USPAS summer 2025, Grad Accelerator Physics

Instructors: Georg Hoffstaetter de Torquat, David Sagain
TAs: Matt Signorelli, Joseph Devlin

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Homework #4 Answers

Exercise (Complex potentials)
When the coordinates $w = x + iy$ and $\bar{w} = x - iy$ are used, the Laplace operator has been derived to be $\vec{\nabla}^2 = 4\partial_w\partial_{\bar{w}} + \partial_z^2$.
(a) Check that this is correct.
Answer:
(a)

$$w = x + iy \ , \quad \bar{w} = x - iy \ , \quad \partial_x = \partial_w + \partial_{\bar{w}} \ , \quad \partial_y = i\partial_w - i\partial_{\bar{w}} \ . \tag{1}$$

Therefore

$$\vec{\nabla}^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 = (\partial_w + \partial_{\bar{w}})^2 - (\partial_w - \partial_{\bar{w}})^2 + \partial_z^2 = 4\partial_w\partial_{\bar{w}} + \partial_z^2 \ . \tag{2}$$

(b) The static magnetic field in a charge free space is given by $\vec{B} = -\vec{\nabla}\psi$. Writing the magnetic field in x and y direction in complex notation as $B = B_x + iB_y$, derive a formula that expresses B and B_z in terms of $\Psi(w, \bar{w}, z)$ and only ∂_w, $\partial_{\bar{w}}$, and ∂_z.
Answer:
(b) $B = B_x + iB_y$ is a complex number, not a vector,

$$\begin{aligned} B &= -(\partial_x + i\partial_y)\psi = -[(\partial_w + \partial_{\bar{w}}) - (\partial_w - \partial_{\bar{w}})]\psi = -2\partial_{\bar{w}}\psi \ , && (3) \\ B_z &= -\partial_z\psi \ . && (4) \end{aligned}$$

(c) Given the vector potential in complex notation as $A = A_x + iA_y$ and A_z, derive a formula that expresses B and B_z given by $\vec{B} = \nabla \times \vec{A}$, again only using ∂_w, $\partial_{\bar{w}}$, and ∂_z and A, A_z.
Answer:

$$\begin{aligned} B &= B_x + iB_y = (\partial_y A_z - \partial_z A_y) + i(\partial_z A_x - \partial_x A_z) && (5) \\ &= -i(\partial_x + i\partial_y)A_z + i\partial_z(A_x + iA_y) = -i2\partial_{\bar{w}}A_z + i\partial_z A \ , && (6) \\ B_z &= \partial_x A_y - \partial_y A_x = (\partial_w + \partial_{\bar{w}})A_y - i(\partial_w - \partial_{\bar{w}})A_x && (7) \\ &= -i\partial_w A + i\partial_{\bar{w}}\bar{A} = 2\mathrm{Im}(\partial_w A) \ . && (8) \end{aligned}$$

Exercise (Rotational field symmetries)
(a) The field in a bending magnet has usually two symmetries: Midplane symmetry since the upper and lower part of the magnet are built identically, and a mirror symmetry with respect to the vertical plane, since each pole is build with right/left symmetry when viewed along the beam pipe. Which multipoles, in addition to the main dipole component, satisfy this symmetry and can therefore be associated with such a bending magnet.
Answer:
We introduce the matrix $\underline{S}$ that describes a reflection on the midplane. The components of $\underline{S}\vec{x}$ are therefore $(x, -y, z)$. Similarly we introduce the matrix $\underline{V}$ that describes reflection on the vertical plane so that the components of $\underline{V}\vec{x}$ are $(-x, y, z)$. The symmetry conditions on the magnetic potential are

$$\psi(\underline{S}\vec{x}) = -\psi(\vec{x}) \ , \quad \psi(\underline{V}\vec{x}) = \psi(\vec{x}) \ . \tag{9}$$

The C_1 symmetry simply states that rotating the field by 2π creates the same field, which does not add a constraint. Inserting the mid plane symmetric multipole expansion $\psi(\vec{x}) = \sum_{n=0}^{\infty} \Psi_n \text{Im}\{(x - iy)^n\}$ in the main field region into these symmetry constraints yields

$$\Psi_n \text{Im}\{\bar{w}^n\} = \Psi_n \text{Im}\{(-x - iy)^n\} = (-1)^n \Psi_n \text{Im}\{w^n\} = -(-1)^n \Psi_n \text{Im}\{\bar{w}^n\} \ . \tag{10}$$

This requires that $\Psi_n = 0$ when n is an even number. The first multipole error that can occur in a dipole is therefore a sextuple.
(b) Similarly, a focusing magnet has C_2 and midplane symmetry. Which multipoles, in addition to the main quadrupole term, satisfy this symmetry and can therefore appear when such a magnet is built.
Answer:
For the additional C_2 symmetry we introduce a matrix $\underline{Q}$ which rotates by π so that $\underline{Q}\vec{x}$ has the components $(-x, -y)$, and requiring $\psi(\underline{Q}\vec{x}) = \psi(\vec{x})$, we therefore have the condition

$$\Psi_n \text{Im}\{\bar{w}^n\} \quad = \quad \Psi_n \text{Im}\{(-x + iy)^n\} = (-1)^n \Psi_n \text{Im}\{\bar{w}^n\} \ . \tag{11}$$

This requires that $\Psi_n = 0$ when n is an odd number. The first multipole error that can occur in a quadrupole is therefore an octupole.
(c) Generalize your observation to a magnet which is built with exact C_n symmetry and midplane symmetry. Which multipole terms can the field have?
Answer:
For C_n symmetry the potential has to be identical after a rotation by $2\pi/n$.

$$\Psi_\nu \text{Im}\{\bar{w}^\nu\} \quad = \quad \Psi_\nu \text{Im}\{r^n e^{-i\nu(\phi + \frac{2\pi}{n})}\} \ . \tag{12}$$

This requires that $\Psi_\nu = 0$ whenever ν is not divisible by n. The first multiple error that can appear in a C_n symmetric winding is therefore a $2n$-pole.

Exercise (Solenoid)
Consider a box shape solenoid field. On the central axis, the solenoid field is given by

$$\vec{B}(z) = \begin{cases} B_0\vec{e}_z \text{ for } z \in [0, L] \\ 0 \text{ else} \end{cases} \tag{13}$$

(a) A particle flies into the solenoid parallel to the central axis. Describe its trajectory after the solenoid.
Answer:
Before entering the solenoid the particle has the coordinates $w = x_0$ and the slope $\dot{w} = 0$. In the rotating coordinate system we have $w_r = we^{i\int_0^t g dt}$ and $g = \frac{qB_z}{2m\gamma}$. Before entering the solenoid, the coordinates in this coordinate system are $w_r = x_0$ and $\dot{w}_r = 0$ since $g = 0$ outside the solenoid.

The differential equation in the rotated coordinate system $\ddot{w}_r = -g^2\dot{w}_r$ has to be solved. We first solve it inside the solenoid where $g = \frac{qB_z}{2m\gamma}$ is constant.

$$w_r(t) = w_r(0)\cos(gt) + \dot{w}_r(0)\frac{1}{g}\sin(gt) = x_0\cos(gt) \ . \tag{14}$$

$$\dot{w}_r(t) = -gw_r(0)\sin(gt) + \dot{w}_r(0)\cos(gt) = -gx_0\sin(gt) \ . \tag{15}$$

After the time T spent in the solenoid, we have as starting values for the trajectory after the solenoid: $w_r(T) = x_0\cos(gT)$ and $\dot{w}_r(T) = -gx_0\sin(gT)$. Since after the solenoid, $\ddot{w}_r = 0$, we obtain for $t > T$

$$w_r(t) = w_r(T) + (t-T)\dot{w}_r(T)t = x_0\cos(gT) - (t-T)gx_0\sin(gT) \ . \tag{16}$$

In the section after the solenoid, the coordinates in the non-rotating region is computed by $w(t) = w_r(t)e^{-i\int_0^t g dt} = w_r(t)e^{-igT}$.

The motion after the solenoid is thus described by

$$w_r(t) = [x_0\cos(gT) - (t-T)gx_0\sin(gT)]e^{-igT} \ . \tag{17}$$

This trajectory after the solenoid describes a line that lies in a plane that contains the central axis and has a angle $-gT$ to the x-axis. This line starts at a distance $x_0\cos(gT)$ from the axis and has the transverse velocity $-gx_0\sin(gT)$ in that plane.
(b) If it touches the central axis somewhere after the solenoid, where would that be? How does the focal length depend on the field B_0 and the length L?
Answer:
The particle will touch the central axis after the solenoid if $gx_0\sin(gT) > 0$. If its total velocity is v, its longitudinal velocity is $v_z = \sqrt{v^2 - [gx_0\sin(gT)]^2} \approx v$. Quadratic terms in x_0 have been neglected in setting up the linearized equation of motion and are therefore also neglected here. It will therefore touch the axis a distance f after the solenoid with $f = \frac{v}{g}\cot(gT)$.

To determine the time T spent inside the solenoid, the transverse velocity is needed for $t < T$ from $\dot{w}(t) = \dot{w}_r(t)e^{igt} + igw_r(t)e^{igt}$. The squared transverse velocity is therefore $|\dot{w}_r(t)|^2 + g^2|w_r(t)|^2 + 2\Im\{\dot{w}_r(t)\bar{w}_r(t)\} = x_0^2g^2$. The longitudinal velocity inside the solenoid is therefore $v_z = \sqrt{v^2 - x_0^2 g^2} \approx v$, and $T = L/v$ to first order.

The focal length is then

$$f = \frac{v}{g} \cot(\frac{Lg}{v}) \; . \tag{18}$$

For very short solenoids the focal strength is equal to $f \approx \frac{v^2}{g^2}\frac{1}{L}$. With increasing L the focal length decreases so that it is not justified to say that only the fringe field of a solenoid focuses.

(c) Show that the magnetic field $\vec{B} = \{\frac{x}{2}\Psi_0'' \, , \; \frac{y}{2}\Psi_0'' \, , \; -\Psi_0'\}$ can be derived from the vector potential $\vec{A} = \{\frac{y}{2}\Psi_0' \, , \; -\frac{x}{2}\Psi_0' \, , \; 0\}$, where Ψ_0 is a function of z.

Alternatively we can write $A = -i\frac{w}{2}\Psi_0'$, $A_z = 0$ and use the equations $B = -i2\partial\bar{w}A_z + i\partial_z A = \frac{w}{2}\Psi_0''$ and $B_z = 2\Im\{\partial_w A\} = -\Psi_0'$.

Answer:

Because Ψ_0 is only a function of z, we have

$$\vec{\nabla} \times \vec{A} = (\partial_x \vec{e}_x + \partial_y \vec{e}_y + \partial_z \vec{e}_z) \times \left(\frac{y}{2}\Psi_0' \vec{e}_x - \frac{x}{2}\Psi_0' \vec{e}_y \right) \tag{19}$$

$$= \frac{x}{2}\Psi_0'' \vec{e}_x + \frac{y}{2}\Psi_0'' \vec{e}_y - \Psi_0' \vec{e}_z \tag{20}$$

$$= \vec{B} \tag{21}$$

(d) Show that during this motion, the particle's angular momentum around the z-axis is not conserved. Also show that the z component of its canonical angular momentum $L_z = \left\{ \vec{r} \times (\vec{p} + e\vec{A}) \right\}_z$ is conserved. To do this, you can show that $\frac{dL_z}{dt} = 0$.

Answer:

$$L_z = x(p_y + eA_y) - y(p_x + eA_x) = \gamma m(x\dot{y} - y\dot{x}) - \frac{e}{2}\Psi_0'(x^2 + y^2) \tag{22}$$

$$\frac{d}{dt}L_z = \gamma m(x\ddot{y} - y\ddot{x}) - \frac{e}{2}\Psi_0''\dot{z}(x^2 + y^2) - e\Psi_0'(x\dot{x} + y\dot{y}) \tag{23}$$

$$= \gamma m x \left(\ddot{y} + \frac{eB_z'\dot{z}}{2\gamma m}x + \frac{eB_z}{\gamma m}\dot{x} \right) - \gamma m y \left(\ddot{x} - \frac{eB_z'\dot{z}}{2\gamma m}y - \frac{eB_z}{\gamma m}\dot{y} \right) \tag{24}$$

From the equation of motion we know that

$$\ddot{x} = \frac{eB_z'\dot{z}}{2\gamma m}y + \frac{eB_z}{\gamma m}\dot{y} \tag{25}$$

$$\ddot{y} = -\frac{eB_z'\dot{z}}{2\gamma m}x - \frac{eB_z}{\gamma m}\dot{x} \tag{26}$$

Thus we have

$$\frac{d}{dt}L_z = 0 \qquad \Longrightarrow \qquad L_z = \text{const.} \tag{27}$$

(e) Given a proton beam of $E_k = 5\text{keV}$, how many turns of a 100A current is approximately needed for a 10cm coil to have a 1 meter focal length.
From

$$f = \frac{v}{g}\cot(\frac{g}{v}L) \approx \frac{v^2}{g^2}\frac{1}{L} \tag{28}$$

we have for $f = 1\text{m}$, $v = \sqrt{\frac{2E_k}{m_p}}$ and $L = 10\text{cm}$

$$g^2 = \frac{v_z^2}{fL} = \frac{2E_k}{m_p f L} \qquad \Longrightarrow \qquad g = 3.093 \times 10^6$$
$$B_z = \frac{2m_p g}{e} = 0.0647\text{T} = \mu_0 n I \qquad \Longrightarrow \qquad n = 515\text{Turns/m}$$

Thus we need approximately 52 turns to have a 1 meter focal length.
Answer:
(f) Show that within the solenoid the particles perform helical motion of radius $\frac{x_0}{2}$ around the axis $x = \frac{x_0}{2}$.
Answer:
The motion in the non-rotating coordinate system inside the solenoid is

$$w(t) = x_0 \cos(gt) e^{-igt} \ . \tag{29}$$

Exercise (Multipoles)
(a) Describe the magnetic field and the magnetic scalar potential in an duodecapole ?
Answer:

$$\psi = \text{Im}\{\Psi_6 \bar{w}^6\} = \Psi_6(-6x^5y + 20x^3y^3 - 6xy^5) \ , \tag{30}$$
$$B = B_x + iB_y = -2\partial_{\bar{w}}\psi = i6\Psi_6\bar{w}^5 \ , \tag{31}$$
$$\begin{pmatrix} B_x \\ B_y \end{pmatrix} = 6\Psi_6 \begin{pmatrix} -\text{Im}\{\bar{w}^5\} \\ \text{Re}\{\bar{w}^5\} \end{pmatrix} = 6\Psi_6 \begin{pmatrix} 5x^4y - 10x^2y^3 + y^5 \\ x^5 - 10x^3y^2 + 5xy^4 \end{pmatrix} . \tag{32}$$

(b) How strong is a duodecapole for which the distance from the central axis to the iron pole is given by a and around each pole is a winding of n wires each having a current I ?
Answer:
Integrating over a line with $x = y$ and back along the line with $y = 0$ leads to a path that contains n wires with current I.

$$\vec{B} \cdot d\vec{l}|_{x=y} = -48\Psi_6 x^6 \frac{1}{r} dr = -6\Psi_6 r^5 dr \ . \tag{33}$$

Since only the line with $x = y$ contributes to the integral we obtain

$$\mu_0 n I = \int \vec{B} \cdot d\vec{l} = -\Psi_6 a^6 = -\frac{d^5 B_y}{dx^5}\frac{a^6}{6!} \ . \tag{34}$$

and

$$k_5 = \frac{q}{p}\frac{d^5 B_y}{dx^5} = 6!\frac{q}{p}\frac{\mu_0 n I}{a^6} \ . \tag{35}$$

(c) : Show what fields are created when a n pole is shifted by a distance Δ in the transverse direction. For example, show that a shifted sextupole has a quadrupole field.

Answer:

The potential of an n pole which is centered at the origin is given by

$$\psi = \mathrm{I}m\{\Psi_n \bar{w}^n\} \ . \tag{36}$$

If it is shifted in x by Δx and in y by Δy, then with $\Delta w = \Delta x + i\Delta y$ its potential is

$$\psi = \mathrm{I}m\{\Psi_n(\bar{w} - \Delta\bar{w})^n\} = \mathrm{I}m\{\Psi_n \bar{w}^n\} - \mathrm{I}m\{n\Delta\bar{w}\Psi_n\bar{w}^{n-1}\} + O^2(\Delta\bar{w}) \ . \tag{37}$$

Therefore a shifted n pole creates, to first order in the shift, has an $n-1$ pole. When the rotation angle out of midplane symmetry of the n pole is given by θ_n with $\Psi_n = |\Psi_n|e^{in\theta_n}$ and is thus $\theta_n = \frac{1}{n}\mathrm{A}rg(\Psi_n)$, this rotation angle of the $n-1$ pole is $\theta_{n-1} = \frac{1}{n-1}\mathrm{A}rg(\Psi_n\Delta\bar{w})$. If a sextupole (n=3) that has midplane symmetry is shifted in a direction that has an angle ϕ out of the horizontal plane, a quadrupole (n=2) is created that has an angle of $-\phi/2$ to the midplane.

1 Lattice Design #4

Report your results for the following exercises of the Ring Design Tutorial exercises section 2.3 number 1, 2, 5, and 6.

1 - Reverse dispersion suppressor: Construct the reverse dispersion suppressor, optimizing the last two quadrupole strengths for $\eta_z = 0$ and $\eta_x' = 0$ at the end for FoDo cells of 90$\circ$ phase advance. How do the two quadrupole values for the reverse dispersion suppressor compare to those obtained for the forward suppressor?

2 - Forward and Reversed Cells: Check that your forward and reverse cells that both start with focusing quads have different periodic beta and alpha functions. Check also that both cell, for the same phase advance of 90 degrees have exactly the same quadrupole strengths. Explain how this can be the correct solution.

3 - Strength of bends: to a good approximation: A dispersion suppressor can be constructed using two arc FODO cells with the first cell having the bend strengths reduced by a factor α and the second cell with the bend strengths reduced by a factor $1-\alpha$. In the case of a 90° FoDo cells, we showed in class that $\alpha = 0.5$, and for a 60° FoDo we found $\alpha = 0$. Find the suitable α for a 72° FoDo. Either determine α analytically or by matching the FoDo in your ring design to 72° phase advance and then finding the α that leads to dispersion suppression.