USPAS summer 2025, Grad Accelerator Physics

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Homework Answers #1

Exercise 1 (Rutherford scattering): Bringing a nucleus with Z_1 protons a distance d to a nucleus with Z_2 protons requires an energy of $Z_1 Z_2 m_e c^2 \frac{r_e}{d}$ (with the classical electron radius $r_e = 2.8$fm and the electron's rest mass $m_e c^2 =$511keV. How close did the 7.7MeV alpha particles in Rutherford's scattering experiment get to the gold nucleus?
Answer: $d = Z_1 Z_2 m_e c^2 r_e / 7.7\text{MeV}$ with $Z_1 = 2$, $Z_2 = 79$ leads to $d = 2 * 79 * 511/7700 r_e = 10.5 r_e = 29$fm.

Exercise 2 (Magnetic rigidity): The momentum of an accelerator is sometimes specified in terms of its magnetic rigidity B-rho ($B\rho$) in units of Tesla meter. Derive a conversion of a particle's B-rho to its energy in GeV for highly relativistic particles.
Answer: $E = pc = qB\rho c$ leads to $E[J] = B\rho[\text{Tm}]\frac{\text{c[m/s]}}{\text{q[Q]}}$. And $E[GeV] = E[eV]10^{-9}$. The units get converted by $E[GeV] = E[J]\frac{1}{\text{e[Q]}}10^{-9}$. And for the charge $q = Ze$ one obtains $E[GeV] = 0.3 Z B\rho[\text{Tm}]$.

Exercise 3 (Isocyclotron): Find the radial dependence of the magnetic field $B_z(r)$ in an isocyclotron with angular frequency ω_z.
Answer: In an isocylcotron we have the following:

$$\omega_z = \frac{q}{\gamma m_0} B_z = \frac{qc^2}{E} B_z \quad \Longrightarrow \quad E = \frac{qc^2}{\omega_z} B_z$$

$$r(E) = \frac{P}{qB_z} = \frac{\sqrt{E^2 - m_0^2 c^4}}{qBc} \quad \Longrightarrow \quad B_z(r) = \frac{m_0 \omega_z}{q\sqrt{1 - \frac{\omega_z^2}{c^2} r^2}}$$

Exercise 4 (Microtron): Consider a microtron with one accelerating cavity ($l = 1$m, $g = 30$MV/m) and $\omega_{RF} = 2\pi \cdot 1.3 \times 10^9$Hz. What is the proper value of the magnetic field B?
Answer: In a microtron the energy that a particle gets in one turn is

$$\Delta E = n \frac{qBc^2}{\omega_{RF}} \tag{1}$$

If $n = 1$, the magnetic field we need in this case is

$$B = \frac{\omega_{RF}}{qc^2}\Delta E = \frac{2\pi \cdot 1.3 \times 10^9}{(2.998 \times 10^8)^2} \cdot 30 \times 10^6 = 2.73T \tag{2}$$

For other choices of n one has to use $B = \frac{1}{n}2.73T$.

Exercise 5 (Drift tube linac): In a Wideroe linear accelerator, what is the limit of the drift tube's length as the speed of particles $v \to c$?

Answer: Because the time it takes for the particle passing through one drift tube must half of the RF period, when $v \to c$, the length of the drift tube should be

$$l = \frac{\pi c}{\omega_{RF}} \tag{3}$$

Exercise 6 (LEP at CERN): The main dipole magnets of the Large Electron Positron (LEP) collider had a bending radius of 3096 m.

1. How strong was their magnetic field when LEP accelerated electrons to 105 GeV?

2. This field strength is relatively small, why was the field not increased to increase the energy?

3. The LEP tunnel was about 26.6km long. What fraction of it was used for bending the beam?

4. LEP produced about 20MW of synchrotron radiation when it stored electrons at 100GeV. How much would the same number of electrons have radiated at 200GeV?

Answers:

1.
$$B = \frac{p}{q\rho} = \frac{E}{cq\rho} = \frac{105 \cdot 10^9}{2.9979 \cdot 10^8 3096}\frac{Vs}{m^2} = 0.1131\text{T} . \tag{4}$$

2. Because of
$$P \propto \gamma^4 \tag{5}$$
the synchrotron radiation power would have become too large.

3.
$$f = \frac{2\pi\rho}{L} = 72\% . \tag{6}$$

4.

$$P(\gamma_2) = P(\gamma_1)(\frac{\gamma_2}{\gamma_1})^4 = 320MW \tag{7}$$

That would be about 30% of the output of a modern nuclear power plant, all deposited on a small stripe on the outside of the beam pipe!

Exercise 7 (The Earth Accelerator)

(1) Assume that the earth has an exact dipole magnetic field which is oriented parallel to the rotation axis. The magnetic field at the poles is about $2 \cdot 10^{-5}$T.

(a) Protons of what kinetic energy could you store in a bare vacuum pipe around the equator. What energy of electrons could you store?

Answer:

Using that the filed of a magnetic dipole is given by

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{3\vec{r}(\vec{r}\cdot\vec{\mu}) - r^2\vec{\mu}}{r^5} \ , \tag{8}$$

one obtains that the ratio between the field at the equator and the pole is $B(\theta = \frac{\pi}{2})/B(\theta = 0) = 1/2$, leaving a field of $B = 10^{-5}$T parallel to the surface at the equator. The momentum in this field would be $p = qBR$ with the radius of the earth being about $R = \frac{4}{2\pi} \cdot 10^7$m. The kinetic energy would then be $E = c\sqrt{p^2 + (mc)^2} - mc^2$, which for protons and electrons is

$$E_p = 18.2\text{G}eV \ , \quad E_e = 19.1\text{G}eV \ . \tag{9}$$

(b) How much of their energy would these electrons lose during one turn?

Answer:

The power that the electron radiates is given by

$$P_{rad} = \frac{c}{6\pi\epsilon_0}\frac{e^2}{R^2}\gamma^4 \ , \tag{10}$$

which leads to the energy that it radiates per turn

$$\Delta E = \frac{1}{3\epsilon_0}\frac{e^2}{R}(\frac{p}{mc})^4 = 1.84\text{k}eV \ . \tag{11}$$

(2) Consider a storage ring built around the 40 Mm circumference of the earth, where 100% of the tunnel were used for bending particles on a circular trajectory.

(a) How large would the energy be for protons when the LHC magnets with a magnetic filed of 8.7 T were used? Could one produce the highest proton energies of the universe in this way?

Answer:

$$E = pc = \rho Bqc = \frac{4 \cdot 10^7}{2\pi} 8.7 \cdot 3.0 \cdot 10^8 e \frac{\text{Tm}^2}{\text{s}} = 16616\text{TeV} \tag{12}$$

The highest proton energies detected in the universe have more than 10^8 TeV however.

(b) How much energy would such a proton loose per turn, i.e. how much energy would accelerating sections have to provide per turn for this partice? What accelerating field would be required if it were continuous around the equator?
Answer:
The energy loss per turn is

$$\Delta E = P\frac{2\pi\rho}{c} = \frac{4\pi r_e}{3}m_e c^2\frac{(\beta\gamma)^4}{\rho} = \frac{4\pi r_e m_e c^2}{3}\frac{(qBc)^4\rho^3}{(m_p c^2)^4} = 93\text{TeV} \quad (13)$$

The accelerating voltage to compensate this would be

$$\text{Field} = \frac{\Delta E}{q2\pi\rho} = \frac{2r_e m_e c^2}{3q}\frac{(qBc)^4\rho^2}{(m_p c^2)^4} = 2.3MV/m\ . \quad (14)$$

(c) How much power of synchrotron radiation would they approximately produce for the same current as in LEP (scaled from the LEP data given above)? Do not forget to scale so that the current stays the same, the number of particles in the ring is then not the same as in LEP.
Answer:
The current for N particles of charge q that happen to be in bending magnets with radius ρ is given by $I = Nqc/(2\pi\rho)$. The power radiated by this current is

$$P = \frac{c}{6\pi\epsilon_0}N\frac{q^2}{\rho^2}\gamma^4 = \frac{\pi}{3\pi\epsilon_0}\frac{qI}{\rho}\gamma^4\ . \quad (15)$$

For two rings with the same current one thus has the following scaling:

$$P_2 = P_1\frac{\rho_1}{\rho_2}(\frac{m_1}{m_2})^4(\frac{E_2}{E_1})^4 = 20\cdot 10^6\frac{2\pi\cdot 3096}{4\cdot 10^7}\frac{1}{1835^4}(\frac{16616}{0.1})^4\text{W} = 654\text{GW} \quad (16)$$

(d) How large would the electron energy in this tunnel be if its synchrotron radiation load per length of the tunnel should be the same as that in LEP when the same current is stored (scaled from the LEP data given above)? Again, the number of particles in the ring is not the same as in LEP.
Answer:

$$\frac{dP_2}{dl} = \frac{dP_1}{dl}(\frac{\gamma_2}{\gamma_1})^4(\frac{\rho_1}{\rho_2})^2\frac{I_2}{I_1}\ , \quad (17)$$

with $\frac{dP_2}{dl} = \frac{dP_1}{dl}$ this leads to

$$E_2 = E_1(\frac{\rho_2}{\rho_1})^{\frac{1}{2}} = 100GeV(\frac{4\cdot 10^4}{2\pi\cdot 3096})^{\frac{1}{2}} = 4.53\text{TeV}\ . \quad (18)$$

Exercise 8 (Energy in Rings and Linacs): A circular accelerator with dipoles of 100m bending radius stores an electron current of 0.1A at 5GeV. How much power is required to compensate for the emission of

synchrotron radiation? How much power would be required to accelerate this electron current to 5GeV in a linear accelerator?
Answer: When a ring accelerator is said to be at 5GeV, this means it has a momentum of $5GeV/c$. This convention is used since the magnetic fields in such a ring scale linearly with momentum and not with energy. For electrons at 5GeV the difference is benign, but for proton– and ion accelerators it is often significant. If this accelerator has the length L and the current $I = 0.1$A, then the number of electrons in it are $N = \frac{IL}{ev}$, with the charge e and the velocity v. Per turn, these electrons spend the time $t = \frac{2\pi\rho}{v}$ in the dipoles with bending radius $\rho = 100$m, where they radiate. Since the time for one resolution is $T = \frac{L}{v}$ they radiate with the average power

$$\bar{P} = P_{rad}N = \frac{t}{T}\frac{I}{3\epsilon_0}\frac{e}{\rho}(\frac{p}{m})^4 = 55.29\text{k}W \ . \quad (19)$$

Accelerating this electron current to the energy $E = c\sqrt{p^2 + (mc)^2} - mc^2$ would however require the power

$$\bar{P}_{linac} = I\frac{E}{e} = 0.49995\text{G}W \ . \quad (20)$$

Exercise 9 (Colliders): (1) The PEP-II asymmetric B-Factory at SLAC stores electrons with an energy of 9.0 GeV and positrons with 3.1 GeV.
(a) How much energy is in the center of momentum system when a positron and an electron collide?
Answer:

$$E_{cm}^2 = 2\left(E_1E_2 + p_1cp_2c\right) + \left(m_1c^2\right)^2 + \left(m_2c^2\right)^2 \approx 4E_1E_2 \ . \quad (21)$$

The last simplification is justified since the electron and positron rest masses are much smaller than E_1 and E_2. This leads to $E_{cm} = 2\sqrt{E_1E_2} = 10.6GeV$.
(b) What energy would positrons need to have in order to create the same energy in the center of momentum during a fixed target collision with electrons?
Answer: For a fixed target one has $p_2 = 0$, leading to $E_{cm}^2 = 2E_1m_2c^2$ and therefore $E_1 = \frac{E_{cm}^2}{2m_2c^2} = 109$TeV.
(2) Check that the LHC with its 7TeV protons should be listed for 10^{17}eV on the Livingston Chart. Where should the LEP collider with 100GeV electrons be on this chart? For any collision experiment with center of momentum energy E_{cm}, the Livingston chart shows how much energy a proton or electron would need to create the same center of momentum energy in a collision with a proton at rest.
Answer: The center of momentum energy in a high–energy collider is $E_{cm} = 2\sqrt{E_1E_2}$, while the center of momentum energy when scattering of

a proton at rest is $E_{cm} = \sqrt{2Em_pc^2}$. The energy on the Livingston chart is therefore $E = \frac{2E_1E_2}{m_pc^2}$. For the LHC with $E_1 = E_2 = 7$TeV and for LEP with $E_1 = E_2 = 0.1$TeV this leads to

$$E_{\rm LHC} = 1.04 \cdot 10^{17}{\rm e}V \ , \quad E_{\rm LEP} = 2.13 \cdot 10^{13}{\rm e}V \ . \qquad (22)$$