USPAS summer 2025, Grad Accelerator Physics

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Homework #3 Answers

Exercise 1 (Variation of constants)
Find the time evolution of the driven harmonic oscillator that satisfies the differential equation

$$\frac{d^2}{dt^2}x = -\omega^2 x + \epsilon\cos(\omega_0 t) \tag{1}$$

with the variation of constants method covered in class. What is the matrix $\underline{L}$, what is the perturbation vector $\Delta\vec{f}$, what is the transfer matrix $\vec{M}$, what is the solution $x(t)$ for initial condition x_0 and v_0, where $v = dx/dt$.
Answers: First we transform the second order differential equation into two first order equations:

$$\frac{d}{dt}x = v\ ,\quad \frac{d}{dt}v = -\omega^2 x + \epsilon\cos(\omega_0 t)\ .$$

In the matrix form from class this is written as

$$\vec{z} = \begin{pmatrix} x \\ v \end{pmatrix}\ ,\quad \frac{d}{dt}\vec{z} = \underline{L}\vec{z} + \Delta\vec{f}\ ,\quad \underline{L} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\ ,\quad \Delta\vec{f}(t) = \begin{pmatrix} 0 \\ \epsilon\cos(\omega_0 t) \end{pmatrix}$$

Next we solve the homogenious differential equation $\frac{d}{dt}\vec{z}_H = \underline{L}\vec{z}$ with the solution

$$\vec{z}_H(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix}\begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$$

Now we are ready to evaluate the variation of constant equation

$$\vec{z}(t) = \vec{z}_H(t) + \int_0^t \underline{M}(t-\hat{t})\Delta\vec{f}(\hat{t})d\hat{t}\ .$$

The left-hand integral computes the effect of the drive oscillation

$$\int_0^t \begin{pmatrix} \cos(\omega(t-\hat{t})) & \frac{1}{\omega}\sin(\omega(t-\hat{t})) \\ -\omega\sin(\omega(t-\hat{t})) & \cos(\omega(t-\hat{t})) \end{pmatrix}\begin{pmatrix} 0 \\ \epsilon\cos(\omega_0\hat{t}) \end{pmatrix}d\hat{t}\ .$$

The top line describes the contribution to $x(t)$ iand on the bottom line that to the velocity $v(t)$. In the following, the top line of the integral is computed to determine $x(t)$:

$$\begin{aligned}
\int_0^t \begin{pmatrix} \cos(\omega(t-\hat{t})) & \frac{1}{\omega}\sin(\omega(t-\hat{t})) \\ -\omega\sin(\omega(t-\hat{t})) & \cos(\omega(t-\hat{t})) \end{pmatrix} \begin{pmatrix} 0 \\ \epsilon\cos(\omega_0\hat{t}) \end{pmatrix} d\hat{t} &= \\
\epsilon\int_0^t \begin{pmatrix} \frac{1}{\omega}\sin(\omega(t-\hat{t}))\cos(\omega_0\hat{t}) \\ \cos(\omega(t-\hat{t}))\cos(\omega_0\hat{t}) \end{pmatrix} d\hat{t} &= \\
\epsilon\int_0^t \begin{pmatrix} \frac{1}{2\omega}(\sin(\omega(t-\hat{t})+\omega_0\hat{t})+\sin(\omega(t-\hat{t})-\omega_0\hat{t})) \\ \cos(\omega(t-\hat{t})+\omega_0\hat{t})+\cos(\omega(t-\hat{t})-\omega_0\hat{t}) \end{pmatrix} d\hat{t} &= \\
\frac{\epsilon}{2}\left[\begin{pmatrix} \frac{1}{\omega}(\frac{1}{\omega-\omega_0}\cos(\omega t-(\omega-\omega_0)\hat{t})+\frac{1}{\omega+\omega_0}\cos(\omega t-(\omega+\omega_0)\hat{t})) \\ -\frac{1}{\omega-\omega_0}\sin(\omega t-(\omega-\omega_0)\hat{t})-\frac{1}{\omega+\omega_0}\sin(\omega t-(\omega+\omega_0)\hat{t}) \end{pmatrix}\right]_0^t &= \\
\frac{\epsilon}{2}\begin{pmatrix} \frac{1}{\omega}(\frac{1}{\omega-\omega_0}\cos(\omega_0 t)+\frac{1}{\omega+\omega_0}\cos(\omega_0 t)) \\ -\frac{1}{\omega-\omega_0}\sin(\omega_0 t)+\frac{1}{\omega+\omega_0}\sin(\omega_0 t) \end{pmatrix} - \begin{pmatrix} \frac{1}{2\omega}(\frac{1}{\omega-\omega_0}\cos(\omega t)+\frac{1}{\omega+\omega_0}\cos(\omega t)) \\ -\frac{1}{\omega-\omega_0}\sin(\omega t)-\frac{1}{\omega+\omega_0}\sin(\omega t) \end{pmatrix} &= \\
\frac{\epsilon}{\omega^2-\omega_0^2}\begin{pmatrix} \cos(\omega_0 t)-\cos(\omega t) \\ -\omega_0\sin(\omega_0 t)+\omega\sin(\omega_0 t) \end{pmatrix} & \qquad (2)
\end{aligned}$$

The full solution for $x(t)$ is therefore

$$x(t) = \cos(\omega t)x_0 + \frac{1}{\omega}\sin(\omega t)v_0 + \frac{\epsilon}{\omega^2-\omega_0^2}(\cos(\omega_0 t)-\cos(\omega t))$$

Exercise 2 (FODO Twiss parameters)

(a) Find the Twiss parameters of the phase space ellipse that is periodic for a FODO Cell in thin lens approximation, i.e. all particles that enter a FODO cell on this phase space ellipse exit the cell on the same ellipse. Let the focusing and defocusing quadrupole have the strength k and $-k$. Furthermore, let the cell start with half a focusing quadrupole, and let the distance between quadrupoles be $L/2$ so that the transport matrix of the cell is given by

$$\underline{M} = \underline{Q}(\frac{k}{2})\underline{D}(\frac{L}{2})\underline{Q}(-k)\underline{D}(\frac{L}{2})\underline{Q}(\frac{k}{2}) \ . \qquad (3)$$

The thin lens matrices of drift and quadrupole are

$$\underline{Q}(k) = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \ , \quad \underline{D}(L) = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \ .$$

(b) Characterize how this periodic phase space ellipse changes along the FODO cell by drawing ellipses in phase space at various points along the cell. Do this for the horizontal and the vertical plane separately.

(c) Compute the periodic dispersion (η, η') for this FODO cell, assuming that there is a thin lens dipole with bending angle ϕ in the center between both quadrupoles.

(d) For what betatron phase advance (in degree) along the FODO is the maximum beta function in the FODO the smallest?

Answer:

(a) The matrices are given by

$$\underline{Q}(\frac{kl}{2}) = \begin{pmatrix} 1 & 0 \\ -\frac{kl}{2} & 1 \end{pmatrix} , \qquad (4)$$

$$\underline{D}(\frac{L}{2}) = \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} , \qquad (5)$$

$$\underline{M} = \begin{pmatrix} 1-2(\frac{kl}{2}\frac{L}{2})^2 & L(1+\frac{kl}{2}\frac{L}{2}) \\ -(\frac{kl}{2})^2 L(1-\frac{kl}{2}\frac{L}{2}) & 1-2(\frac{kl}{2}\frac{L}{2})^2 \end{pmatrix} . \qquad (6)$$

From $\cos\mu = \frac{1}{2}\mathrm{Tr}(\underline{M})$ one obtains $\cos\mu = 1-2(\frac{kl}{2}\frac{L}{2})^2$ and thus $|\frac{kl}{2}\frac{L}{2}| = \sin\frac{\mu}{2}$. We use $\xi = \frac{kl}{2}\frac{L}{2}$ and observe that a real phase advance μ for a periodic phase space ellipse can only be found when $|\xi| < 1$. The Twiss parameters are obtained from $\alpha \propto M_{11} - M_{22} = 0$ and $\beta = |M_{12}/\sin\mu| = |\frac{L}{2\xi}|\sqrt{\frac{1+\xi}{1-\xi}}$.

(b) The periodic ellipse is upright, due to $\alpha = 0$. It is sheered downward by the first focusing quadrupole and then sheered to the right by the following drift. The defocusing quadrupole now sheers twice as strongly upward, and the second drift sheers again to the right, so that the last focusing quad returns the ellipse to its original form by a downward sheer.

(c) The dispersion of a thin lens dipole is given by $\vec{D}_0 = (D, D') = (0, \phi)$. To obtain the dispersion of the FODO, one has to transport this dispersion of the first dipole by a drift of length $\frac{L}{4}$ and then through the defocusing quad and the rest of the FODO. To this one has to add the dispersion $\vec{D}_0 = (0, \phi)$ from the second dipole which is transported through a drift $\frac{L}{4}$ and through the last quadrupole. This leads to

$$\vec{D} = \underline{Q}(\frac{kl}{2})\underline{D}(\frac{L}{2})\underline{Q}(-kl)\underline{D}(\frac{L}{4})\vec{D}_0 + \underline{Q}(\frac{kl}{2})\underline{D}(\frac{L}{4})\vec{D}_0 = \begin{pmatrix} L(1-\frac{1}{2}\xi) \\ 2-\xi-\xi^2 \end{pmatrix} . \qquad (7)$$

The periodic dispersion $\vec{\eta}$ must satisfy $\vec{\eta}(L) = \underline{M}\vec{\eta}(0) + \vec{D} = \vec{\eta}(0)$, and it is therefore computed as

$$\vec{\eta}(0) = (\underline{1} - \underline{M})^{-1}\vec{D} = \begin{pmatrix} L\frac{1+\frac{1}{2}\xi}{2\xi^2}\phi \\ 0 \end{pmatrix} . \qquad (8)$$

(d) When $k > 0$, the first quadrupole is focusing and the maximal beta function is located in the center of this quad. The smallest possible value for this beta function is obtained from $\partial_\xi\beta \propto \frac{1}{2}\frac{1}{1+\xi} + \frac{1}{2}\frac{1}{1-\xi} - \frac{1}{\xi} \propto \xi^2 + \xi - 1 = 0$. The positive solution of this equation is $\xi = \frac{\sqrt{5}-1}{2} = \sin\frac{\mu}{2}$, and thus $\mu = 76°$.

Exercise 3 (Symmetric and Asymmetric 4Bumps)

Given a FODO lattice which has the periodic Twiss parameters $\beta_x = \beta_y = 10m$, $\alpha_x = \alpha_y = 0$ at its exit.
(a) If you want to construct a symmetric arrangement of six quadrupoles to design an interaction region with a horizontal beta function of 0.5m and a vertical beta function of 0.05m in its center. How would the transport matrix from the FODO to the interaction point have to lock like?
Answer:
A simple FODO, in which the drift after the focusing and after the defocusing quadrupole have the same length leads to a periodic FODO lattice which is mirror symmetric with respect to the center of each quadrupole, where therefore $\alpha_{x0} = 0$ and $\alpha_{y0} = 0$. For simplicity it is therefore good to choose the center of the last FODO quadrupole as the starting point of the low-beta insertion. At the interaction point there is $\alpha_x^* = 0$ and $\alpha_y^* = 0$. The transport matrix from FoDo to IP is then

$$\underline{M} = \begin{pmatrix} \cos\psi_x & \sqrt{\beta_{x0}\beta_x^*}\sin\psi_x & 0 & 0 \\ \frac{1}{\sqrt{\beta_{x0}\beta_x^*}}\sin\phi_x & \cos\psi_x & 0 & 0 \\ 0 & 0 & \cos\psi_y & \sqrt{\beta_{y0}\beta_y^*}\sin\psi_y \\ 0 & 0 & \frac{1}{\sqrt{\beta_{y0}\beta_y^*}}\sin\psi_y & \cos\psi_y \end{pmatrix} \tag{9}$$

(b) Why are six quadrupoles at fixed locations not sufficient to adjust the two beta functions?
Answer:
For the horizontal plain, there are 2 free parameters that have to be adjusted to the specified values at the interaction point (IP): β_x^*, and $\alpha_x^* = 0$. And for the vertical plain the corresponding 2 parameters have to be adjusted. Therefore 4 free parameters are required to adjust the Twiss parameters at the IP. Since the interaction region is arranged in a symmetric way, the quadrupoles in the second half of the low-beta insertion do not contribute independent parameters. Six quadrupoles therefore do not allow for a sufficient number of free parameters.

(c) Assume there is also a symmetric arrangement of four horizontal corrector coils and that the Twiss parameters at their places are known. Specify the relative strength of these coils so that a closed bump is created that only changes the orbit position at the low beta point, but not orbit angle.
Answer:
Let the 4 corrector coils be located at Twiss parameters β_i and ψ_i, and their corrector angles are θ_i for $i \in \{1, \dots, 4\}$. The symmetric arrangement requires that the beta functions at 1 and 4 are the same and that those at 2 and 3 are the same. For simplicity we locate the origin of the betatron phases at the IP so that $\psi(IP) = 0$. The symmetric arrangement then means for the phases that $\psi_4 = \mu - \psi_1$ and $\psi_3 = \mu - \psi_2$, where we choose ψ_1 and ψ_2 to be positive. Since the first magnet is further from the IP, one has $\psi_1 > \psi_2$.

The bump which changes the orbit position at the IP but not the angle is symmetric, and therefore also the corrector coil angles have to be symmetric, i.e. $\theta_1 = \theta_4$ and $\theta_2 = \theta_3$.

The closed orbit that is created when the four corrector coils are excited can be written as

$$x(s) = \frac{\sqrt{\beta(s)}}{2\sin\frac{\mu}{2}} \sum_i \theta_i \sqrt{\beta_i} \cos(|\psi(s) - \psi_i| - \frac{\mu}{2}) \ . \tag{10}$$

This bump has to be 0 at position s_1, and due to symmetry will then also be 0 at position s_4,

$$\begin{aligned} x(s_1) = \frac{\sqrt{\beta_1}}{2\sin\frac{\mu}{2}} \quad & [\quad \theta_1\sqrt{\beta_1}\cos\frac{\mu}{2} + \theta_2\sqrt{\beta_2}\cos(\psi_1 - \psi_2 - \frac{\mu}{2}) \\ & + \quad \theta_2\sqrt{\beta_2}\cos(\psi_1 + \psi_2 - \frac{\mu}{2}) + \theta_1\sqrt{\beta_1}\cos(2\psi_1 - \frac{\mu}{2}) = 0 \end{aligned} \tag{11}$$

The relation between the bending angle therefore has to be

$$\theta_2 = -\sqrt{\frac{\beta_1}{\beta_2}} \frac{\cos\psi_1}{\cos\psi_2} \theta_1 \tag{12}$$

$$\theta_3 = \theta_2 \ , \tag{13}$$

$$\theta_4 = \theta_1 \ . \tag{14}$$

(d) Specify the relative strength of these coils so that a closed bump is created that only changes the orbit slope at the low beta point, but not the orbit position.

Answer:

Since now the orbit is anti-symmetric, also the corrector angles have to be anti-symmetric, i.e. $\theta_3 = -\theta_2$ and $\theta_4 = -\theta_1$. To have no closed orbit distortion at position s_2 and s_4 it is required that

$$\begin{aligned} x(s_1) = \frac{\sqrt{\beta_1}}{2\sin\frac{\mu}{2}} \quad & [\quad \theta_1\sqrt{\beta_1}\cos\frac{\mu}{2} + \theta_2\sqrt{\beta_2}\cos(\psi_1 - \psi_2 - \frac{\mu}{2}) \\ & - \quad \theta_2\sqrt{\beta_2}\cos(\psi_1 + \psi_2 - \frac{\mu}{2}) - \theta_1\sqrt{\beta_1}\cos(2\psi_1 - \frac{\mu}{2}) = 0 \end{aligned} \tag{15}$$

The relation between the bending angle therefore has to be

$$\theta_2 = -\sqrt{\frac{\beta_1}{\beta_2}} \frac{\sin\psi_1}{\sin\psi_2} \theta_1 \tag{16}$$

$$\theta_3 = \theta_2 \ , \tag{17}$$

$$\theta_4 = \theta_1 \ . \tag{18}$$

1 Lattice Design #3

Report your results for the following exercises of the Ring Design Tutorial exercises section 2.3 number 1, 2, 5, and 6.

1 - Reverse dispersion suppressor: Construct the reverse dispersion suppressor, optimizing the last two quadrupole strengths for $\eta_z = 0$ and $\eta'_x = 0$ at the end for FoDo cells of 90$\circ$ phase advance. How do the two quadrupole values for the reverse dispersion suppressor compare to those obtained for the forward suppressor?

2 - Forward and Reversed Cells: Check that your forward and reverse cells that both start with focusing quads have different periodic beta and alpha functions. Check also that both cell, for the same phase advance of 90 degrees have exactly the same quadrupole strengths. Explain how this can be the correct solution.

3 - Strength of bends: to a good approximation: A dispersion suppressor can be constructed using two arc FODO cells with the first cell having the bend strengths reduced by a factor α and the second cell with the bend strengths reduced by a factor $1-\alpha$. In the case of a 90° FoDo cells, we showed in class that $\alpha = 0.5$, and for a 60° FoDo we found $\alpha = 0$. Find the suitable α for a 72° FoDo. Either determine α analytically or by matching the FoDo in your ring design to 72° phase advance and then finding the α that leads to dispersion suppression.