synchrotron
- 1946: Goward and Barnes build the first synchrotron (using a betatron magnet)
- 1949: Wilson et al. at Cornell are first to store beam in a synchrotron
  (later 300MeV, magnet of 80 Tons)
- 1949: McMillan builds a 320MeV electron synchrotron

- Many smaller magnets instead of one large magnet
- Only one acceleration section is needed, with

$$R = \frac{p(t)}{qB(R,t)} = \text{const.}$$

$$\omega = 2\pi \frac{v_{\text{particle}}}{L} n$$

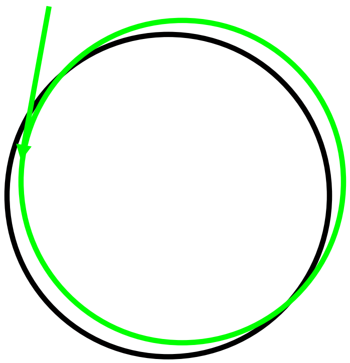
for an integer n called the harmonic number

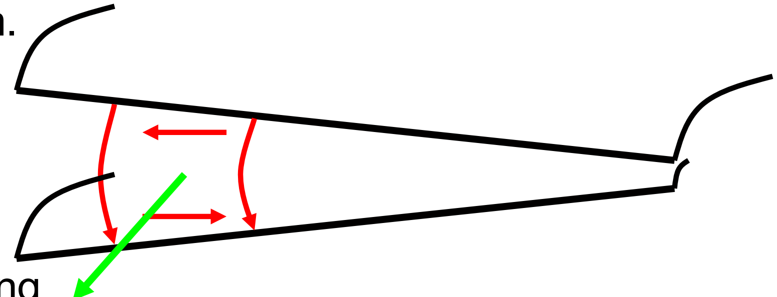


The weak focusing synchrotron

- 1952: Operation of the Cosmotron, 3.3 GeV proton synchrotron at Brookhaven: Beam pipe height: 15cm.

Natural ring focusing:



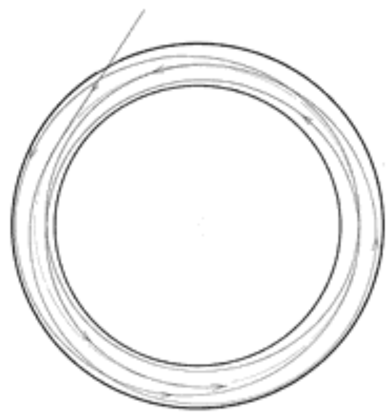


Vertical focusing

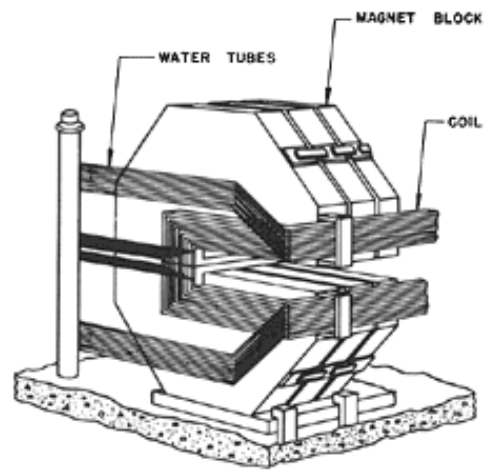
+ Horizontal defocusing + ring focusing

Focusing in both planes

The Cosmotron

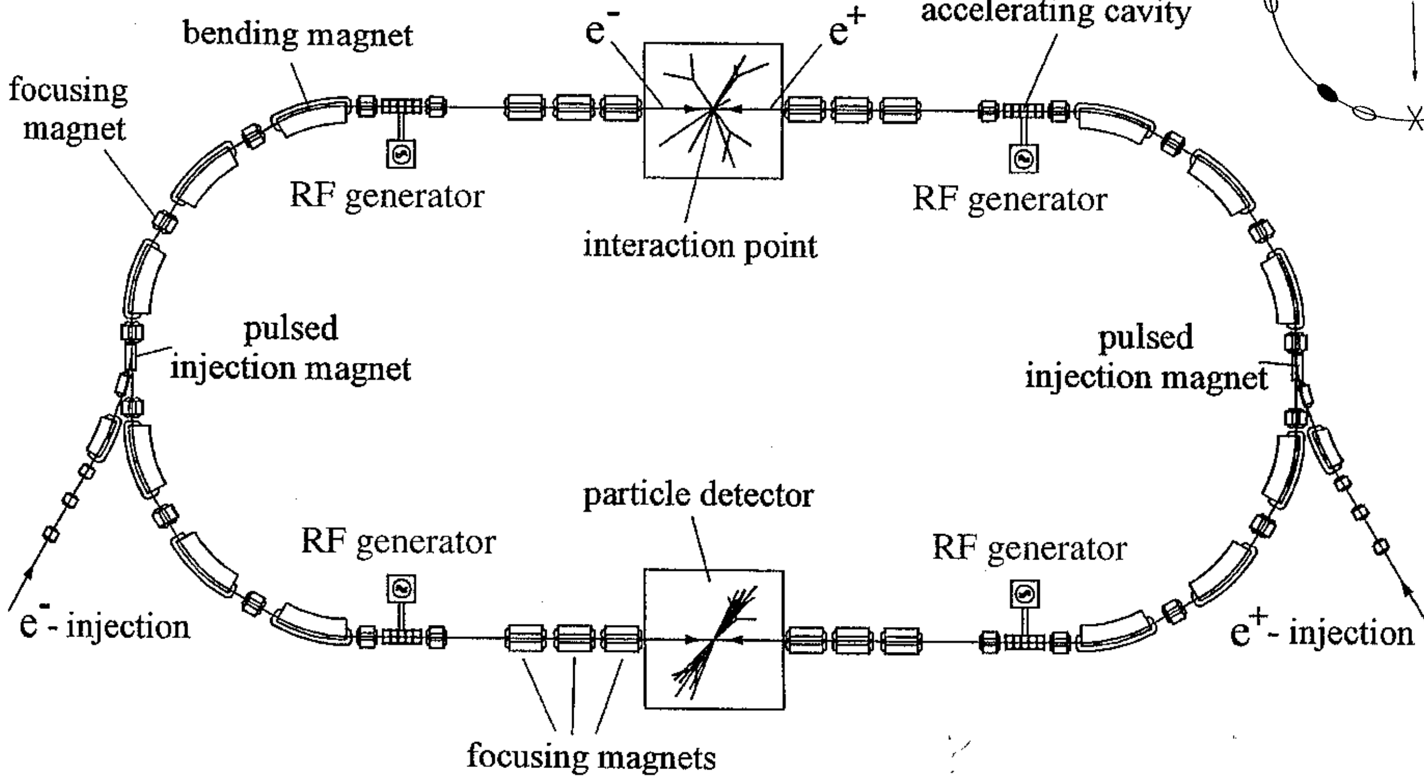


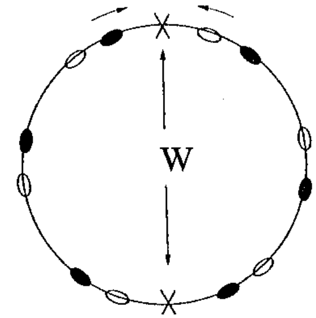
Weak focusing accelerator



Elements of a collider

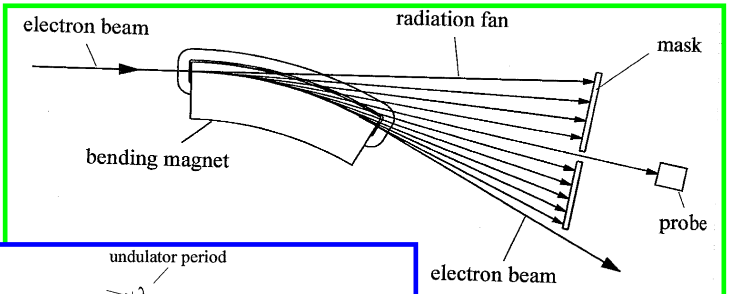
- Saving one beam while injection another
- Avoiding collisions outside the detectors.
- Compensating the forces between e^+ and e^- beams

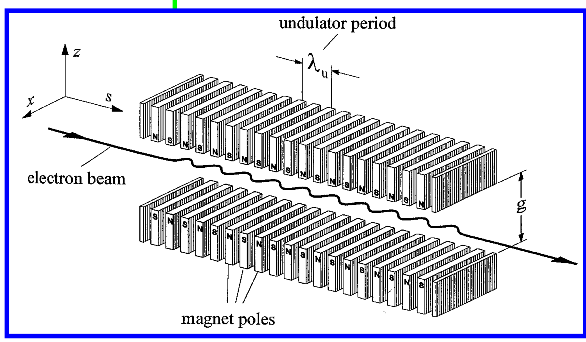


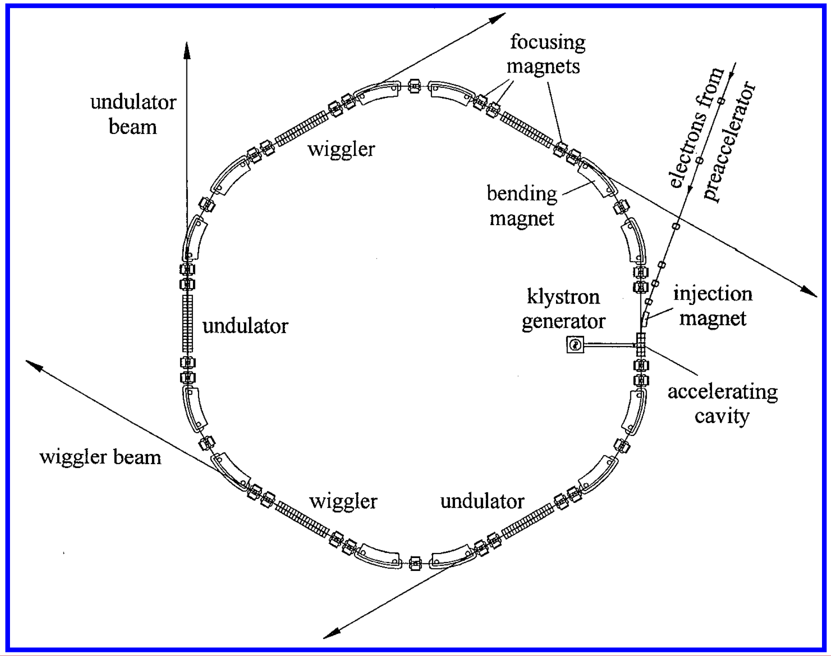


4 Generations of Light Sources

- 1st Genergation (1970s): Many HEP rings are parasitically used for X-ray production
- 2nd Generation (1980s): Many dedicated X-ray sources (light sources)
- 3rd Generation (1990s): Several rings with dedicated radiation devices (wigglers and undulators)
- Today (4th Generation): Construction of Free Electron Lasers (FELs) driven by LINACs



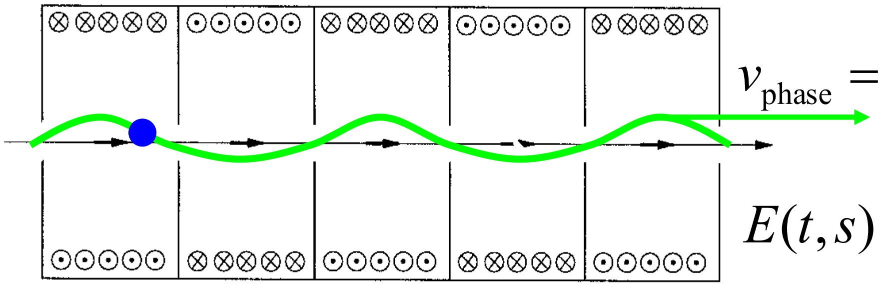




Accelerating cavities

- 1933: J.W. Beams uses resonant cavities for acceleration

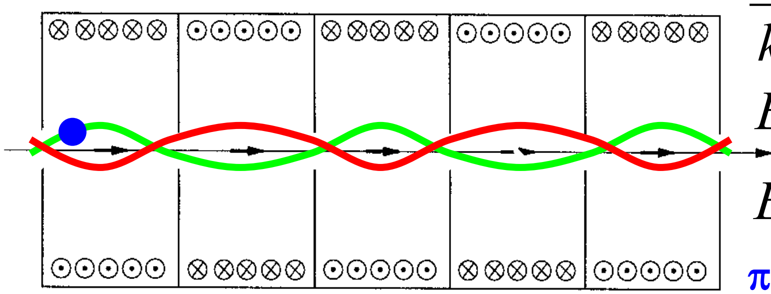
Traveling wave cavity:



$$v_{\text{phase}} = \frac{\omega}{k} = v_{\text{particle}}$$ Here v=c for electrons

$$E(t,s) \approx E_{\max} \sin(\omega t - ks)$$

Standing wave cavity:



$$\frac{\omega}{k} = v_{\text{particle}}$$

$$E(t,s) \approx E_{\max} \sin(\omega t)\sin(ks)$$

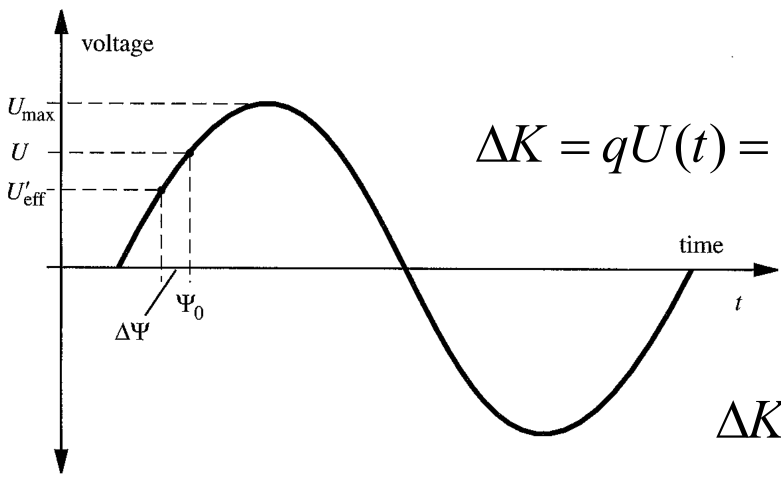
$$E(\tfrac{s}{v_{\text{particle}}}, s) \approx E_{\max} \sin^2(ks)$$

π or the 1/2βλ mode

Transit factor (for this example): $\langle E\rangle = \frac{1}{\lambda_{RF}} \int_0^{\lambda_{RF}} E(\tfrac{s}{v_{\text{particle}}}, s)\, ds = \tfrac{1}{2} E_{\max}$

Phase focusing

- 1945: Veksler (UDSSR) and McMillan (USA) realize the importance of phase focusing



$$\Delta K = qU(t) = qU_{\max}\sin(\omega(t - t_0) + \psi_0)$$

Longitudinal position in the bunch:

$$\sigma = s - s_0 = -v_0(t - t_0)$$

$$\Delta K(\sigma) = qU_{\max}\sin(-\tfrac{\omega}{v_0}(s - s_0) + \psi_0)$$

$$\Delta K(0) > 0$$ (Acceleration)

$$\Delta K(\sigma) < \Delta K(0) \text{ for } \sigma > 0 \Rightarrow \tfrac{d}{d\sigma}\Delta K(\sigma) < 0$$ (Phase focusing)

$$\left.\begin{array}{r} qU(t) > 0 \\ q\tfrac{d}{dt}U(t) > 0 \end{array}\right\} \quad \psi_0 \in (0, \frac{\pi}{2})$$

Phase focusing is required in any RF accelerator.

Longitudinal phase space in a Linac

Other particles: $$\frac{dE}{ds} = \hat{E}\cos\Phi$$

Reference particle: $$\frac{dE_0}{ds} = \hat{E}\cos\Phi_0$$

$$\phi = \Phi - \Phi_0 = \omega(t - t_0)$$

$$\frac{d\delta}{ds} = \tfrac{\hat{E}}{E_0}\left(\cos(\Phi_0 + \phi) - \cos\Phi_0\right) \approx -\phi\tfrac{\hat{E}}{E_0}\sin\Phi_0$$

$$\frac{d\phi}{ds} = \omega\left(\frac{1}{v} - \frac{1}{v_0}\right) \approx \omega\left(\frac{1}{v_0 + \frac{dv}{d\delta}\big|_0\delta} - \frac{1}{v_0}\right) \approx -\omega\frac{\frac{dv}{d\delta}\big|_0}{v_0^2}\delta = -\omega\frac{c^2}{v_0^3\gamma_0^2}\delta$$

$$\frac{dv}{d\delta} = E\frac{dv}{dE} = \gamma\frac{dv}{d\gamma} = c\gamma\frac{d\sqrt{1 - \frac{1}{\gamma^2}}}{d\gamma} = \frac{c}{\gamma^2\beta}$$

$$\frac{d^2\phi}{ds^2} \approx -\omega\frac{c^2}{v_0^3\gamma_0^2}\frac{d\delta}{ds} \approx \tfrac{\hat{E}}{E_0}\sin\Phi_0\,\omega\frac{c^2}{v_0^3\gamma_0^2}\phi$$

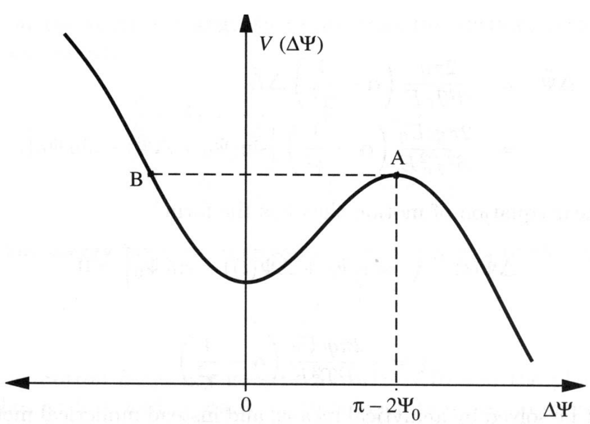
Stability for small phases when the factor on the right-hand side is negative.

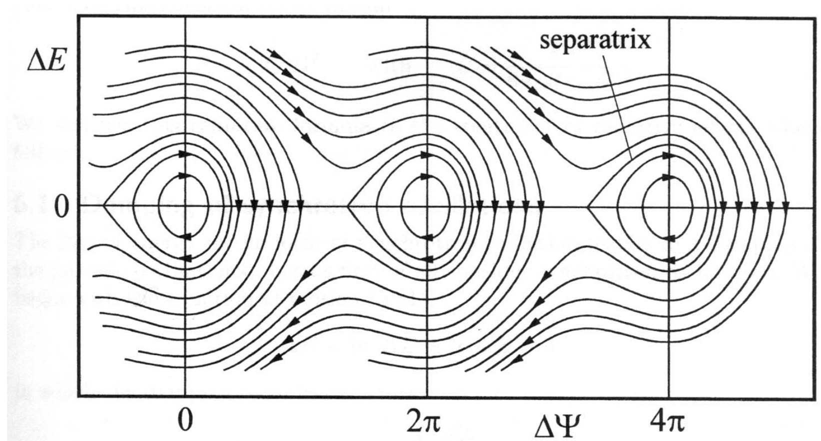
For not very small phases one cannot linearize.

$$\frac{d\delta}{ds} = \tfrac{\hat{E}}{E_0}\left(\cos(\Phi_0 + \phi) - \cos\Phi_0\right) \qquad \frac{d\phi}{ds} \approx -\omega\frac{c^2}{v_0^3\gamma_0^2}\delta$$

$$H(\phi,\delta) = -\tfrac{q\overline{E}_s}{K_0}\left(\sin(\Phi_0 + \phi) - \phi\cos\Phi_0\right) - \omega\frac{c^2}{v_0^3\gamma_0^2}\delta^2$$

$$\tfrac{d}{dt}\phi = \tfrac{\partial}{\partial\delta}H\ , \quad \tfrac{d}{dt}\delta = -\tfrac{\partial}{\partial\phi}H$$





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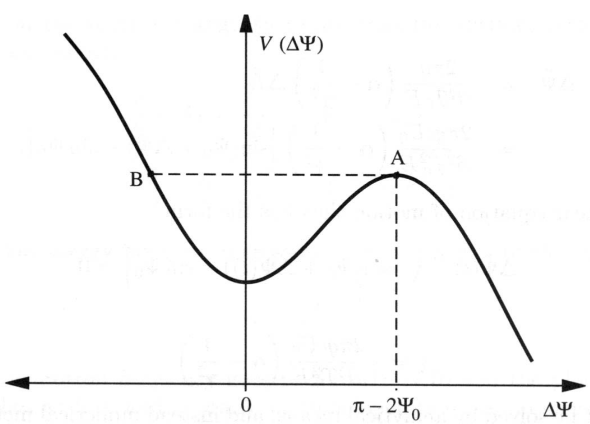
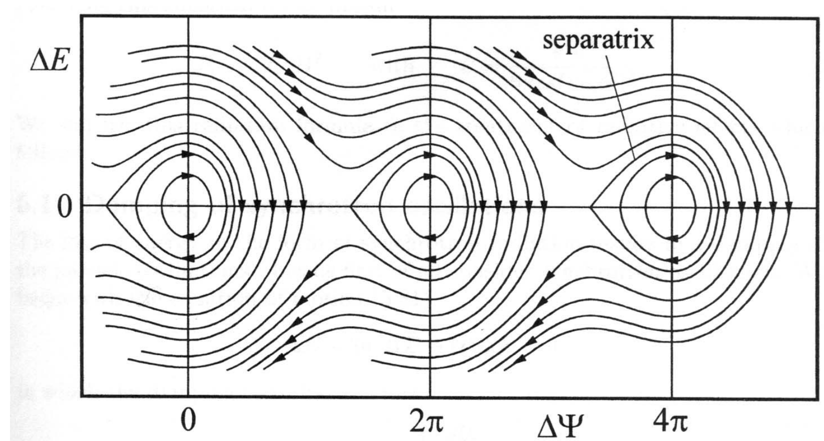
$$\frac{d\delta}{ds} = \frac{q\overline{E}_s}{K_0}\left(\cos(\Phi_0+\phi)-\cos\Phi_0\right) \approx -\phi\frac{q\overline{E}_s}{K_0}\sin\Phi_0$$

$$\frac{d^2\phi}{ds^2} \approx -\omega\frac{c^2}{v_0^3\gamma_0^2}\frac{d\delta}{ds} = -\omega\frac{c^2}{v_0^3\gamma_0^2}\frac{q\overline{E}_s}{K_0}\left(\cos(\Phi_0+\phi)-\cos\Phi_0\right)$$

$$= -\frac{d}{d\phi}\underline{\omega\frac{c^2}{v_0^3\gamma_0^2}\frac{q\overline{E}_s}{K_0}\left(\sin(\Phi_0+\phi)-\phi\cos\Phi_0\right)}$$

Effective potential





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Longitudinal phase space in rings

Reference particle: $\frac{dE_0}{ds} = \hat{E}\cos\Phi_0$

Other particles: $\frac{dE}{ds} = \hat{E}\cos\Phi$

$$\phi = \Phi - \Phi_0 = \omega(t - t_0)$$

$$\frac{d\delta}{ds} = \frac{\hat{E}}{E_0}\left(\cos(\Phi_0 + \phi) - \cos\Phi_0\right) \approx -\phi\frac{\hat{E}}{E_0}\sin\Phi_0$$

Momentum compaction $\alpha = \frac{p}{L}\frac{dL}{dp}$, closed orbit length change with momentum.

$$\frac{d\phi}{ds} = \omega(t - t_0) = \omega\left(\frac{L}{v} - \frac{L_0}{v_0}\right) = \omega\left(\frac{L_0}{v} + \alpha\frac{L_0}{p_0\, v_0}dp - \frac{L_0}{v_0}\right) = \frac{\omega L_0}{v_0}\left(\alpha\frac{dp}{p_0} - \frac{dv}{v_0}\right) = \frac{\omega L_0}{v_0\,\beta_0^2}\left(\alpha - \frac{1}{\gamma^2}\right)\delta$$

Stability for small phases when the factor on the right-hand side is negative.

$$\frac{d^2\phi}{ds^2} = -\phi\frac{\omega L_0}{v_0\,\beta_0^2}\left(\alpha - \frac{1}{\gamma^2}\right)sin\phi_0$$

Transition energy and phase focusing

Stability for small phases when the factor on the right-hand side is negative.

$$\frac{d^2\phi}{ds^2} = -\phi \frac{\omega L_0}{v_0\, \beta_0^2}\left(\alpha - \frac{1}{\gamma^2}\right) sin\phi_0$$

For only natural ring focusing, $\rho \propto p \Rightarrow \alpha = 1$. For stronger focusing, $\alpha < 1$.

For small γ, $\alpha - \frac{1}{\gamma^2} < 0$ and $\phi_0 \epsilon [0, \pi]$ leads to a stable phase.

For large γ, $\alpha - \frac{1}{\gamma^2} < 0$ and $\phi_0 \epsilon [\pi, 2\pi]$ leads to a stable phase.

At the transition energy, one must jump the RF phase by π. This jump doers not have to happen very quickly, because at that energy, phases hardly change because of $\frac{d^2\phi}{ds^2} \approx 0$.

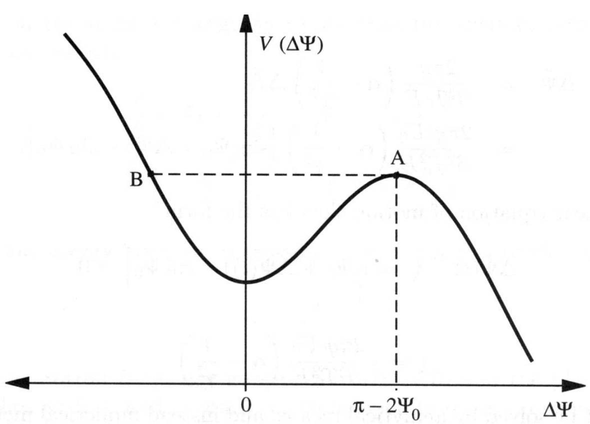
Accelerating longitudinal phase space

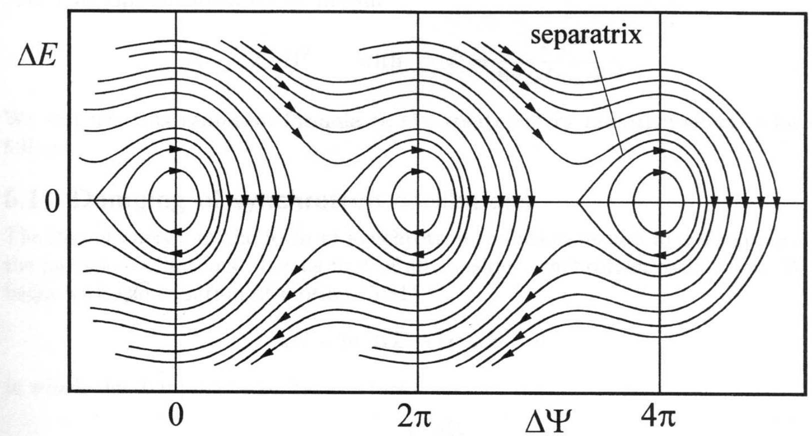
For not very small phases one cannot linearize.

$$\frac{d\delta}{ds} = \frac{\hat{E}}{E_0}\left(\cos(\Phi_0 + \phi) - \cos\Phi_0\right) \qquad \frac{d\phi}{ds} = -\frac{\omega L_0}{v_0\,\beta_0^2}\left(\alpha - \frac{1}{\gamma^2}\right)\delta$$

$$H(\phi,\delta) = -\frac{qE}{E}\left(\sin(\Phi_0 + \phi) - \phi\cos(\Phi_0)\right)) - \frac{\omega L_0}{2\,v_0\,\beta_0^2}\left(\alpha - \frac{1}{\gamma^2}\right)\delta^2$$

$$\frac{d}{dt}\phi = \frac{\partial}{\partial\delta}H\,, \qquad \frac{d}{dt}\delta = -\frac{\partial}{\partial\phi}H$$





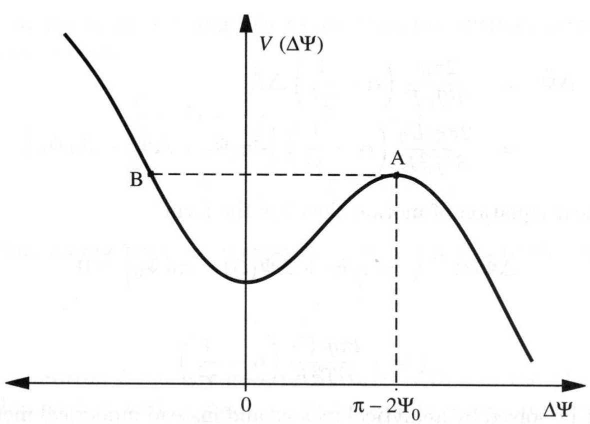
The effective accelerating potential

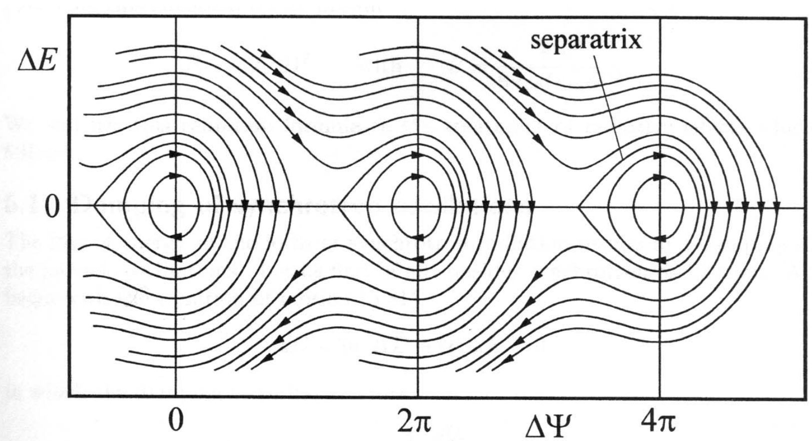
$$\frac{d\delta}{ds} = \frac{\hat{E}}{E_0}\left(\cos(\Phi_0 + \phi) - \cos\Phi_0\right) \qquad \frac{d\phi}{ds} = -\frac{\omega L_0}{v_0\,\beta_0^2}\left(\alpha - \frac{1}{\gamma^2}\right)\delta$$

$$\frac{d^2\phi}{ds^2} = -\frac{\omega L_0}{v_0\,\beta_0^2}\left(\alpha - \frac{1}{\gamma^2}\right)\frac{\hat{E}}{E_0}(\cos(\Phi_0 + \phi) - \cos(\Phi_0))$$

$$\frac{d^2\phi}{ds^2} = -\frac{\partial}{\partial\phi}\frac{\omega L_0}{v_0\,\beta_0^2}\left(\alpha - \frac{1}{\gamma^2}\right)\frac{\hat{E}}{E_0}(\sin(\Phi_0 + \phi) - \phi\cos(\Phi_0))$$

Effective potential





Transport maps of cavities

(1) Linearization: $E_r(r,z,t) =_1 -\frac{r}{2}\partial_z E_z(0,z,t) \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{E} = 0$

$$B_\varphi(r,z,t) =_1 \frac{1}{c^2}\frac{r}{2}\partial_t E_z(0,z,t) \quad \Rightarrow \quad \vec{\nabla}\times\vec{B} = \frac{1}{c^2}\partial_t\vec{E}$$

(2) Equation of motion:

$$a = \frac{p_x}{p_0}$$

$$a' = \frac{1}{p_0 v}(F_x - aF_z) = -\frac{q}{p_0 v}[\frac{r}{2}(\partial_z + \frac{v}{c^2}\partial_t)E_z + aE_z]$$

$$= -\frac{q}{p_0 v}[\frac{r}{2}(\frac{d}{dz} - \frac{1}{v}(1-\frac{v^2}{c^2})\partial_t)E_z + aE_z] \approx -\frac{1}{p_0}[r\frac{1}{2}p_0'' + ap_0']$$

$$u = r\sqrt{p}$$

p denotes p_0 for simplicity

Focusing !

$$u' = a\sqrt{p} + r\sqrt{p}\,\frac{p'}{2p}$$

$$u'' \approx -\frac{1}{\sqrt{p}}(r\frac{1}{2}p'' + ap') + a\sqrt{p}\,\frac{p'}{p} + r(\frac{p''}{2\sqrt{p}} - \frac{p'}{4\sqrt{p}^3}) = -u\left(\frac{p'}{2p}\right)^2$$

Transport maps of cavities

(3) Average focusing over one period with relatively little energy change:

$$u'' \approx -u\frac{\Delta^2/4}{p^2}\,,\quad \Delta = \sqrt{\langle p'^2\rangle}$$

(4) Continuous energy change:

$$p' \approx \Omega\,,\quad \Omega = \langle p'\rangle$$

$$\frac{d^2}{dp^2}u \approx \frac{1}{\Omega^2}u'' \approx -u\frac{(\Delta/\Omega)^2}{4p^2}$$

$$\frac{d^2}{dp^2}(r\sqrt{p}) = \frac{d^2}{dp^2}r\sqrt{p} + \frac{d}{dp}r\frac{1}{\sqrt{p}} - r\frac{1}{4\sqrt{p}^3} \approx -r\frac{(\Delta/\Omega)^2}{4\sqrt{p}^3}$$

$$\frac{d^2}{dp^2}r + \frac{d}{dp}r\frac{1}{p} \approx -r\frac{(\Delta/\Omega)^2-1}{4p^2} = -r\frac{\varepsilon^2}{p^2}$$

$$r(p) = \eta(-\ln(p)) \quad\Rightarrow\quad \frac{d^2}{dp^2}r = \frac{1}{p^2}\eta' + \frac{1}{p^2}\eta'' = \frac{1}{p^2}\eta' - \eta\frac{\varepsilon^2}{p^2}$$

Transport maps of traveling wave cavities

$$\eta'' = -\varepsilon^2\eta\,,\quad \eta(\xi) = A\cos(\varepsilon\xi) - B\sin(\varepsilon\xi)$$

$$\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{p'}{p} \end{pmatrix}\begin{pmatrix} \cos(\varepsilon\ln(p)) & \sin(\varepsilon\ln(p)) \\ -\sin(\varepsilon\ln(p)) & \cos(\varepsilon\ln(p)) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{p'}{p} \end{pmatrix}\begin{pmatrix} \cos(\varepsilon\ln(\frac{p}{p_i})) & \sin(\varepsilon\ln(\frac{p}{p_i})) \\ -\sin(\varepsilon\ln(\frac{p}{p_i})) & \cos(\varepsilon\ln(\frac{p}{p_i})) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{p_i}{p'} \end{pmatrix}\begin{pmatrix} r_i \\ a_i \end{pmatrix}$$

$$E_z = \sum_{n=0}^{\infty} g_n \cos(n\tfrac{2\pi}{L}z + \alpha_n)\cos(\omega t - kz + \varphi_0)$$

$$= \sum_{n=0}^{\infty} g_n \cos(n\tfrac{2\pi}{L}z + \alpha_n)\cos(\varphi_0)\,,\quad \alpha_0 = 0$$

$$\langle p' \rangle = g_0\cos(\varphi_0)\,, \langle p'^2 \rangle = \left(g_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} g_n^2\right)\cos^2(\varphi_0) \quad \Rightarrow \quad \varepsilon = \frac{1}{2}\sqrt{\frac{1}{2}\sum_{n=1}^{\infty}\frac{g_n^2}{g_0^2}}$$

Transport maps of standing wave cavities

$$\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{p'}{p} \end{pmatrix} \begin{pmatrix} \cos(\varepsilon \ln(\frac{p}{p_i})) & \sin(\varepsilon \ln(\frac{p}{p_i})) \\ -\sin(\varepsilon \ln(\frac{p}{p_i})) & \cos(\varepsilon \ln(\frac{p}{p_i})) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{p_i}{p'} \end{pmatrix} \begin{pmatrix} r_i \\ a_i \end{pmatrix}$$

$$E_z = \sum_{n=-\infty}^{\infty} g_n e^{in\frac{2\pi}{L}z} \cos(kz)\cos(\omega t + \varphi_0) \ , \quad \omega t = kz \, , k = m\tfrac{\pi}{L} \, , \quad g_{-n} = g_n^*$$

$$= \tfrac{1}{4} \sum_{n=-\infty}^{\infty} g_n [e^{i(n+m)\frac{2\pi}{L}z+\varphi_0} + e^{i(n-m)\frac{2\pi}{L}z-\varphi_0} + e^{in\frac{2\pi}{L}z+\varphi_0} + e^{in\frac{2\pi}{L}z-\varphi_0}]$$

$$= \tfrac{1}{4} \sum_{n=-\infty}^{\infty} [g_{n-m} e^{i\varphi_0} + g_{n+m} e^{-i\varphi_0} + 2 g_n \cos(\varphi_0)] e^{in\frac{2\pi}{L}z} = \sum_{n=-\infty}^{\infty} f_n e^{in\frac{2\pi}{L}z}$$

$$\left.\begin{aligned} \langle p' \rangle &= f_0 \, , \\ \langle p'^2 \rangle &= \sum_{n=0}^{\infty} |f_n|^2 \end{aligned}\right\} \varepsilon = \tfrac{1}{2} \sqrt{\sum_{n=0}^{\infty} \left|\tfrac{f_n}{f_0}\right|^2 - 1}$$

Phase space preservation in cavities

Average focusing over one period with relatively little energy change:

$$\begin{pmatrix} r \\ a \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & \frac{p'}{p} \end{pmatrix} \begin{pmatrix} \cos(\varepsilon \ln(\frac{p}{p_i})) & \sin(\varepsilon \ln(\frac{p}{p_i})) \\ -\sin(\varepsilon \ln(\frac{p}{p_i})) & \cos(\varepsilon \ln(\frac{p}{p_i})) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{p_i}{p'} \end{pmatrix}}_{\underline{M}} \begin{pmatrix} r_i \\ a_i \end{pmatrix}$$

$$\det(\underline{M}) = \frac{p_i}{p}$$

Because the determinant is not 1, the phase space volume is no longer conserved but changes by p_0/p.

A new propagation and definition of Twiss parameters is therefore needed:

$$r = \sqrt{2J \frac{1}{\beta_r \gamma_r} \beta} \sin(\psi + \phi_0)$$

Twiss parameters in accelerating cavities

$$\alpha = -\tfrac{1}{2}\beta' , \quad \gamma = \tfrac{1+\alpha^2}{\beta}$$

$$a = r' = \sqrt{2J\tfrac{mc}{p}}\left[-\tfrac{2\alpha+\beta\frac{p'}{p}}{2\sqrt{\beta}}\sin(\psi+\phi_0) + \tfrac{\beta\psi'}{\sqrt{\beta}}\cos(\psi+\phi_0)\right]$$

$$a' \approx -\tfrac{1}{p}\left[r(pK + \tfrac{1}{2}p'') + ap'\right]$$

Quad + cavities

$$a' = -\sqrt{2J\tfrac{mc}{\beta p}}\begin{pmatrix} \frac{(\beta\psi')^2+\alpha^2}{\beta} + \alpha' - \alpha\frac{p'}{p} + \beta\frac{p''}{2p} - \beta\frac{3p'^2}{4p^2} \\ 2\alpha\psi' + \beta\frac{p'}{p}\psi' - \beta\psi'' \end{pmatrix}\begin{pmatrix} \sin(\psi+\phi_0) \\ \cos(\psi+\phi_0) \end{pmatrix}$$

$$= -\sqrt{2J\tfrac{mc}{\beta p}}\begin{pmatrix} \beta(K + \frac{1}{2}\frac{p''}{p}) - (\alpha + \beta\frac{p'}{2p})\frac{p'}{p} \\ \beta\frac{p'}{p}\psi' \end{pmatrix}\begin{pmatrix} \sin(\psi+\phi_0) \\ \cos(\psi+\phi_0) \end{pmatrix}$$

$$\Rightarrow \quad \psi' = \frac{A}{\beta}, \text{ choice} : A = 1$$

$$\alpha' + \gamma = \beta\left[K + \left(\tfrac{p'}{2p}\right)^2\right]$$

$$\begin{pmatrix} r \\ a \end{pmatrix} = \sqrt{2J\tfrac{mc}{p}}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha+\beta\frac{p'}{2p}}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi+\phi_0) \\ \cos(\psi+\phi_0) \end{pmatrix}$$

Beta functions in accelerating cavities

$$\begin{pmatrix} r \\ a \end{pmatrix} = \sqrt{2J_n \frac{mc}{p}} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\tilde{\alpha}}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos(\psi + \phi_0) \\ \sin(\psi + \phi_0) \end{pmatrix} \quad , \quad \tilde{\alpha} = \alpha + \beta \frac{p'}{2p}$$

For systems with changing energy one uses the normalized Courant-Snyder invariant **J_n = J b_r g_r**

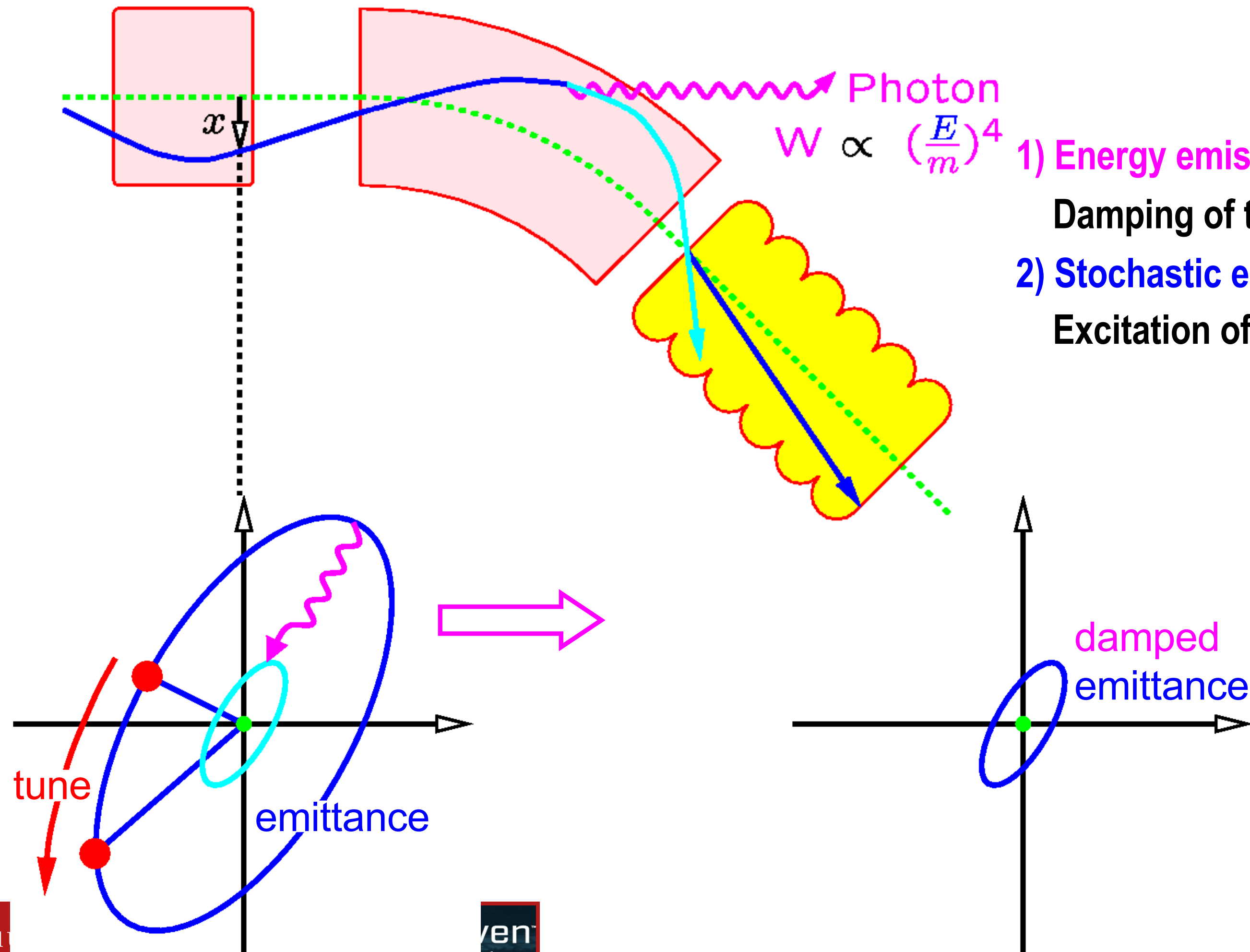
$$\begin{pmatrix} r & a \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta}} & \frac{\tilde{\alpha}}{\sqrt{\beta}} \\ 0 & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\tilde{\alpha}}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} r \\ a \end{pmatrix} \frac{p}{2mc} = \begin{pmatrix} r & a \end{pmatrix} \begin{pmatrix} \frac{1+\tilde{\alpha}^2}{\beta} & \tilde{\alpha} \\ \tilde{\alpha} & \beta \end{pmatrix} \begin{pmatrix} r \\ a \end{pmatrix} \frac{p}{2mc} = J_n$$

Reasons:

(1) **J** is the phase space amplitude of a particle in **(x , a)** phase space, which is the area in phase space (over 2p) that its coordinate would circumscribe during many turns in a ring. However, **a=p_x/p_0** is not conserved when p0 changes in a cavity. Therefore J is not conserved.

(2) **J_n = J p_0/mc** is therefore proportional to the corresponding area in **(x , p_x)** phase space, and is thus conserved.

Radiative damping of the transverse emittance



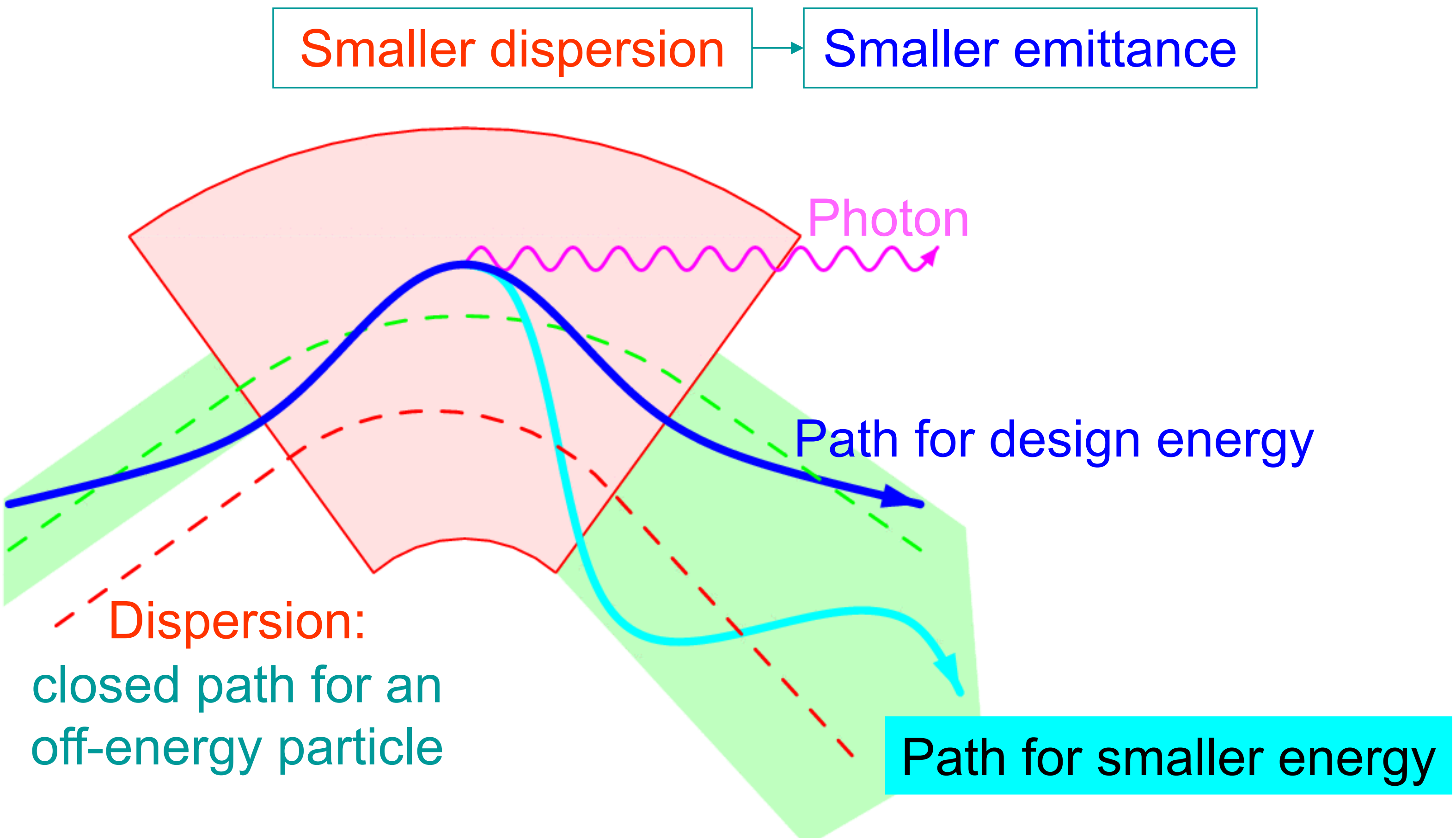
1) Energy emission:

Damping of the beam size

2) Stochastic emission:

Excitation of the beam size

Radiative excitation of the transverse emittance



Radiative damping of the longitudinal emittance

1) Energy emission: Particles with larger energy radiate more, leading them closer to the average energy.

2) Stochastic emission: Random noise in the energy of emitted photons lead to an energy spread.

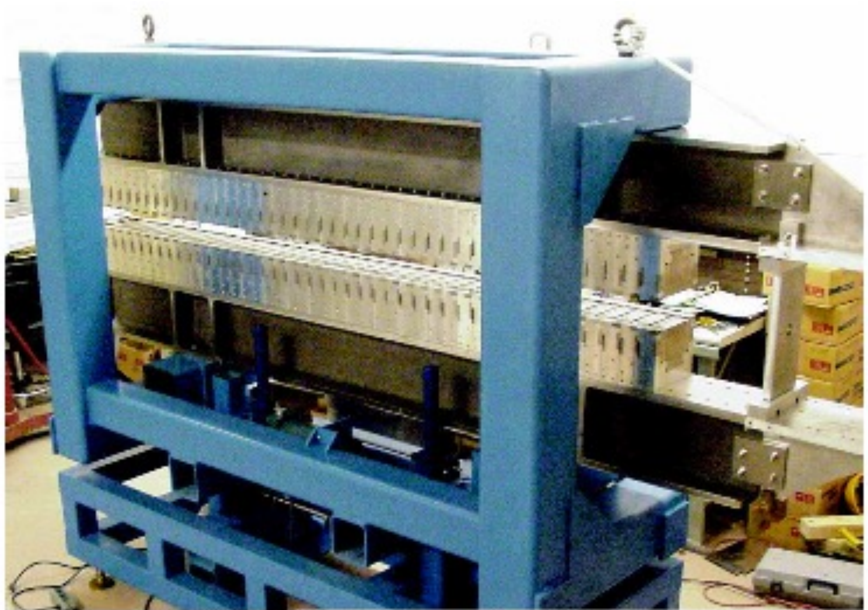
$\Rightarrow$

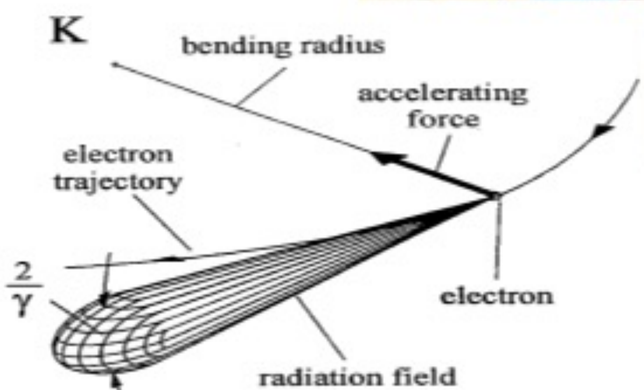
The equilibrium of the two effects leads to an equilibrium longitudinal emittance.

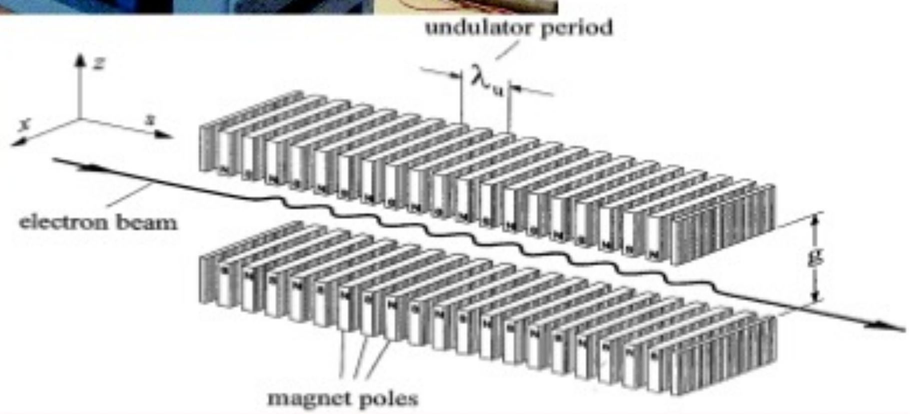
The damping time is the time it takes to radiate off the energy of the beam, while it is kept at constant energy with RF cavities. It is usually a few 100 revolutions.

During this damping time the beam forgets its history, particle coordinates are reshuffled within the beam.

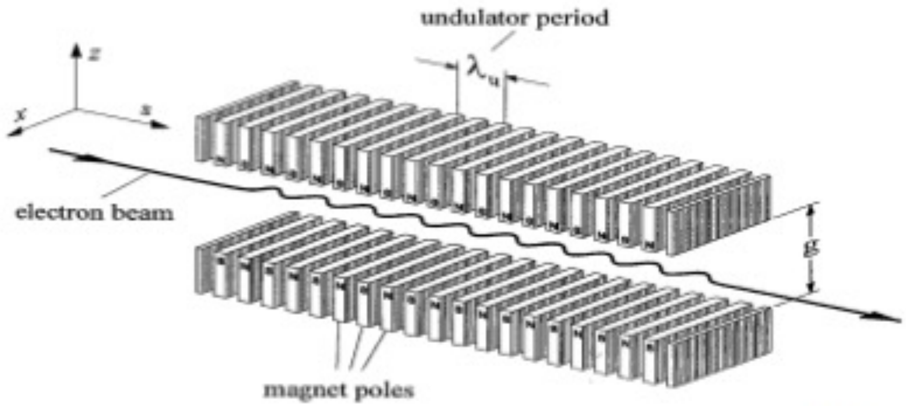
Radiative production by electrons



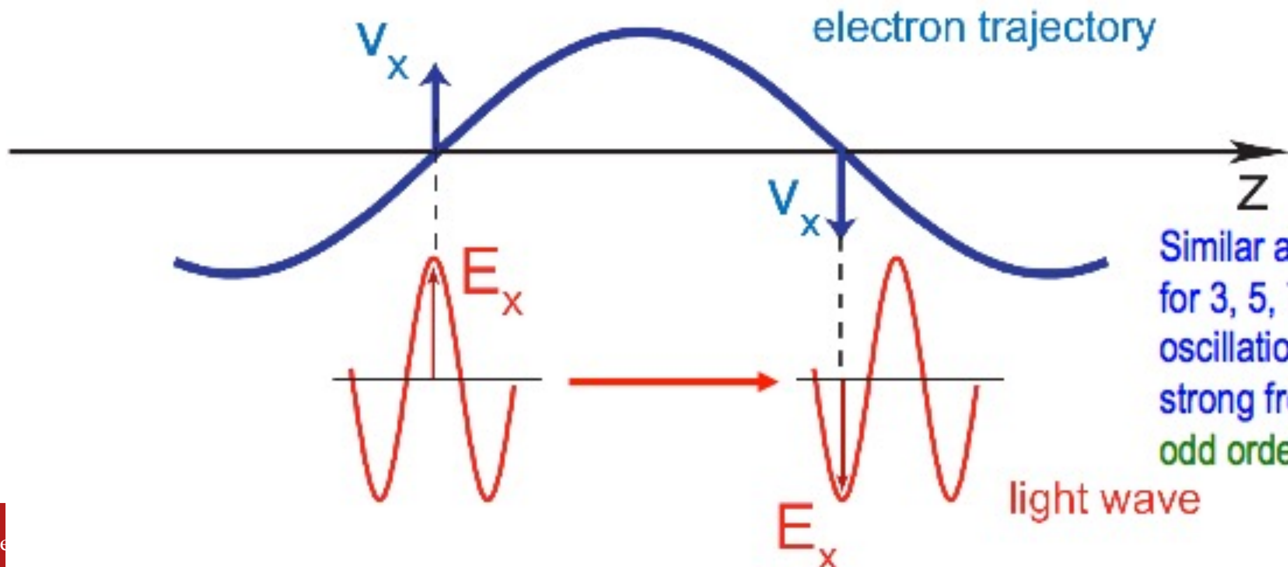




Radiative production in undulators

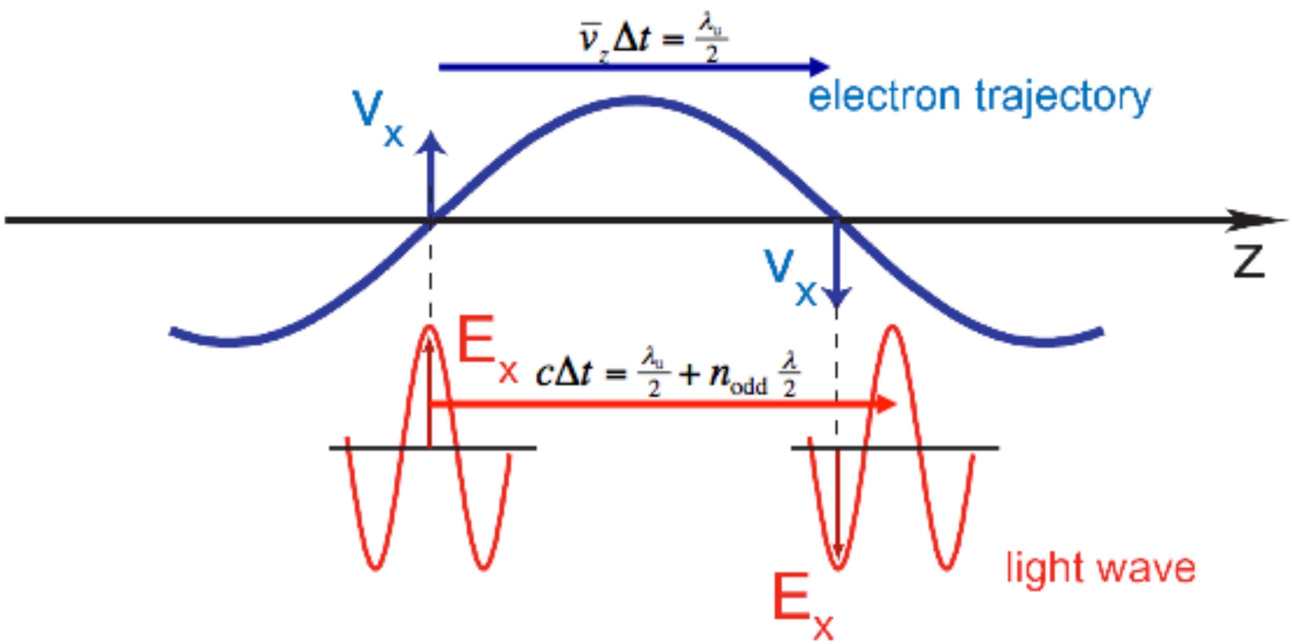


When the light wave passes the electron beam by half a wavelength per half beam oscillation, the radiation from each beam oscillation adds.



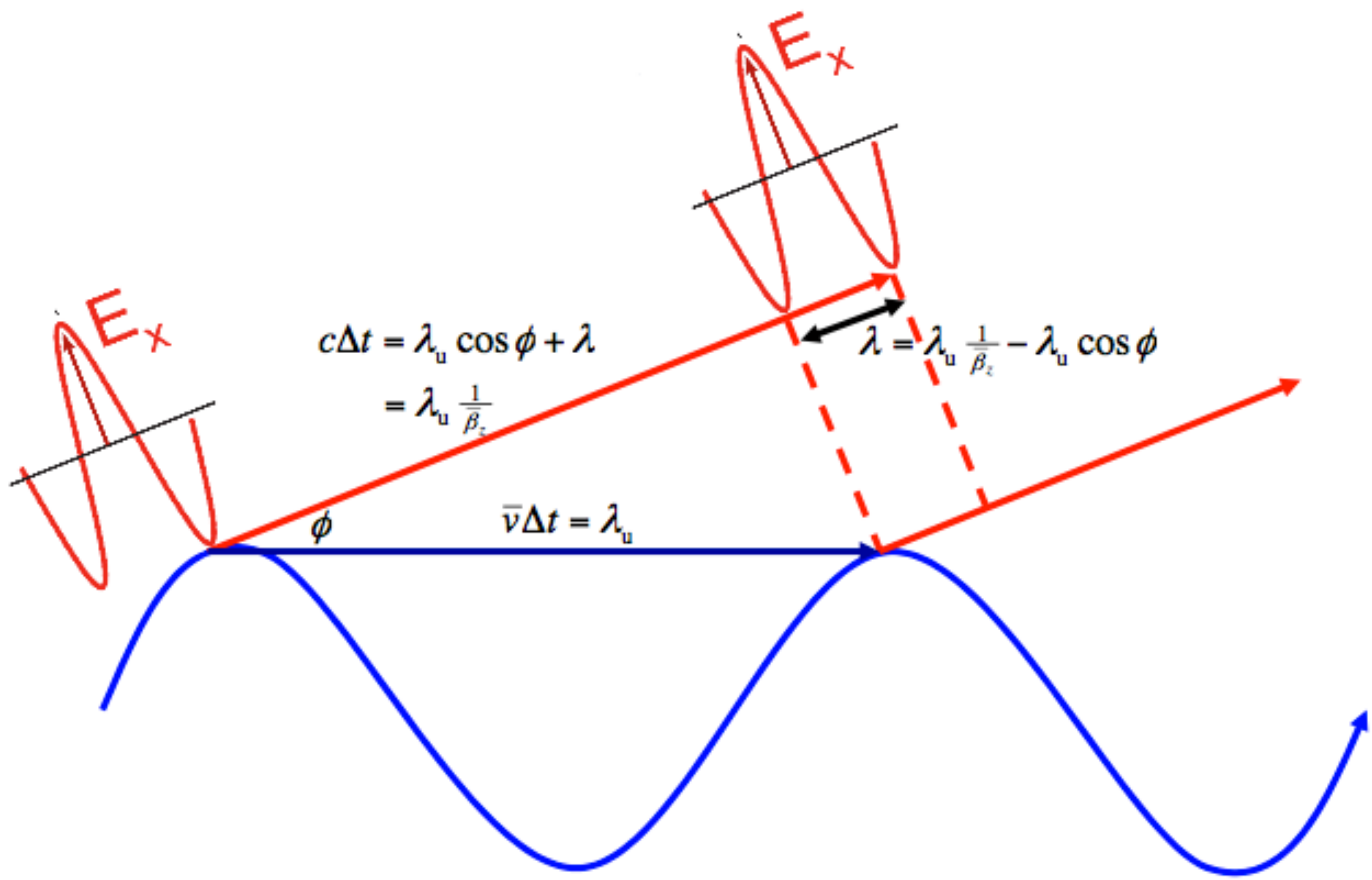
Similar adding can happen for 3, 5, 7, etc. half beam oscillations. For every strong frequency there are odd order harmonics.

Coherent addition of radiation



$$\frac{c}{\bar{v}_z}\frac{\lambda_u}{2} = \frac{\lambda_u}{2} + n_{odd}\frac{\lambda}{2} \quad \Rightarrow \quad \boxed{\lambda = \frac{1}{n_{odd}}\lambda_u\left(\frac{1}{\bar{\beta}_z} - 1\right)}$$

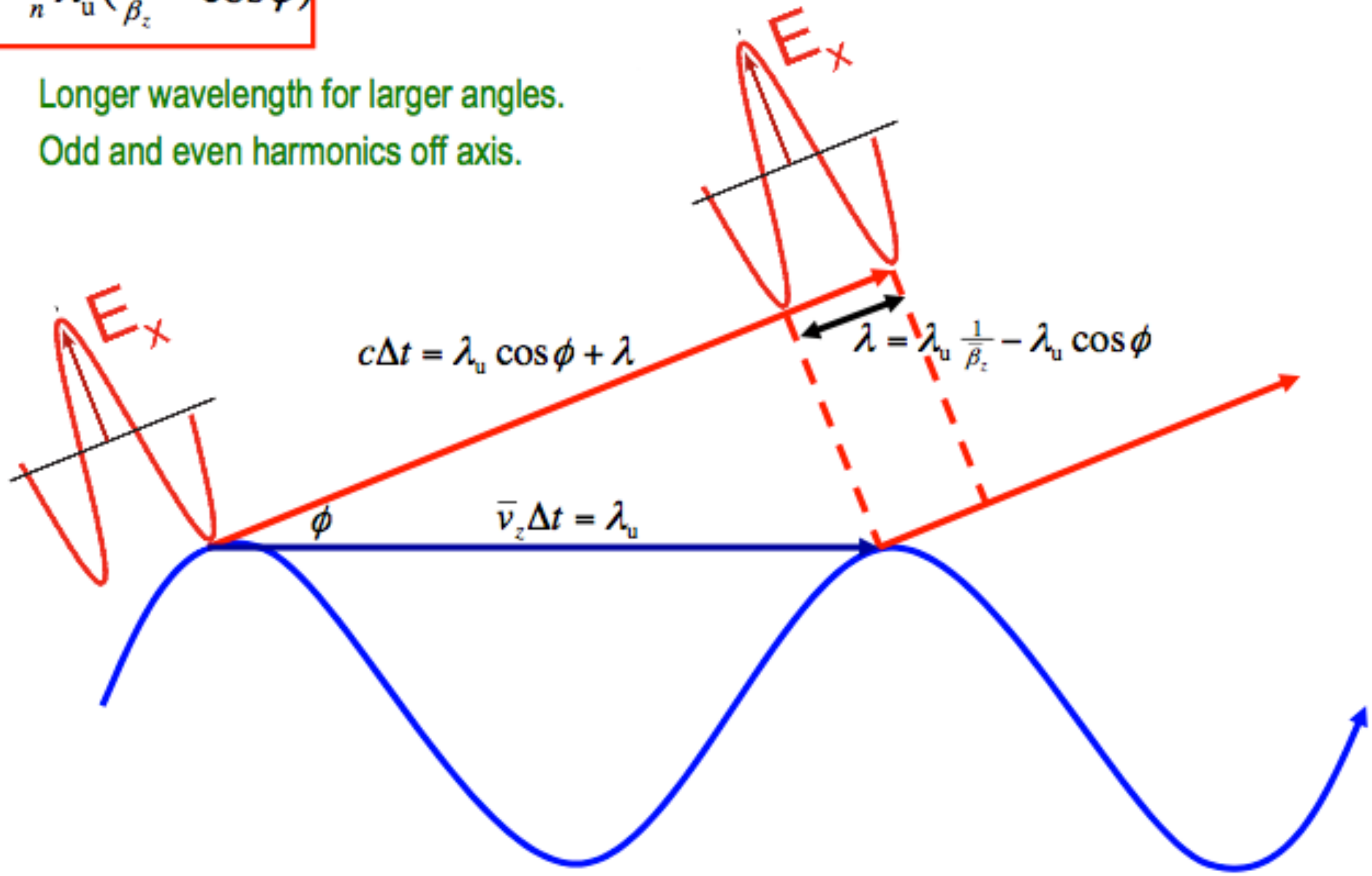
Coherent addition at angles



Radiation production at angles

$$\lambda = \frac{1}{n}\lambda_u\left(\frac{1}{\beta_z} - \cos\phi\right)$$

1) Longer wavelength for larger angles.
2) Odd and even harmonics off axis.



Lasing at the JLAB FEL

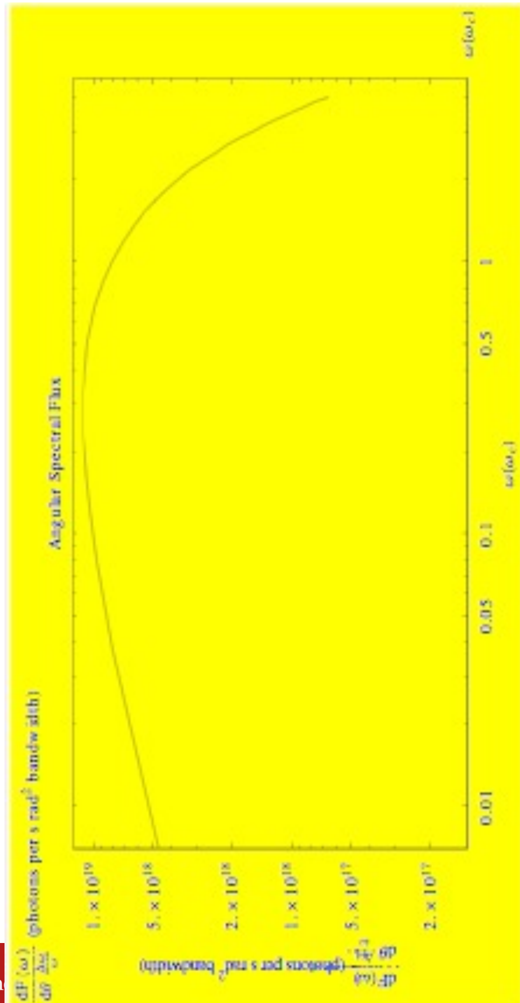
Wiggler gap

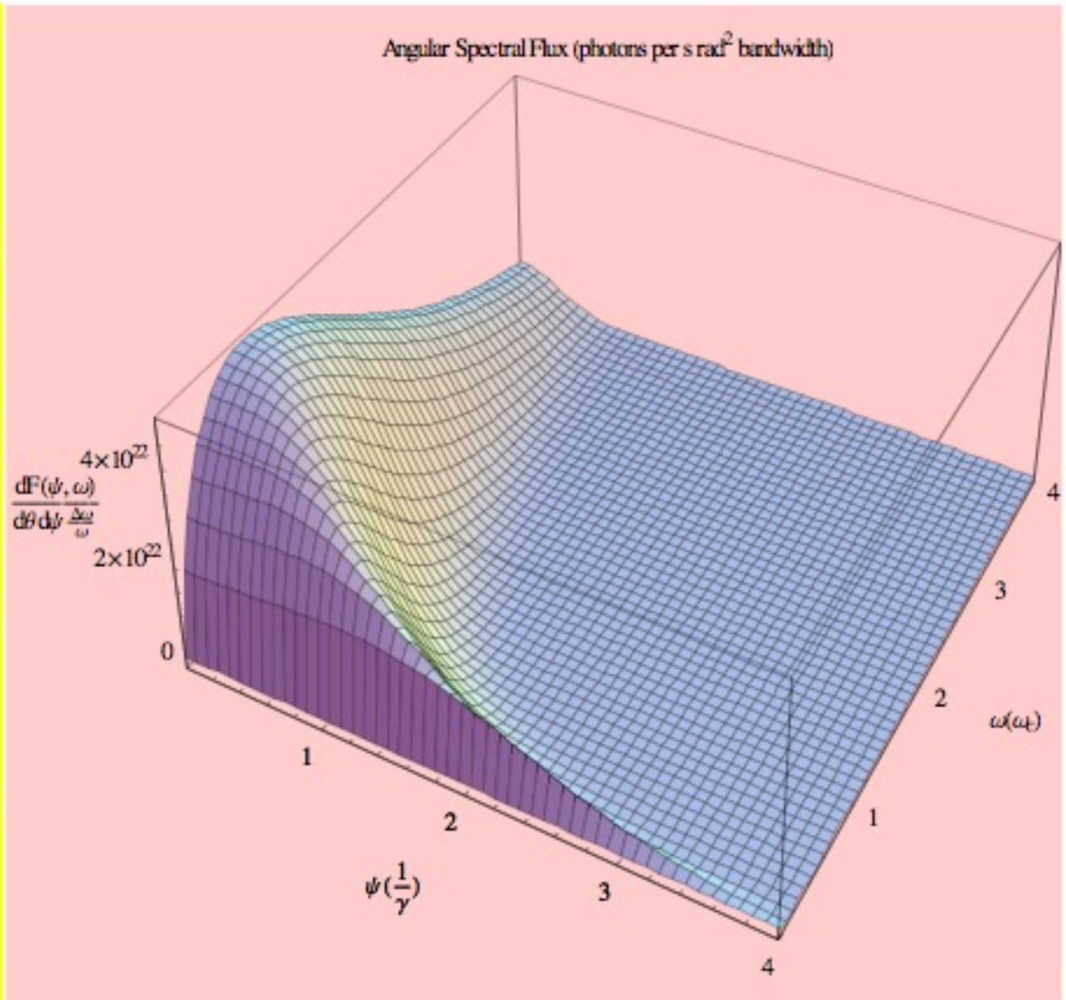
High Reflector

Hole Outcoupler

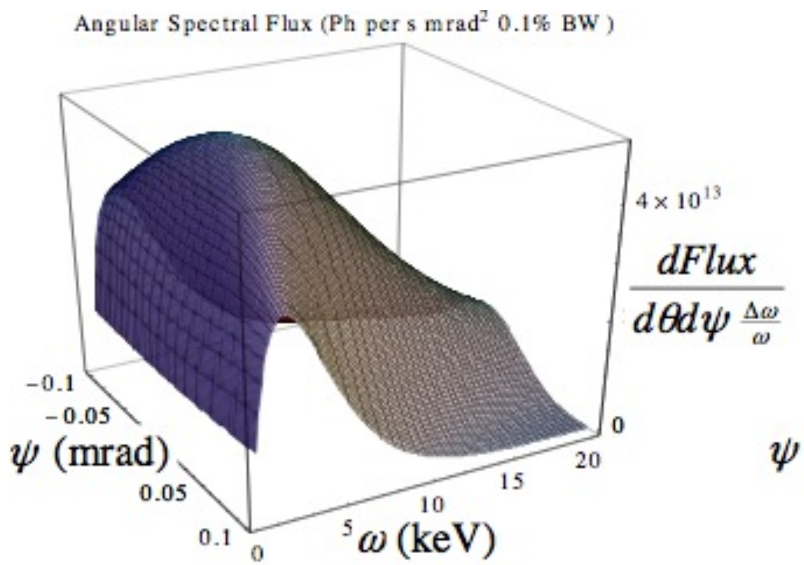
Beam in control room

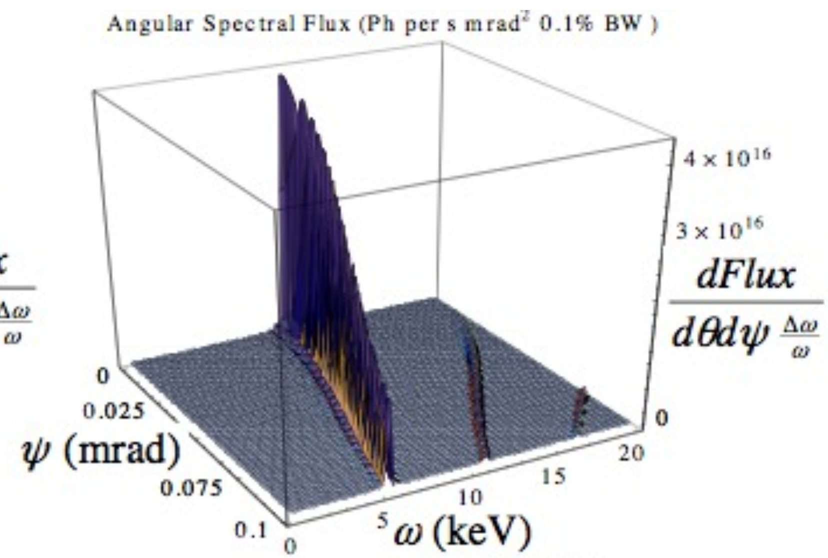
Radiative from bending magnets

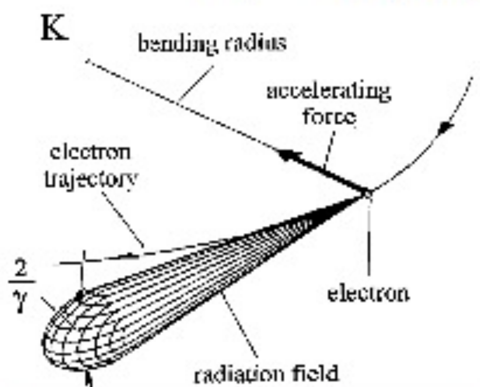


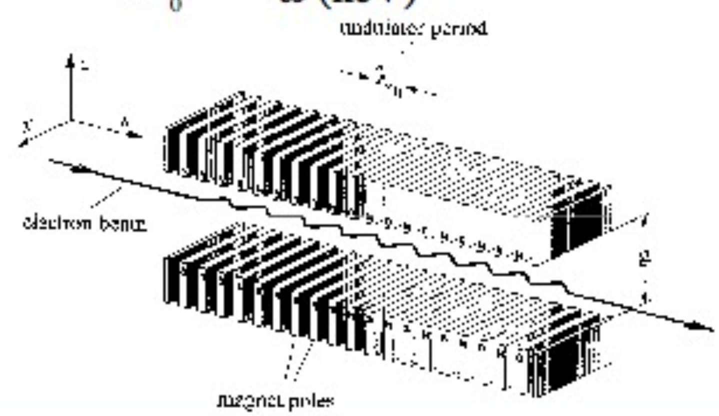


Photon flux from bends and undulators

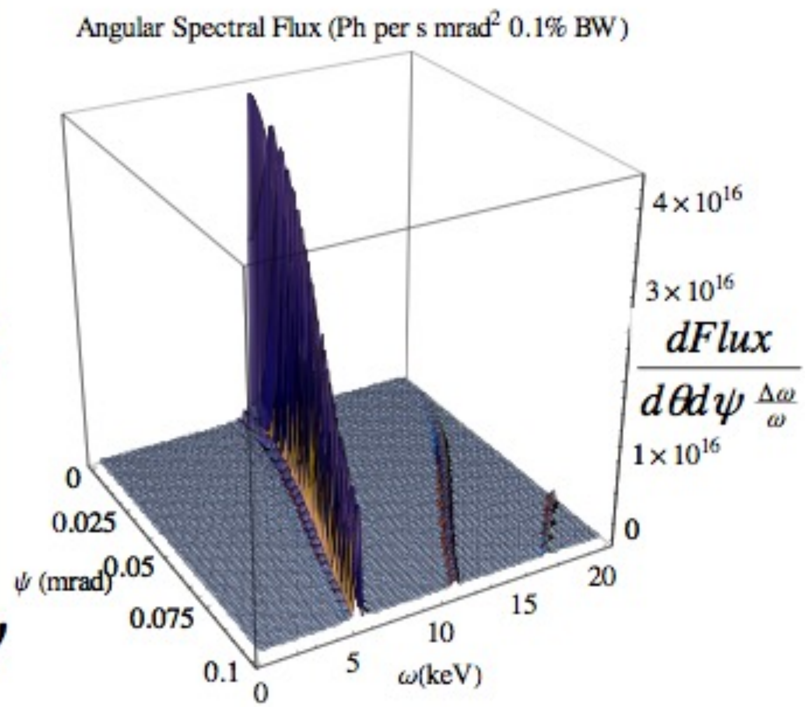








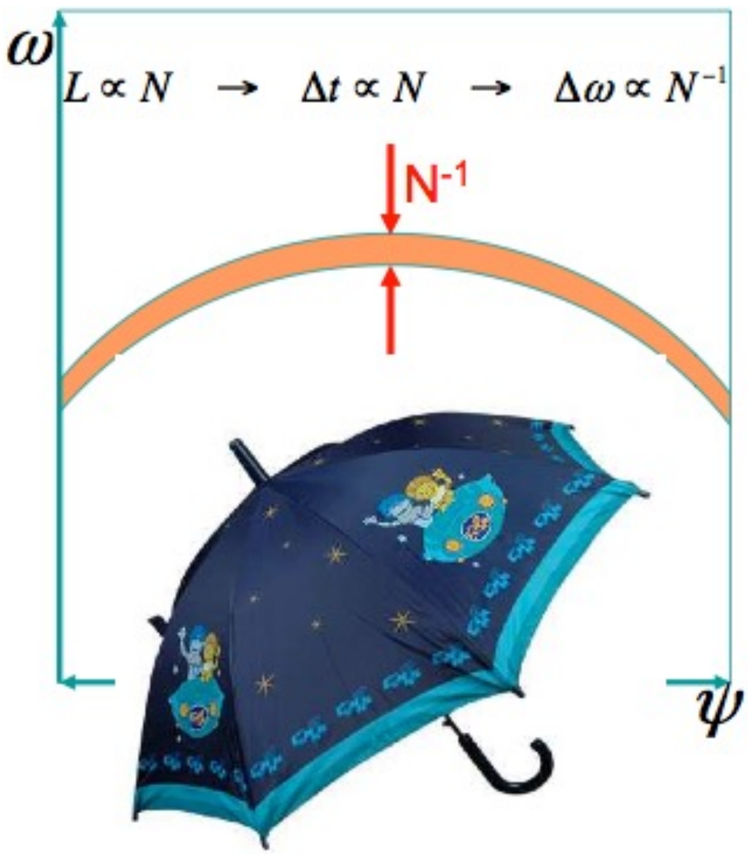
The umbrella of N-pole undulator radiation

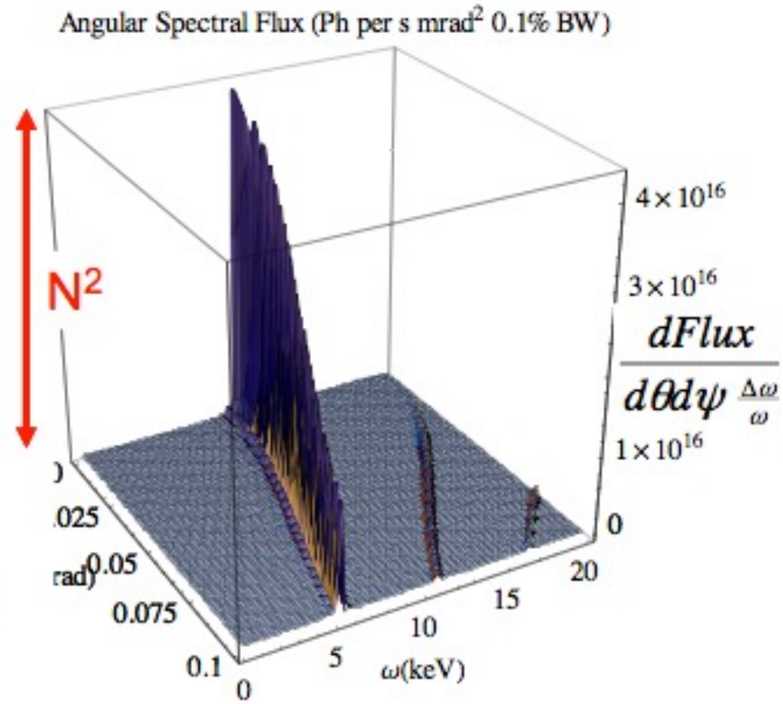


$$F_{total} \propto N$$

Flux from N poles is N times the flux from one pole

The umbrella of N-pole undulator radiation





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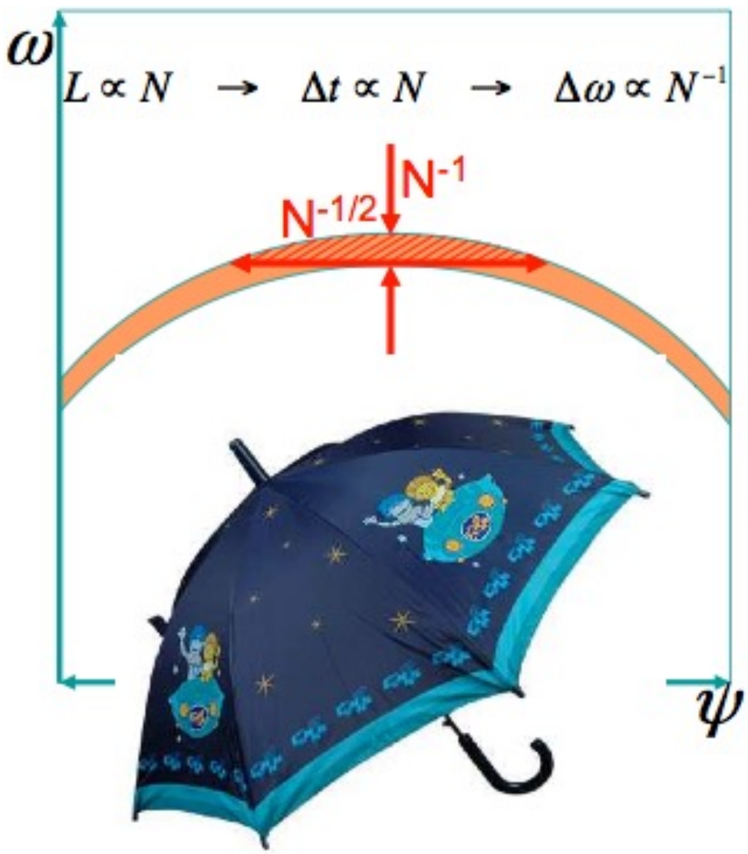
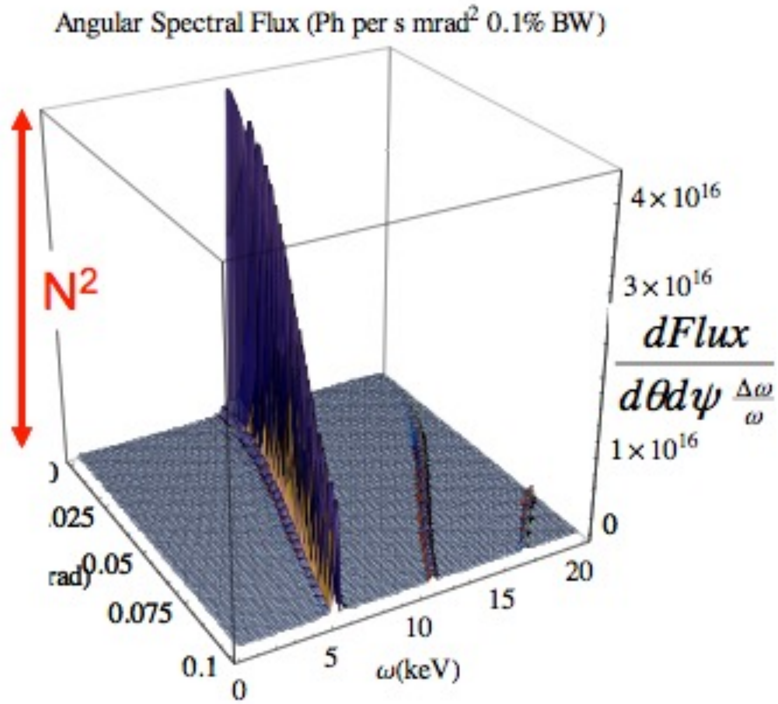
The umbrella of N-pole undulator radiation



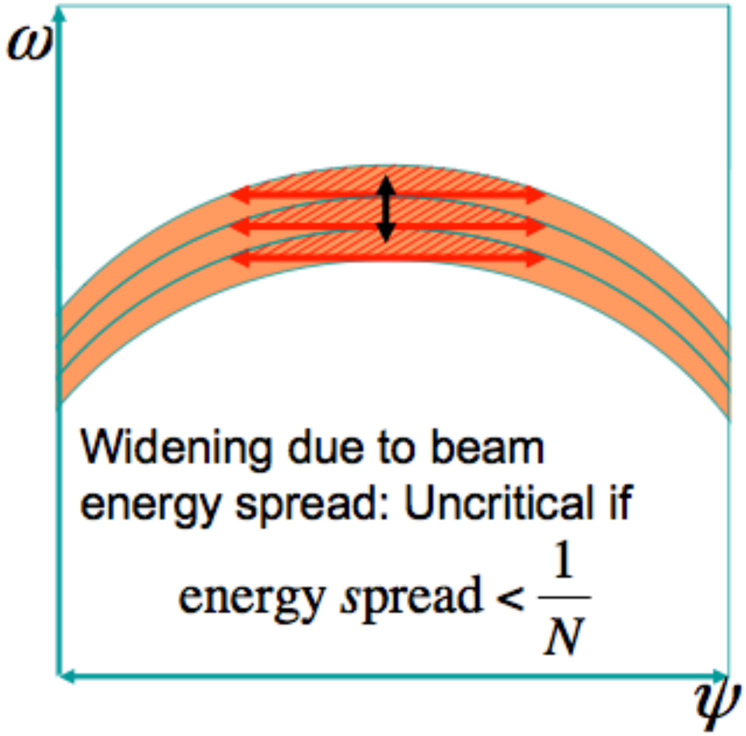


$F_{total} \propto N$

The power in the central cone is in dependent of N

Flux from N poles is N times the flux from one pole

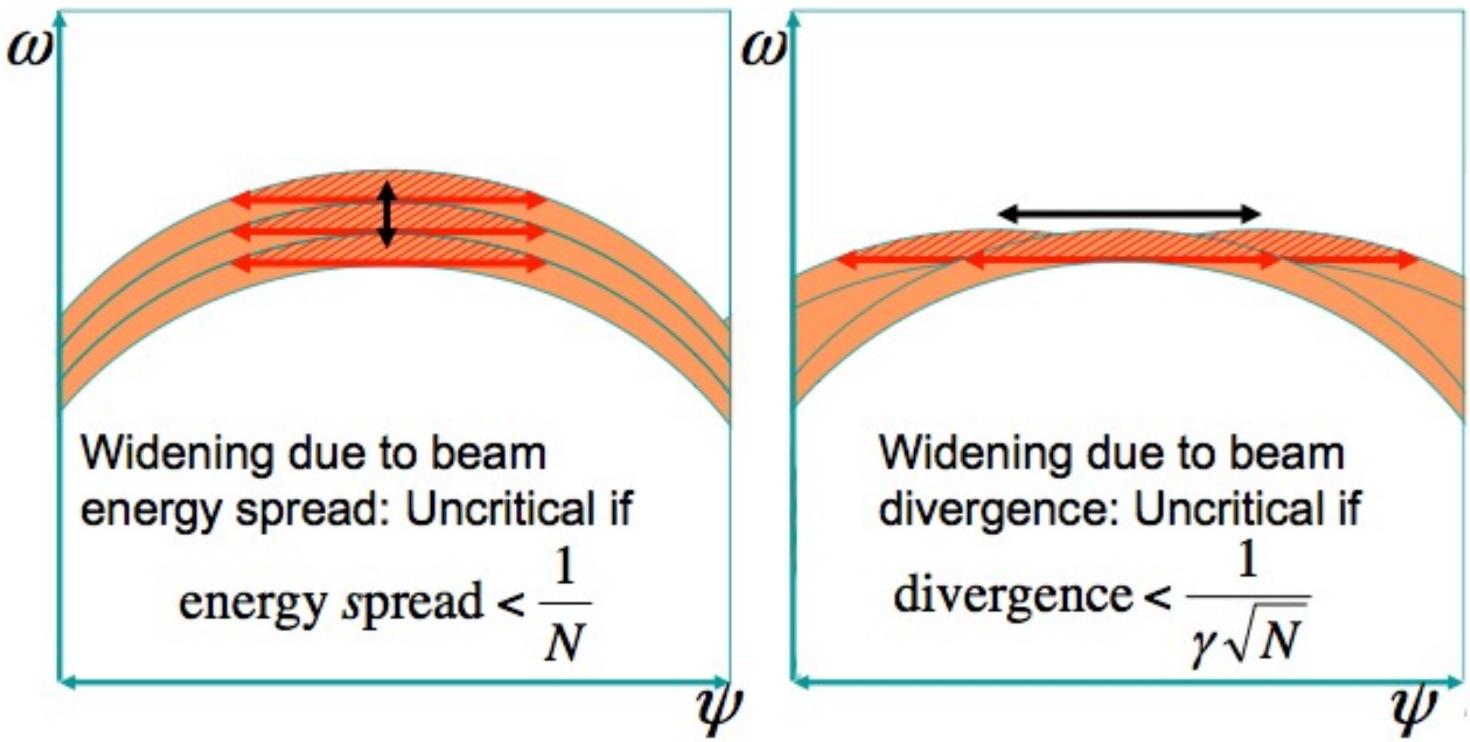
Brightness reduction by beam properties



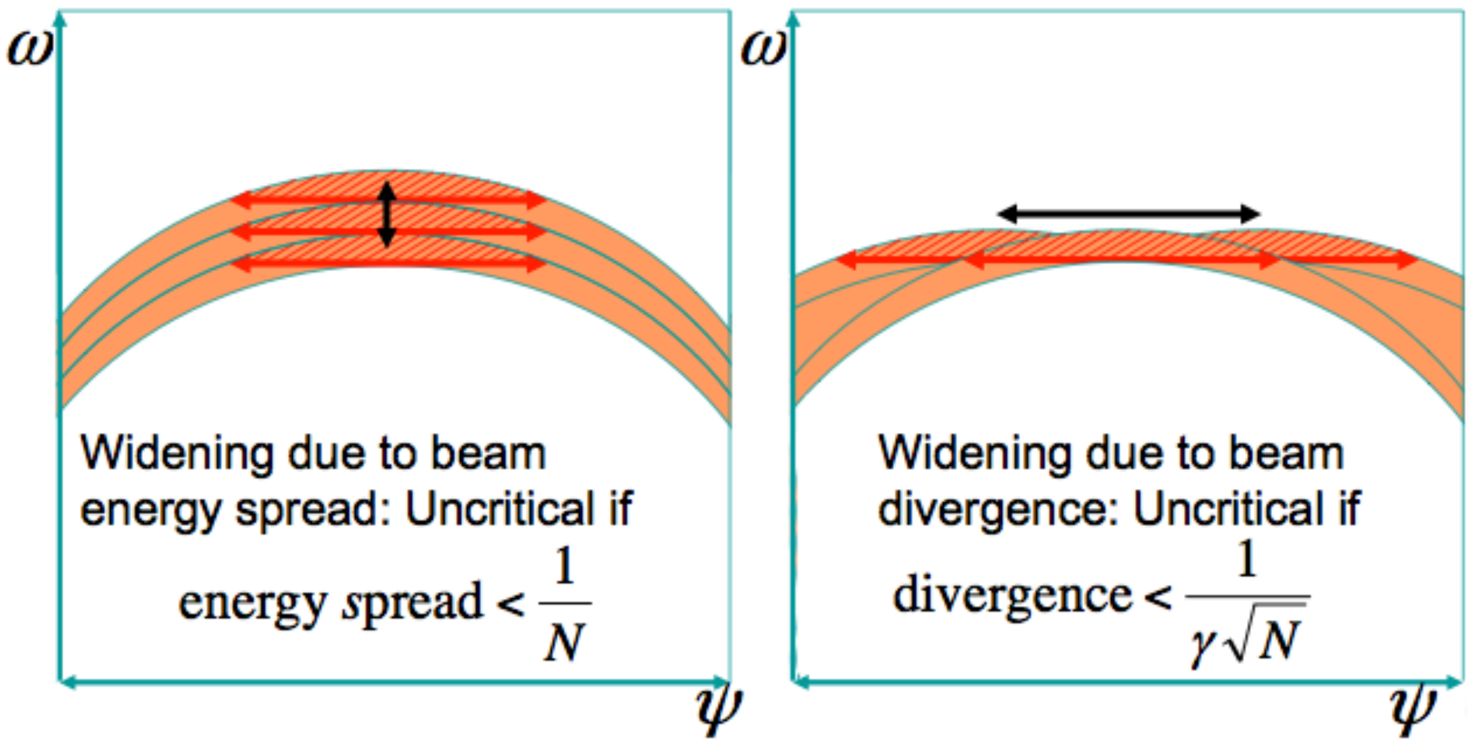
Widening due to beam energy spread: Uncritical if

$$\text{energy spread} < \frac{1}{N}$$

Brightness reduction by beam properties



Brightness reduction by beam properties

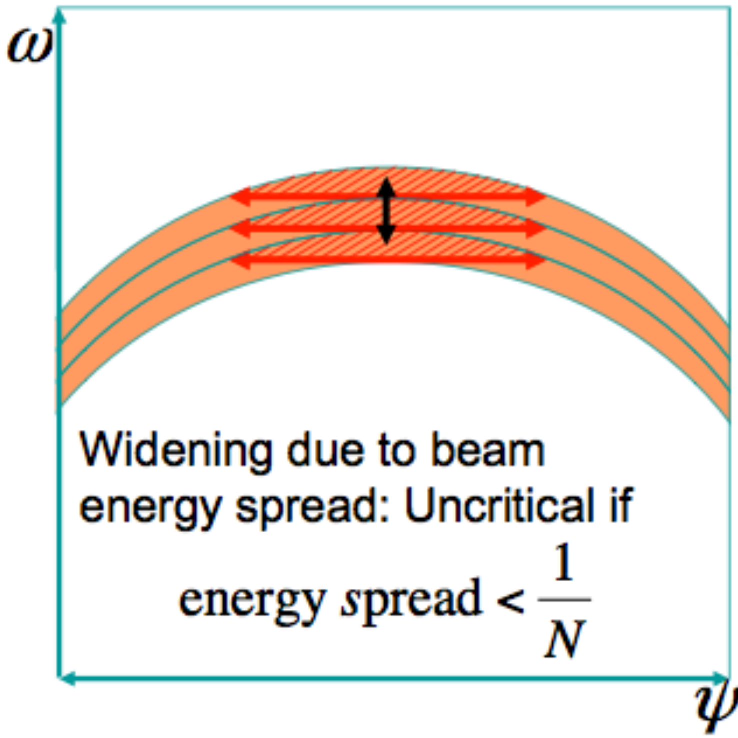


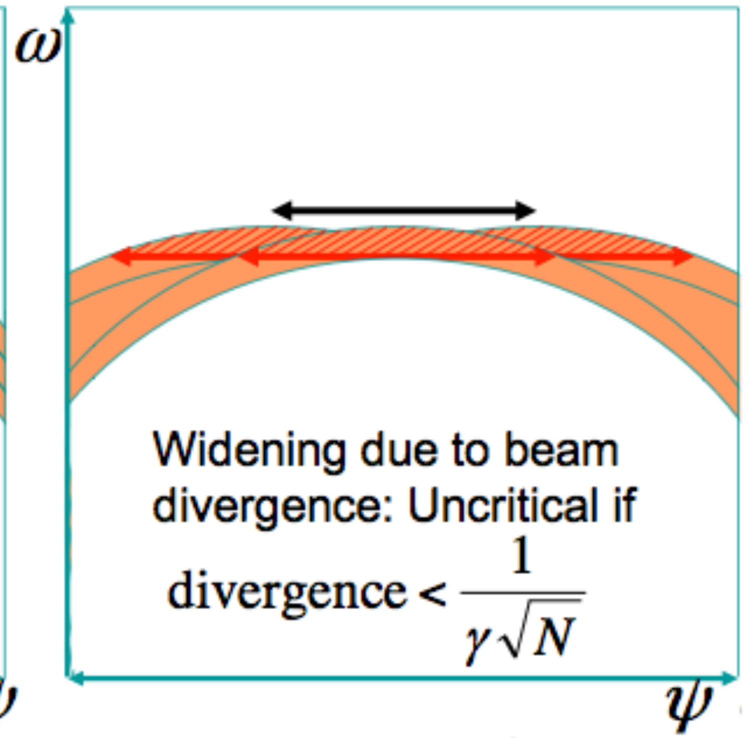
Field from a single electron cannot be distinguished from field from a spot with:

$$\text{spot size} < \frac{\lambda}{\text{divergence}}$$

Cornell

Brightness reduction by beam properties





Field from a single electron cannot be distinguished from field from a spot with:

$$\text{spot size} < \frac{\lambda}{\text{divergence}}$$

To take advantage of many undulator poles, the electron beam needs to have little energy spread, little divergence, and small beam size.

Dynamical Systems

$\vec{z}(s) = \vec{M}(s; s_0, \vec{z}_0)$ dynamical variable $\vec{z}$

Flow, transport map $\vec{M}$

By referring to a reference trajectory, transport maps in accelerators become origin preserving: $\vec{M}(s; s_0, \vec{0}) = \vec{0}$

Flows build a group under concatenation:

$$\vec{M}(s; s_1, \cdot) \circ \vec{M}(s_1; s_0, \vec{z}_0) = \vec{M}(s; s_1, \vec{M}(s_1; s_0, \vec{z}_0)) = \vec{M}(s; s_0, \vec{z}_0)$$

1) Identity element: $\vec{M}(s; s_0, \vec{z}) = \vec{z}$

2) Inverse element of $\vec{M}(s; s_0, \vec{z}) = \vec{M}^{-1}(s; s_0, \vec{z})$ is $\vec{M}(s_0; s, \vec{z})$

In physics, the flow is often given as a solution of a first order ODE $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z}, s)$

(Note that an nth order ODE can be rewritten as an n-dimensional first order ODE.)

Uniqueness

Note that not all ODEs $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z},s)$

have a unique solution $\vec{z}(s)$

through a given point $\vec{z}(0) = \vec{z}_0$

Picard-Lindeloef:

A unique solution through $(\vec{z}_0, s_0)$ exists for $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z},s)$ if $\vec{f}(\vec{z},s)$ is Lipschitz continuous and bounded,

i.e. it is continuous, bounded, and there is a number $\boldsymbol{N}$ such that

$$|\vec{f}(\vec{z}_1,s) - \vec{f}(\vec{z}_2,s)| < N|\vec{z}_1 - \vec{z}_2|$$

Example: $H = \frac{1}{2}p^2 + V(q)\,,\quad V(q) = -8\sqrt{|q|}^3 \quad \Rightarrow \quad \dot{q} = p\,,\quad \dot{p} = 12\sqrt{|q|}$

<u>There are two solutions through the point (q,p,t)=(0,0,0)</u>

1. $(q(t), p(t)) = (0,0)$
2. $(q(t), p(t)) = (t^4, 4t^3) \quad \Rightarrow \quad (\dot{q}, \dot{p}) = (4t^3, 12t^2)$

In our following treatments we do require uniqueness of solutions.

Linear Systems

Linear ODEs in N dimesions $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z}, s)$ have $\vec{f}(\lambda\vec{z}, s) = \lambda\vec{f}(\vec{z}, s)$

$$\frac{d}{ds}\vec{z} = \underline{L}(s)\vec{z}$$

There are N linearly independent solutions. For example $\vec{z}_n(s)$ going through $(\vec{z}_0, s_0)$
With $z_{0i} = 0$ for $i \neq n$ and $z_{0n} = 1$

$$\vec{z}_0 = (0, \ldots, 1, \ldots, 0)^T \quad \Rightarrow \quad \vec{z}_n(s)$$

One speaks of N fundamental solutions.

Superposition for linear ODEs:
If z1 is a solution and z2 is a solution, then
any linear combination Az1 +Bz2 is also a solution

$$\frac{d}{ds}\vec{z}_1 = \underline{L}(s)\vec{z}_1 \quad \& \quad \frac{d}{ds}\vec{z}_2 = \underline{L}(s)\vec{z}_2 \quad \Rightarrow \quad \frac{d}{ds}(A\vec{z}_1 + B\vec{z}_2) = \underline{L}(s)(A\vec{z}_1 + B\vec{z}_2)$$

Therefore any solution through $(\vec{z}_0, s_0)$ can be written as $\vec{z}(s) = \sum_{n=1}^{N} \vec{z}_n(s) z_{0n}$

$$\vec{z}(s) = \vec{M}(s; s_0, \vec{z}_0) = \underline{M}(s, s_0)\vec{z}_0 \qquad \underline{M}(s, s_0) = (\vec{z}_1(s), \ldots, \vec{z}_N(s))$$

Nonlinear Systems

Noninear ODEs in N dimesions $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z}, s)$

Have no fundamental solutions. Each solution has to be determined separately for each initial condition.

Examples: Plasma, Galaxies

$$H(\ldots,\vec{r}_j,\ldots,\ldots,\vec{p}_j,\ldots) = \sum_j \frac{\vec{p}_j^{\,2}}{2m_j} + \sum_{k\neq j} \frac{q_j q_k}{|\vec{r}_j - \vec{r}_k|}$$

Finding a general solution, flow, or transport map can be very hard.
This has not even been possible for the 3 body problem.

$$\vec{z}(s) = \vec{M}(s; s_0, \vec{z}_0)$$

Weakly Nonlinear Systems

Weakly nonlinear ODEs $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z},s)$

Have a right hand side that can be

approximated well by a truncated Taylor expansion

$$\vec{f}(\vec{z},s) \approx \underline{L}(s)\vec{z} + \sum_{j,k} \vec{f}_{jk} z_j z_k + \sum_{j,k,l} \vec{f}_{jkl} z_j z_k z_l + \ldots + \sum_{\vec{k},\text{order O}} \vec{f}_{\vec{k}} \vec{z}^{\vec{k}} + \ldots$$

$$\vec{z}^{\vec{k}} = \prod_{n=1}^{N} z_n^{k_n} , \quad \sum_{\vec{k},\text{order O}} \ldots = \sum_{n=1}^{N} \sum_{k_n} \ldots \quad \text{with} \quad \sum_{n=1}^{N} k_n = O$$

By solving the Taylor expanded ODE one tries to find a Taylor expansion of the transport map: $\vec{M}(s;s_0,\vec{z}_0) \approx \underline{M}(s,s_0)\vec{z}_0 + \ldots + \sum_{\vec{k},\text{order O}} \vec{M}_{\vec{k}} \vec{z}_0^{\vec{k}} + \ldots$

Note:

While this approach is usually chosen, it is not certain that a transport map of the Taylor expanded ODE is a Taylor expansion of the transport map of the original ODE. One therefore often speaks of “formally” finding the Taylor expansion of the transport map.

Aberrations and Sensitivities

$$\vec{M}(s;s_0,\vec{z}_0) \approx \underline{M}(s,s_0)\vec{z}_0 + \ldots + \sum_{\vec{k},\text{order } O} \vec{M}_{\vec{k}} \vec{z}_0^{\,\vec{k}} + \ldots$$

The Taylor coefficients are called aberrations of order O and are denoted by

$$(z_i, z_1^{k_1} \ldots z_6^{k_6}) \equiv M_{\vec{k},i}\ , \quad \text{order} \quad O = \sum_{n=1}^{6} k_n$$

Parameter dependences lead to sensitivities:

$$\tfrac{d}{ds}\vec{z} = \vec{f}(\vec{z},s,\varepsilon) \quad \Rightarrow \quad \vec{z}(s) = \vec{M}(s,\varepsilon;s_0,\vec{z}_0)$$

$$\vec{M}(s,\varepsilon;s_0,\vec{z}_0) \approx \underline{M}(s,s_0)\vec{z}_0 + \underline{M}^1(s,s_0)\vec{z}_0\varepsilon + \ldots + \sum_{\vec{k},n,\text{order } O} \vec{M}_{\vec{k}}^{\,`n} \vec{z}_0^{\,\vec{k}} \varepsilon^n$$

$$(z_i, z_1^{k_1} \ldots z_6^{k_6}\varepsilon^n) \equiv M_{\vec{k},i}^{n}\ , \quad \text{order} \quad O = n + \sum_{j=1}^{6} k_j$$

How can all these Taylor coefficients be computed?

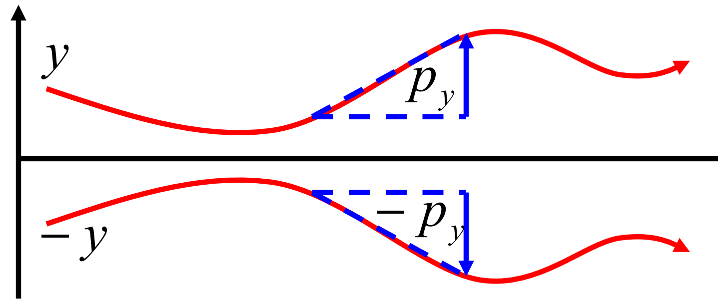
Horizontal Midplane Symmetry

This is the most important symmetry in nearly all accelerators.

$$\vec{r}^{\oplus} = (x, -y, z)$$

$$\vec{p}^{\oplus} = (p_x, -p_y, p_z)$$

$$\tfrac{d}{dt}\vec{p} = \vec{F}(\vec{r}, \vec{p}) \quad \Rightarrow \quad \tfrac{d}{dt}\vec{p}^{\oplus} = \vec{F}(\vec{r}^{\oplus}, \vec{p}^{\oplus})$$



$$\vec{z} = (x, a, y, b, \tau, \delta) \qquad \vec{z}(s) = \vec{M}(s, \vec{z}_0)$$

$$\vec{z}^{\oplus} = (x, a, -y, -b, \tau, \delta) \qquad \vec{z}^{\oplus}(s) = \vec{M}(s, \vec{z}_0^{\oplus})$$

$$M_i(s, \vec{z}_0^{\oplus}) = M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{1,2,5,6\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = -M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{3,4\}$$

$$(x, x_0^{k_1} \dots \delta_0^{k_6}) = 0 \quad \text{for} \quad k_3 + k_4 \text{ is odd}$$ similarly for a, τ and δ

$$(y, x_0^{k_1} \dots \delta_0^{k_6}) = 0 \quad \text{for} \quad k_3 + k_4 \text{ is even}$$ similarly for b

Double Midplane Symmetry

In addition to midplane symmetry, some elements are symmetric around the vertical plane, e.g. quadrupoles, glass lenses

$$\vec{z} = (x, a, y, b, \tau, \delta) \qquad \vec{z}\,(s) = \vec{M}(s, \vec{z}_0)$$

$$\vec{z}^{\oplus} = (x, a, -y, -b, \tau, \delta) \qquad \vec{z}^{\oplus}(s) = \vec{M}(s, \vec{z}_0^{\oplus})$$

$$\vec{z}^{\otimes} = (-x, -a, y, b, \tau, \delta) \qquad \vec{z}^{\otimes}(s) = \vec{M}(s, \vec{z}_0^{\otimes})$$

$$M_i(s, \vec{z}_0^{\oplus}) = M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{1,2,5,6\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = -M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{3,4\}$$

$$M_i(s, \vec{z}_0^{\otimes}) = -M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{1,2\}$$

$$M_i(s, \vec{z}_0^{\otimes}) = M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{3,4,5,6\}$$

$$(x, x_0^{k_1} \ldots \delta_0^{k_6}) = 0 \quad \text{for} \quad k_1 + k_2 \text{ is even or } k_3 + k_4 \text{ is odd}$$ similarly for a, τ and δ

$$(y, x_0^{k_1} \ldots \delta_0^{k_6}) = 0 \quad \text{for} \quad k_1 + k_2 \text{ is odd or } k_3 + k_4 \text{ is even}$$ similarly for b

Rotational Symmetry

Some optical elements are completely rotationally symmetric in the x-y plane, e.g. solenoid magnets, many glass lenses

$$w = x + iy\,,\quad \alpha = a + ib$$

$$\vec{z} = (w, \overline{w}, \alpha, \overline{\alpha}, \tau, \delta)$$

$$\vec{z}^{\oplus} = (e^{i\varphi}w, e^{-i\varphi}\overline{w}, e^{i\varphi}\alpha, e^{-i\varphi}\overline{\alpha}, \tau, \delta)$$

$$\vec{z}(s) = \vec{M}(s, \vec{z}_0)$$

$$\vec{z}^{\oplus}(s) = \vec{M}(s, \vec{z}_0^{\oplus})$$

$$M_i(s, \vec{z}_0^{\oplus}) = e^{i\varphi} M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{1,3\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = e^{-i\varphi} M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{2,4\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{5,6\}$$

$$(w, w^{k_1} \ldots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq 1$$ similarly for α

$$(\overline{w}, w^{k_1} \ldots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq -1$$ similarly for α*

$$(\tau, w^{k_1} \ldots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq 0$$ similarly for δ

$$(w, |w_0|^2 w_0), (\overline{\alpha}, |w_0|^2 \overline{w}_0), (\alpha, w_0^2 \overline{\alpha}_0), (\tau, |w_0|^2)$$ can all be non-zero

C_n Symmetry

Some optical elements have C_n symmetric in the x-y plane, e.g. C_2 for quadrupole, C_3 for sextupoles, etc.

$$w = x + iy\,,\quad \alpha = a + ib$$

$$\vec{z} = (w, \overline{w}, \alpha, \overline{\alpha}, \tau, \delta)$$

$$\vec{z}^{\oplus} = (e^{i\frac{2\pi}{n}} w, e^{-i\frac{2\pi}{n}} \overline{w}, e^{i\frac{2\pi}{n}} \alpha, e^{-i\frac{2\pi}{n}} \overline{\alpha}, \tau, \delta)$$

$$\vec{z}(s) = \vec{M}(s, \vec{z}_0)$$

$$\vec{z}^{\oplus}(s) = \vec{M}(s, \vec{z}_0^{\oplus})$$

$$M_i(s, \vec{z}_0^{\oplus}) = e^{i\frac{2\pi}{n}} M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{1,3\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = e^{-i\frac{2\pi}{n}} M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{2,4\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{5,6\}$$

$(w, w^{k_1} \ldots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq jn + 1$ similarly for α

$(\overline{w}, w^{k_1} \ldots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq jn - 1$ similarly for α^*

$(\tau, w^{k_1} \ldots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq jn$ similarly for δ

$(w, \overline{w}_0), (\overline{\alpha}, |w_0|^2 w_0), (\alpha, \overline{w}_0^2 \alpha_0), (\tau, |w_0|^2)$ can all be non-zero for C_2

Symplecticity

$$[\vec{\partial}\vec{M}^T]^T \underline{J}[\vec{\partial}\vec{M}^T] = \underline{J}$$

Symplecticity leads to the requirement that sums over certain products of aberrations must be either 0 or 1.

Separation into linear an nonlinear part of the map: $\vec{M}(\vec{z}) = \underline{M}_1(\vec{z} + \vec{N}(\vec{z}))$

$$\underline{M}(\vec{z}) = [\vec{\partial}\vec{M}^T]^T = [(\underline{1} + \vec{\partial}\vec{N}^T)\underline{M}_1^T]^T = \underline{M}_1(\underline{1} + \underline{N}(\vec{z}))$$

$$(1 + \underline{N})^T \underline{M}_1^T \underline{J}\underline{M}_1(1 + \underline{N}) = \underline{J} \quad \Rightarrow \quad \underline{M}_1^T \underline{J}\underline{M}_1 = \underline{J}\,, \quad \underline{N}^T \underline{J} + \underline{J}\underline{N} = -\underline{N}^T \underline{J}\underline{N}$$

For the leading order n-1 (the first order that appears in N): $\underline{N}^T \underline{J} + \underline{J}\underline{N} = 0 + O^n$

N is a Hamiltonian matrix up to order n and can thus be written up to order n as: $\vec{N}(\vec{z}) = \underline{J}\vec{\partial} f(\vec{z}) + O^{n+1}$

$$w(s_f) = (x, x_0)\partial_a f + (x, y_0)\partial_b f + i[(y, x_0)\partial_a f + (y, y_0)\partial_b f]$$

$$= (w, x_0)\partial_a f + (w, y_0)\partial_b f$$

$$= \tfrac{1}{2}[(w, w_0) + (w, \overline{w}_0)][\partial_\alpha f + \partial_{\overline{\alpha}} f] - \tfrac{1}{2}[(w, w_0) - (w, \overline{w}_0)][\partial_\alpha f - \partial_{\overline{\alpha}} f]$$

$$= (w, w_0)\partial_{\overline{\alpha}} f + (w, \overline{w}_0)\partial_\alpha f = (w, w_0)\partial_{\overline{\alpha}} f$$

Special Aberrations

Dispersion (for δ as parameter of 4-dimensional motion) $\vec{z} = \underline{M}(s)\vec{z}_0 + \vec{D}(s)\delta$

Chromatic aberrations $(x,\ldots\delta^n)\,,\quad n\neq 0$

Geometric aberrations $(x, x^{k_1}a^{k_2}y^{k_3}b^{k_4}\ldots)\,,\quad \sum_{i=1}^{4}k_i\neq 0$

Purely Geometric aberrations $(x,\ldots\delta^n)\,,\quad n=0$

Opening aberrations $(x, x^{k_1}\ldots y^{k_2}\ldots)\,,\quad k_1+k_2=0$

Field aberrations $(x, x^{k_1}\ldots y^{k_2}\ldots)\,,\quad k_1+k_2\neq 0$

Spherical imaging systems: $(w,\alpha)=0$

Spherical aberration for rotational symmetry $(w,\alpha|\alpha|^2)$

Coma line $(w,w|\alpha|^2)$

Coma circle $(w,\overline{w}\alpha^2)$

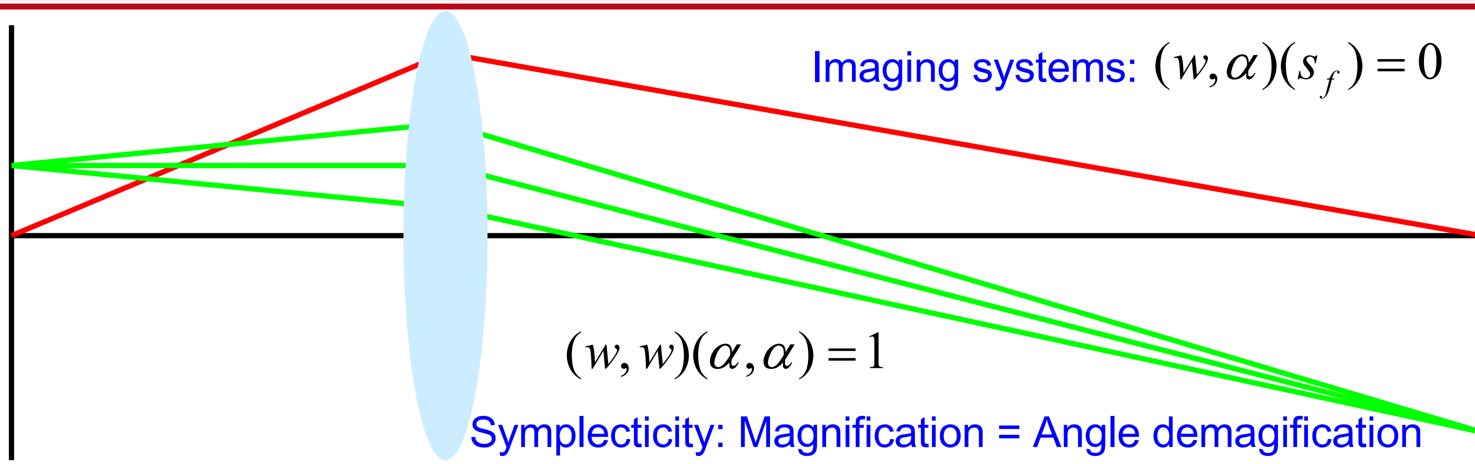
Astigmatism $(w,w^2\overline{\alpha})$

Curvature of Image $(w,|w|^2\alpha)$

Distortion $(w,w|w|^2)$

Aberrations for rotational symmetry

Imaging systems: $(w,\alpha)(s_f) = 0$

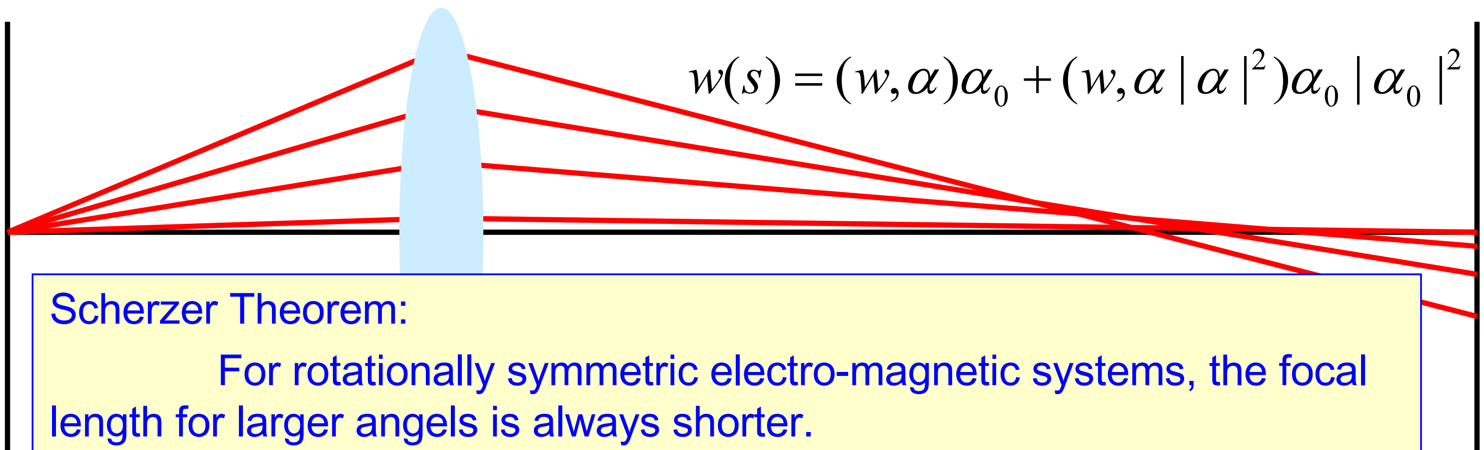


$$(w,w)(\alpha,\alpha) = 1$$

Symplecticity: Magnification = Angle demagification

Spherical aberration for rotational symmetry $(w,\alpha\,|\,\alpha\,|^2)$

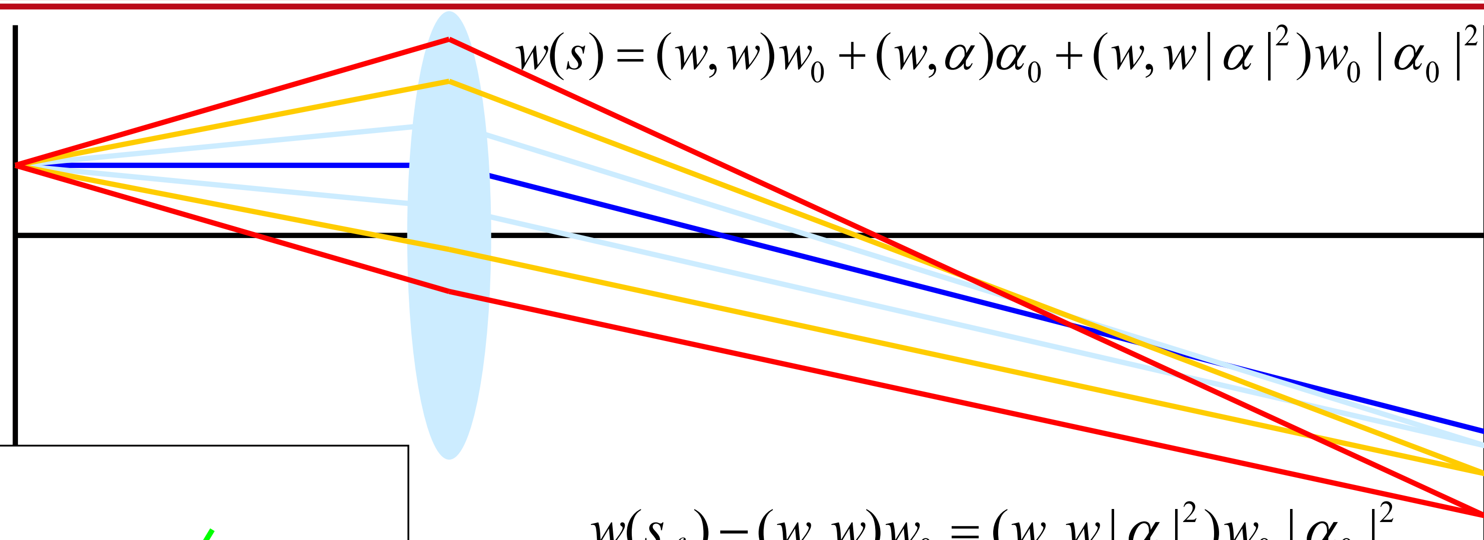
$$w(s) = (w,\alpha)\alpha_0 + (w,\alpha\,|\,\alpha\,|^2)\alpha_0\,|\,\alpha_0\,|^2$$



Scherzer Theorem:

For rotationally symmetric electro-magnetic systems, the focal length for larger angels is always shorter.

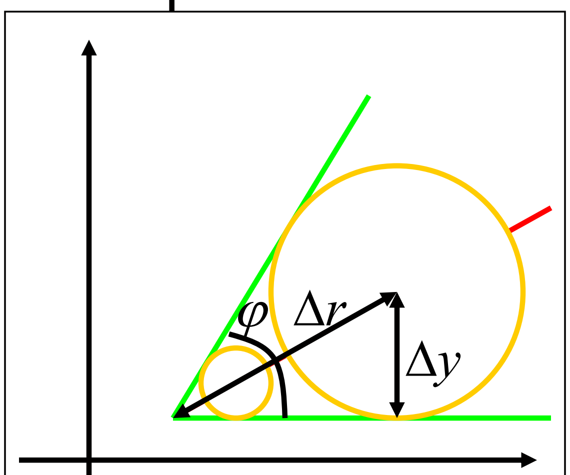
Koma Line and Koma Circle



$$w(s) = (w,w)w_0 + (w,\alpha)\alpha_0 + (w,w|\alpha|^2)w_0|\alpha_0|^2$$

$$w(s_f) - (w,w)w_0 = (w,w|\alpha|^2)w_0|\alpha_0|^2$$
$$+ (w,\overline{w}\alpha^2)\overline{w}_0\alpha_0^2$$

$$\varphi = 2\arcsin(\tfrac{\Delta y}{\Delta r}) = 2\arcsin(\tfrac{(w,\overline{w}\alpha^2)}{(w,w|\alpha|^2)})$$



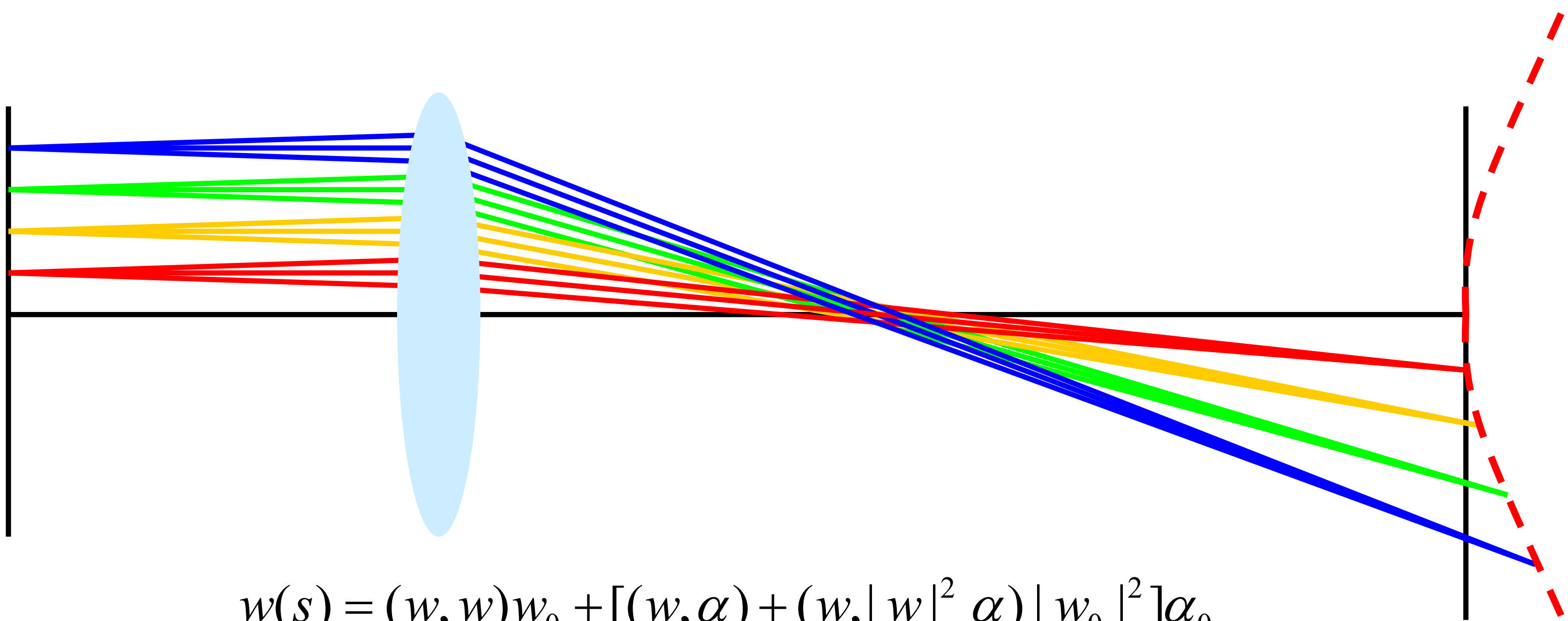
Symplecticity yields:

$$(w,w|\alpha|^2) = 2(w,\overline{w}\alpha^2) \quad \Rightarrow \quad \varphi = 60°$$

Since:

$$w(s_f) = (w,w_0)\partial_{\overline{\alpha}}[\ldots + \mathrm{Re}\{Kw\alpha\overline{\alpha}^2\}] = (w,w_0)[\ldots + Kw\alpha\overline{\alpha} + \tfrac{1}{2}\overline{K}\overline{w}\alpha^2]$$

Curvature of Image

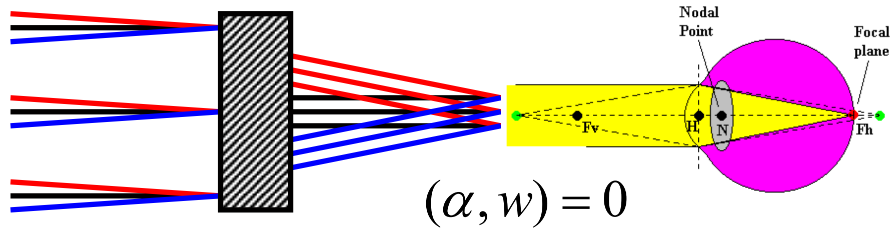


$$w(s) = (w,w)w_0 + [(w,\alpha) + (w,|w|^2\alpha)|w_0|^2]\alpha_0$$

The focus occurs at $(w,\alpha)(s_f) + (w,|w|^2\alpha)(s_f)|w_0|^2 = 0$

Other special systems

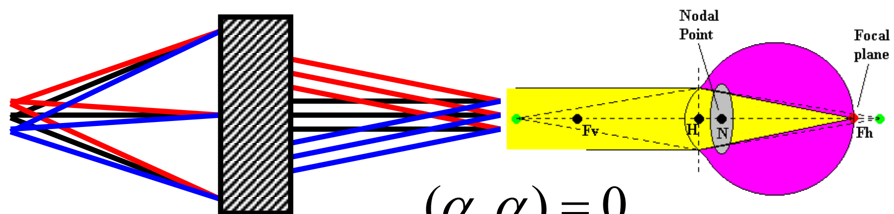
Telescope:
parallel to parallel system



$$(\alpha, w) = 0$$

Nonlinearly corrected telescope: $(\alpha, w^n) = 0$

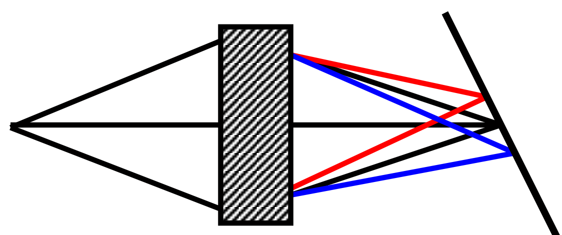
Microscope:
point to parallel system



$$(\alpha, \alpha) = 0$$

Nonlinearly corrected microscope: $(\alpha, \alpha^n) = 0$

Spectrograph:
point to parallel system



(x, δ) large
$(x, a) = 0$
(x, x) small

Nonlinearly corrected spectrograph: $(x, a^n b^m) = 0$

Tilt of focal plane: $(x, a\delta) \neq 0$ the focus is at $(x,a)(s_f) + (x, a\delta)(s_f)\delta = 0$

Variation of constants

$$\vec{z}' = \vec{f}(\vec{z}, s)$$

$$\vec{z}' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s)$$ Field errors, nonlinear fields, etc can lead to $\Delta\vec{f}(\vec{z}, s)$

$$\vec{z}_H' = \underline{L}(s)\vec{z}_H \quad \Rightarrow \quad \vec{z}_H(s) = \underline{M}(s)\vec{z}_{H0} \quad \text{with} \quad \underline{M}'(s)\vec{a} = \underline{L}(s)\underline{M}(s)\vec{a}$$

$$\vec{z}(s) = \underline{M}(s)\vec{a}(s) \quad \Rightarrow \quad \vec{z}'(s) = \underline{M}'(s)\vec{a} + \underline{M}(s)\vec{a}'(s) = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s)$$

$$\vec{a}(s) = \vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\, d\hat{s}$$

$$\vec{z}(s) = \underline{M}(s)\left\{\vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\, d\hat{s}\right\}$$

$$= \vec{z}_H(s) + \int_0^s \underline{M}(s, \hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\, d\hat{s}$$

Perturbations are propagated from s to s’

Iteration of Aberrations

$$\vec{z}(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vec{z}_1(s) = \vec{z}_H(s)$$

$$\vec{z}_2(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_1(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vdots$$

$$\vec{z}_n(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_{n-1}(\hat{s}),\hat{s})\,d\hat{s}$$

Taylor expansions: $\Delta\vec{f}(\vec{z},s) = \Delta\vec{f}_2(\vec{z},s) + \Delta\vec{f}_3(\vec{z},s) + \ldots\ , \quad \Delta f_O = \sum_{\vec{k}\text{ ,order } O} \vec{f}_{\vec{k}}\,\vec{z}^{\,\vec{k}}$

$$\vec{z}_1(s) = \underline{M}(s)\vec{z}_0$$

$$\vec{z}_2(s) = \underline{M}(s)\vec{z}_0 + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}_2(\vec{z}_1(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vec{z}_3(s) = \underline{M}(s)\vec{z}_0 + \int_0^s \underline{M}(s,\hat{s})\{[\Delta\vec{f}_2(\vec{z}_2(\hat{s}),\hat{s})]_3 + \Delta\vec{f}_3(\vec{z}_1(\hat{s}),\hat{s})\}\,d\hat{s}$$

$$\vdots$$

Poisson Bracket

The Poisson Bracket is defined as

$$[f(\vec{z}), g(\vec{z})] = \sum_i \partial_{q_j} f \, \partial_{p_j} g - \partial_{p_j} f \, \partial_{q_j} g = \vec{\partial}^T f \, \underline{J} \, \vec{\partial} g$$

The Poisson Bracket can be viewed as a product on the vector space of phase space functions. It is:

1) Linear: $[f, ag] = [af, g] = a[f, g], \quad a \in IR$
2) Distributive: $[f, g + h] = [f, g] + [f, h]$

This turns the vector space into an algebra.

The multiplication is furthermore:

1) Anti-commutative: $[f, g] = -[g, f]$
2) Has a Jacobi-identity: $[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$

   as can be proven by the product rule: $[f, gh] = g[f, h] + [f, g]h$

This turns the algebra into a Lie algebra.

Example: $\vec{a} \times \vec{b}$ turns IR^3 into a Lie algebra.

Map computation by Lie Algebra

The Poisson-Bracket operator of f, $:g:$ is defined as $:g:h=[g,h]$

$$:H:g=[H,g]=-[g,H]=-\vec{\partial}^T g \underline{J}\,\vec{\partial}H=-\vec{\partial}^T g\,\tfrac{d}{ds}\vec{z}=-\tfrac{d}{ds}g(\vec{z})$$

$$\tfrac{d}{ds}\vec{z}=\vec{f}(\vec{z},s)\quad\Rightarrow\quad -:H:z_j=\tfrac{d}{ds}z_j=f_j(\vec{z},s)\,,\quad -:H:f_j=\tfrac{d}{ds}f_j-\tfrac{\partial}{\partial s}f_j$$

In the main field region where $\tfrac{d}{ds}\vec{z}=\vec{f}(\vec{z})\quad\Rightarrow\quad -:H:f_j=\tfrac{d}{ds}f_j=\tfrac{d^2}{ds^2}z_j$

$$\text{If } g(\vec{z})=\tfrac{d^n}{ds^n}z_j \text{ then } -:H:g=\tfrac{d}{ds}g=\tfrac{d^{n+1}}{ds^{n+1}}z_j\quad\Rightarrow\quad (-:H:)^n z_j=\tfrac{d^n}{ds^n}z_j$$

Propagator: $e^{-\Delta s:H:}\vec{z}=\sum_{n=0}^{\infty}\frac{(-\Delta s:H:)^n}{n!}\vec{z}=\sum_{n=0}^{\infty}\frac{\Delta s^n}{n!}\frac{d^n}{ds^n}\vec{z}=\vec{M}(s+\Delta s;s,\vec{z})$

$$\vec{M}_2\circ\vec{M}_1(\vec{z}_0)=\vec{M}_2(\Delta s_2,\vec{z}(\Delta s_1))=\sum_{n=1}^{\infty}\frac{(-\Delta s_1:H_1:)^n}{n!}\vec{M}_2(\Delta s_2,\vec{z}_0)$$

$$=e^{-\Delta s_1:H_1(\vec{z}_0):}e^{-\Delta s_2:H_2(\vec{z}_0):}\vec{z}_0$$

$$\vec{M}(s;\vec{z}_0)=\vec{M}_n\circ\ldots\circ\vec{M}_2\circ\vec{M}_1(\vec{z}_0)=e^{-\Delta s_1:H_1:}\ldots e^{-\Delta s_n:H_n:}\vec{z}_0$$

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Poisson Bracket Invariance

The Poisson Bracket is invariant under a symplectic transfer map

$$[f(\vec{M}(\vec{z})), g(\vec{M}(\vec{z}))] = \vec{\partial}^T f\Big|_{\vec{M}} \underline{M}\,\underline{J}\,\underline{M}^T \vec{\partial} g\Big|_{\vec{M}} = [f(\vec{z}), g(\vec{z})]\Big|_{\vec{M}(\vec{z})}$$

For nonlinear expansions, one writes the transport map as a linear matrix

and a nonlinear Lie exponent,

$$\vec{M}_1(\vec{z}) = \underline{M}_1 e^{:H_1(\vec{z}):^n} \vec{z} = \underline{M}_1 \sum_{n=0}^{\infty} \frac{:H_1^n:}{n!} \vec{z}$$

since a linear Lie exponent requires infinitely many terms in the power sum, but the nonlinear exponent terminates when a finite order expansion is sought.

$$(\underline{M}_2 e^{:H_2^n(\vec{z}):} \vec{z}) \circ (\underline{M}_1 e^{:H_1^n(\vec{z}):} \vec{z}) = \underline{M}_2 e^{:H_2^n(\underline{M}_1 e^{:H_1^n(\vec{z}):} \vec{z}):} \underline{M}_1 e^{:H_1^n(\vec{z}):} \vec{z}$$

$$= \underline{M}_2 \underline{M}_1 e^{:H_2^n(\underline{M}_1 e^{:H_1^n(\vec{z}):} \vec{z}):} e^{:H_1^n(\vec{z}):} \vec{z} = \underline{M}_2 \underline{M}_1 e^{:H_1^n(\vec{z}):} e^{:H_2^n(\underline{M}_1 \vec{z}):} \vec{z}$$

When these equations are used to compute and manipulate transfer maps, one speaks of the Lie algebraic method.

Computing Taylor Expansions

$$f(x) \approx \sum_{n=1}^{\text{order } O} \tfrac{x^n}{n!} \partial^n f\Big|_0$$

But taking this approach for complicated functions would be very cumbersome:

1. $$f(x) = \frac{1}{1+\sin x} - 1\,, \quad f(0) = 0\,, \quad \partial f\big|_0 = \frac{-\cos x}{(1+\sin x)^2}\bigg|_0 = -1\,,$$

$$\partial^2 f(x) = \frac{\sin x(1+\sin x) + 2\cos^2 x}{(1+\sin x)^3}\bigg|_0 = 2\,, \quad f(x) \approx -x + x^2 + O^3$$

2. $$f(x) = \frac{1}{1+\sin x} - 1\,,$$

$$f(x) \approx \frac{1}{1 + x - \frac{1}{6}x^3 + O^4} - 1$$

$$\approx -(x - \tfrac{1}{6}x^3) + (x - \tfrac{1}{6}x^3)^2 - (x - \tfrac{1}{6}x^3)^3 + O^4$$

$$\approx -x + x^2 - \tfrac{5}{6}x^3 + O^4$$

This approach is formalized in the field of automatic differentiation using a Differential Algebra.

Computations with TPSA(n)

Computation of a function in IR is done by a finite number of elementary operations (+,-,x) and elementary function evaluations (sin, cos, exp, 1/x, …).

$$f(x) = \frac{1}{1+\sin x} - 1$$

1. $x \in IR$
2. $\sin$
3. $1+$
4. $1/$
5. -1

If $g_n(x)$ is the truncated power series of order n of g(x) and $h_n(x)$ is that of h(x)

we can look for elementary operations ("+","-","x") so that

g_n"+"h_n is the TPS(n) of g+h

g_n"-"h_n is the TPS(n) of g-h

g_n"x"h_n is the TPS(n) of gxh

Similarly we can look for elementary functions ("sin","cos","exp","1/x",…) so that

"sin"(g_n) is the TPS(n) of sin(g), "exp"(g_n) is the TPS(n) of exp(g), etc.

Evaluating all elementary operations and elementary functions in f(x) in terms of "+","…" starting with the TPS(n) of x, leads to the TPS(n) of f(x).

Automatic Differentiation with TPSA(n)

Example: computing the TPS(3) of $f(x)=\dfrac{1}{1+\sin x}-1$

1. TPS(3) of x is x
2. TPS(3) of "sin" x is $x-\frac{1}{6}x^3$
3. $1\text{"+"}x-\frac{1}{3}x^3=1+x-\frac{1}{6}x^3$
4. $i(x)=\frac{1}{1+x}, \text{"}i\text{"}(x)=1-x+x^2-x^3, \text{"}i\text{"}(x-\frac{1}{6}x^3)=1-x+x^2-\frac{5}{6}x^3$
5. $1-x+x^2-\frac{5}{6}x^3\text{"}-\text{"}1=-x+x^2-\frac{5}{6}x^3$

This automatically (i.e. with a computer) leads to derivatives of f(x):

$$f(0)=0,\ f'(0)=-1,\ f''(0)=2,\ f'''(0)=-4$$

Truncated power series can be added "+" and multiplied "x" and there is a neutral element of multiplication (i.e.1). Therefore the vector space of TPS(n) forms an algebra. It is called the Truncated Power Series Algebra TPSA(n).

The Differential Algebra ${}_1D_1$

An addition and multiplication with a scalar leads to a vector space over IR^2:

$$\{a_0,a_1\},\ \{b_0,b_1\}\in IR^2,\ t\in IR$$

$$\{a_0,a_1\}+\{b_0,b_1\}=\{a_0+b_0,a_1+b_1\}$$

$$t\{a_0,a_1\}=\{ta_0,ta_1\}$$

The introduction of a multiplication $\{a_0,a_1\}\{b_0,b_1\}=\{a_0b_0,a_0b_1+a_1b_0\}$
leads to an algebra if it is:

1) Distribut. $\{a_0,a_1\}(\{b_0,b_1\}+\{c_0,c_1\})=\{a_0,a_1\}\{b_0,b_1\}+\{a_0,a_1\}\{c_0,c_1\}$

2) Has a neutral element: $\{a_0,a_1\}\{1,0\}=\{a_0,a_1\}$

and additionally to a ring if it is

3) Commutative: $\{a_0,a_1\}\{b_0,b_1\}=\{b_0,b_1\}\{a_0,a_1\}$

4) Associative: $\{a_0,a_1\}(\{b_0,b_1\}\{c_0,c_1\})=(\{a_0,a_1\}\{b_0,b_1\})\{c_0,c_1\}$

All these properties are clearly given, since first order power expansion
have this multiplication: $(a_0+a_1x)(b_0+b_1x)=a_0b_0+(a_0b_1+a_1b_0)x+O^2$

The Differential Algebra $_1D_1$

By the introduced addition and multiplication we created an algebra, since the multiplication is commutative and associative we also created a ring, but not a field. Complex numbers are a field since there is a multiplicative inverse for all numbers except 0.

$$\{a_0, a_1\}\{b_0, b_1\} = \{a_0 b_0, a_0 b_1 + a_1 b_0\} \quad \Rightarrow \quad \{a_0, a_1\}\{\tfrac{1}{a_0}, -\tfrac{a_1}{a_0^2}\} = \{1,0\}$$

We further introduce a differentiation: $\partial\{a_0, a_1\} = \{a_1, 0\}$

It is a differentiation since it satisfies a product rule:

$$\partial(\{a_0, a_1\}\{b_0, b_1\}) = \{a_0 b_1 + a_1 b_0, 0\} = (\partial\{a_0, a_1\})\{b_0, b_1\} + \{a_0, a_1\}(\partial\{b_0, b_1\})$$

By adding a differentiation we have created a Differential Algebra (DA).

Differentiation of Polynomials: $f(x) = 2 + x^2 \quad \Rightarrow \quad f'(x) = 2x$

$$f(\{2,1\}) = \{2,0\} + \{4,4\} = \{6,4\} = \{f(2), f'(2)\}$$

Since $\{f, f'\} + \{g, g'\} = \{(f+g), (f+g)'\}$, $\{f, f'\}\{g, g'\} = \{(fg), (fg)'\}$

Every polynomial: $P(\{f, f'\}) = \{P(f), [P(f)]'\}$ and $P(\{x,1\}) = \{P(x), P'(x)\}$

Elementary Functions in ${}_1D_1$

$$e(a_0 + a_1 x) = e(a_0) + e'(a_0) a_1 x + O^2$$

leads to

$$e(\{a_0, a_1\}) = \{e(a_0), a_1 e'(a_0)\}$$

$$\sin(\{a_0, a_1\}) = \{\sin a_0, a_1 \cos a_0\}$$

$$\cos(\{a_0, a_1\}) = \{\cos a_0, -a_1 \sin a_0\}$$

Since $\{f, f'\} + \{g, g'\} = \{(f+g), (f+g)'\}$, $\{f, f'\}\{g, g'\} = \{(fg), (fg)'\}$

and $e(\{f, f'\}) = \{e(f), [e(f)]'\}$

Therefore $F(\{f, f'\}) = \{F(f), [F(f)]'\}$ and $F(\{x, 1\}) = \{F(x), F'(x)\}$

So that automatic differentiation works not only for Polynomials but for any function that is constructed from a finite number of operations and elementary functions.

Computer programs that have differential algebra elements as data types can evaluate any function or algorithm in this data type and obtain derivatives of the function or derivatives of the algorithm.

The Differential Algebra ${}_nD_v$

The concept of ${}_1D_1$ can be extended to truncated power series of order n and to v variables. This leads to the differential algebra ${}_nD_v$. For each coefficient in the nth order expansion there is one dimension in the vectors of ${}_nD_v$.

Power expansions for v variables have extremely many expansion coefficients:

A polynomial of order n in v variables has $\dim({}_nD_v) = \frac{(n+v)!}{n!v!}$ coefficients since

$$\underbrace{\dim({}_nD_v) - \dim({}_{n-1}D_v)}_{z_1^{k_1}\dots z_v^{k_v},\sum_{j=1}^{v} k_j = n} = \underbrace{\dim({}_nD_{v-1})}_{z_1^{k_1}\dots z_{v-1}^{k_{v-1}},\sum_{j=1}^{v} k_j \le n} \quad , \quad \frac{(n+v)!}{n!v!} - \frac{(n-1+v)!}{(n-1)!v!} = \frac{(n+v-1)!}{n!(v-1)!}$$

and iteration of ${}_nD_v$ starts with the correct conditions:

$$\dim({}_nD_1) = n+1 = \tfrac{(n+1)!}{n!}$$

$$\dim({}_0D_v) = 1 = \tfrac{v!}{v!}$$

Example: $\dim({}_{10}D_6) = 8008$

Computer programs that have differential algebra elements as data types produce the nth order power expansion of v-dimensional functions or algorithms automatically.

Equivalence classes in ${}_n D_v$

A TPS(n) of a function f(x) defines the equivalence class of all functions that have the same TPS(n).

$$\text{Def}: \quad f =_n g \quad \text{if} \quad \vec{\partial}^{\vec{k}} f(0) = \vec{\partial}^{\vec{k}} g(0) \quad \forall\, \vec{k} \text{ with order} \le n$$

$=_n$ is an equivalence relation since it has

1) the identity property $f =_n f \quad \forall f$
2) the symmetry property $f =_n g \quad if \quad g =_n f$
3) the transitivity property $f =_n h \quad if \quad f =_n g \; and \; g =_n h$

Equivalence classes: $\text{Def}: \quad [f]_n = \{g \mid g =_n f\}$

Arithmetic of equivalence class:

Those operations generate a differential algebra.

$$\begin{aligned}
[f]_n + [g]_n &\equiv [f+g]_n \\
[f]_n [g]_n &\equiv [f\,g]_n \\
t[f]_n &\equiv [t\,f]_n \\
\partial_j [f]_n &\equiv [\partial_j f]_{n-1} \\
e([f]_n) &\equiv [e(f)]_n
\end{aligned}$$

Concatenation of maps

$$f(x),\, g(x) \quad \text{and} \quad [f]_n, [g_n] \in_n D_1$$

$$[f(g(x))]_0 = [f(g(0))]_0$$

$$[f(g(x))]_1 = [f(g(0)) + g'(0) f'(g(0)) x]_1$$

The composition of two TPS(n) can only be computed if the first one is origin preserving, then $[f]_n \circ [g]_n \equiv [f(g(x))]_n$

If two maps that are know to order n and the first one is origin preserving, then the composition of the maps is known to order n.

$$[\vec{M}_1]_n, [\vec{M}_2]_n \in_n D_v^v$$

$$[\vec{M}_2]_n \circ [\vec{M}_1]_n = [\vec{M}_2(\vec{M}_1(\vec{z}))]_n$$

Therefore the reference trajectory is always chosen as origin for the maps accelerator elements.

Inversion of Maps

The nth order inverse of an origin preserving function can be computed within the differential algebra (DA):

$$\vec{M}(\vec{z}) = \underline{M}_1\vec{z} + \vec{N}(\vec{z})$$

$$\vec{M}\circ\vec{M}^{-1}(\vec{z}) = \underline{M}_1\vec{M}^{-1} + \vec{N}\circ\vec{M}^{-1} = \vec{z}$$

$$\vec{M}^{-1} = \underline{M}_1^{-1}(\vec{z} - \vec{N}\circ\vec{M}^{-1})$$

$$[\vec{M}^{-1}]_n = \underline{M}_1^{-1}[\vec{z} - \vec{N}\circ\vec{M}^{-1}]_n = \underline{M}_1^{-1}(\vec{z} - [\vec{N}]_n\circ[\vec{M}^{-1}]_{n-1})$$

Iterative computation of the inverse:

$$[\vec{M}^{-1}]_1 = [\underline{M}_1^{-1}\vec{z}]_1$$

$$[\vec{M}^{-1}]_2 = \underline{M}_1^{-1}(\vec{z} - [\vec{N}]_2\circ[\underline{M}_1^{-1}\vec{z}]_1)$$

$$[\vec{M}^{-1}]_3 = \underline{M}_1^{-1}(\vec{z} - [\vec{N}]_3\circ(\underline{M}_1^{-1}(\vec{z} - [\vec{N}]_2\circ[\underline{M}_1^{-1}\vec{z}]_1)))$$

Generating Functions

The motion of particles can be represented by Generating Functions

Each flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

With a Jacobi Matrix : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = \left(\vec{\partial}_0 \vec{M}^T\right)^T$

That is Symplectic: $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

Can be represented by a Generating Function:

$$F_1(\vec{q}, \vec{q}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1 \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1$$

$$F_2(\vec{p}, \vec{q}_0, s) \quad \text{with} \quad \vec{q} = \vec{\partial}_p F_2 \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_2$$

$$F_3(\vec{q}, \vec{p}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_3 \quad , \quad \vec{q}_0 = -\vec{\partial}_{p_0} F_3$$

$$F_4(\vec{p}, \vec{p}_0, s) \quad \text{with} \quad \vec{q} = \vec{\partial}_q F_4 \quad , \quad \vec{q}_0 = -\vec{\partial}_{p_0} F_4$$

6-dimensional motion needs only one function ! But to obtain the transport map this has to be inverted.

Computation of Generating Functions

For any map for which the TPS(n) is know, a TPS(n+1) of a generating function that produces this map can be computed. For example, looking for

$$F_1(\vec{q},\vec{q}_0,s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1(\vec{q},\vec{q}_0,s) \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s)$$

$\vec{z} = \vec{M}(\vec{z}_0)$ is given as TPS(n)

$$\begin{pmatrix} \vec{q} \\ \vec{q}_0 \end{pmatrix} = \begin{pmatrix} \vec{M}_q(\vec{z}_0) \\ \vec{q}_0 \end{pmatrix} = \vec{l}(\vec{z}_0) , \quad \begin{pmatrix} \vec{p}_0 \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \vec{p}_0 \\ \vec{M}_p(\vec{z}_0) \end{pmatrix} = \vec{h}(\vec{z}_0) = \underline{J}\left[\vec{\partial} F_1(\vec{q},\vec{q}_0)\right]_{\vec{l}(\vec{z}_0)}$$

$$\vec{\partial} F_1 = -\underline{J}\vec{h} \circ \vec{l}^{\,-1} \quad \Rightarrow F_1 = -\underline{J} \int_0^{(\vec{q},\vec{q}_0)} \vec{h} \circ \vec{l}^{\,-1}(\vec{Q})\, d\vec{Q}$$

$$[\vec{M}]_n \quad \Rightarrow [\vec{l}\,]_n, [\vec{h}]_n \quad \Rightarrow [\vec{l}^{\,-1}]_n, [\vec{h}]_n \circ [\vec{l}^{\,-1}]_n$$

$$[F_1]_{n+1} = -\underline{J} \int_0^{(\vec{q},\vec{q}_0)} [\vec{h}]_n \circ [\vec{l}^{\,-1}]_n \, d\vec{Q}$$

Particle coordinates (q0,p0) are propagated by such generating functions when zeros of the following equations are found numerically:

$$\vec{p} + \vec{\partial}_q F_1(\vec{q},\vec{q}_0,s) = \vec{0} \quad \text{and} \quad \vec{p}_0 - \vec{\partial}_{q_0} F_1(\vec{q},\vec{q}_0,s) = \vec{0}$$

The next Generation of Accelerator builders

Cornell University