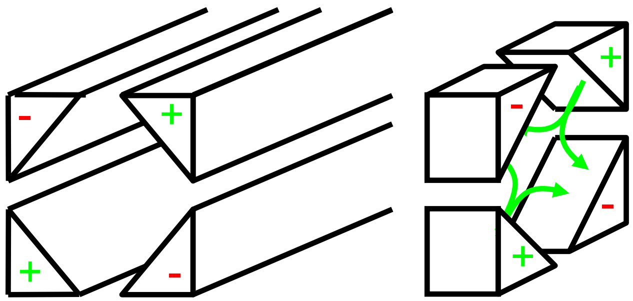
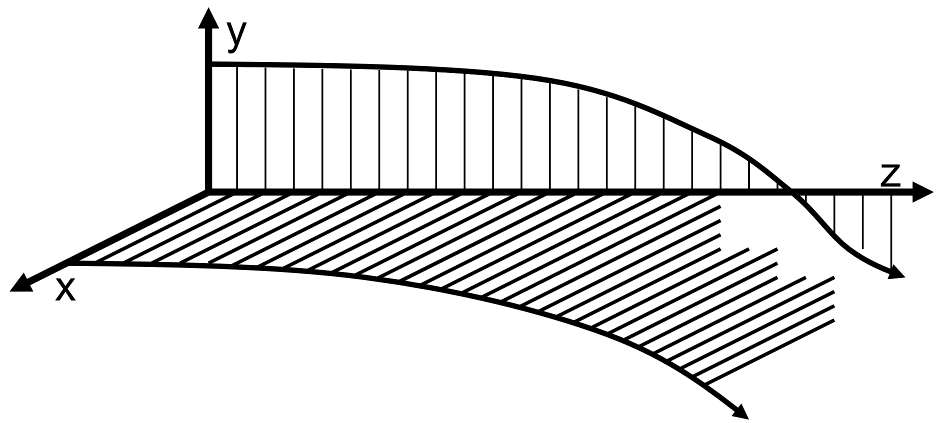
Quadrupole Focusing

Magnetic fields can be computed from magnetic scalar potentials, because $\vec{\nabla}\times\vec{B} = 0$: $\vec{B} = -\vec{\nabla}\psi$





In a quadrupole particles are focused in one plane and defocused in the other plane. Other modes of strong focusing are not possible.

The field is

1) zero in the center.

2) Increases linearly in x an y.

3) When x and y motion is not coupled, B_x depends only on y and y depends only on x.

$$\vec{B} = -\vec{\nabla}\psi = -\vec{\nabla}\,\psi_2\,x\,y = -\psi_2\begin{pmatrix} y \\ x \end{pmatrix}$$

Linearized equations of motion in quadrupoles

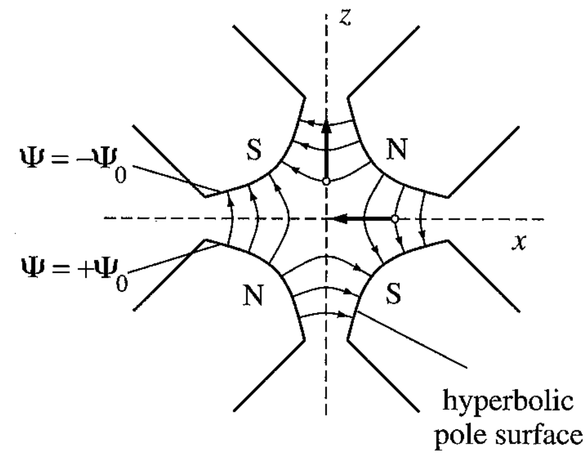
Equation of motion in x and y from the Lorence Force equation $\frac{\mathrm{d}\vec{p}}{dt} = q\vec{v}X\vec{B}$ for

(1) Transverse magnetic fields: $\frac{\mathrm{d}\vec{p}}{dt} = q\vec{v}X\vec{B}$

(2) Linearization for small coordinates

(3) No coupling of x-motion to y motion

(4) Straight motion through the center of the magnetic element



$m_0\gamma\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix}$ = q $\begin{pmatrix}-\dot{z}B_y\\ \dot{z}B_x\end{pmatrix}$ using $\frac{d}{dt} = \dot{z}\frac{d}{dz}$ this leads to $\begin{pmatrix}x''\\ y''\end{pmatrix} = \frac{\mathrm{q}}{m_0\gamma\dot{z}}\begin{pmatrix}-x\partial_x B_y\\ y\partial_y B_x\end{pmatrix} = -\frac{\mathrm{q}}{p_0}\psi_2\begin{pmatrix}-x\\ y\end{pmatrix}$

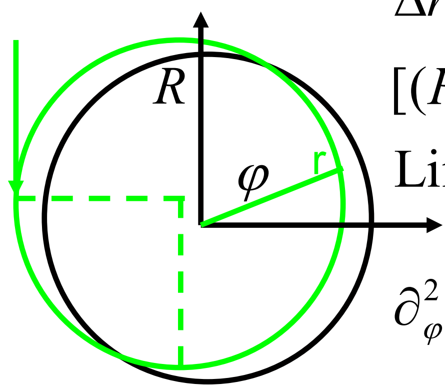
$$x'' = -k\,x$$
$$y'' = k\,y$$

For positive k, focusing in x and defocusing in y.

Linearized motion in a bend

$$\rho = \frac{p}{qB}$$

Weak focusing from natural ring focusing:



$$\Delta r = r - R$$

$$[(R+\Delta r)\cos\varphi - \Delta x_0]^2 + [(R+\Delta r)\sin\varphi - \Delta y_0]^2 = R^2$$

$$\text{Linearization in } \Delta: \quad \Delta r = (\cos\varphi\,\Delta x_0 + \sin\varphi\,\Delta y_0)$$

$$\partial_\varphi^2 \Delta r = -\Delta r \quad \Rightarrow \quad \Delta\ddot{r} = -\dot{\varphi}^2 \Delta r = -\left(\frac{v}{\rho}\right)^2 \Delta r = -\left(\frac{qB}{m\gamma}\right)^2 \Delta r$$

$$\frac{d^2\Delta r}{ds^2} = \frac{d^2\Delta r}{dt^2}\frac{1}{v^2} = -\frac{1}{\rho^2}\Delta r$$

Focusing strength: $\frac{1}{\rho^2}$

Linear equation of motion for bends and quadrupoles:

$$x'' = -(k + \frac{1}{\rho^2})\,x$$
$$y'' = ky$$

Betatron formalism for linear motion

$$x'' = -x\,K$$
$$y'' = \ \ y\,k$$

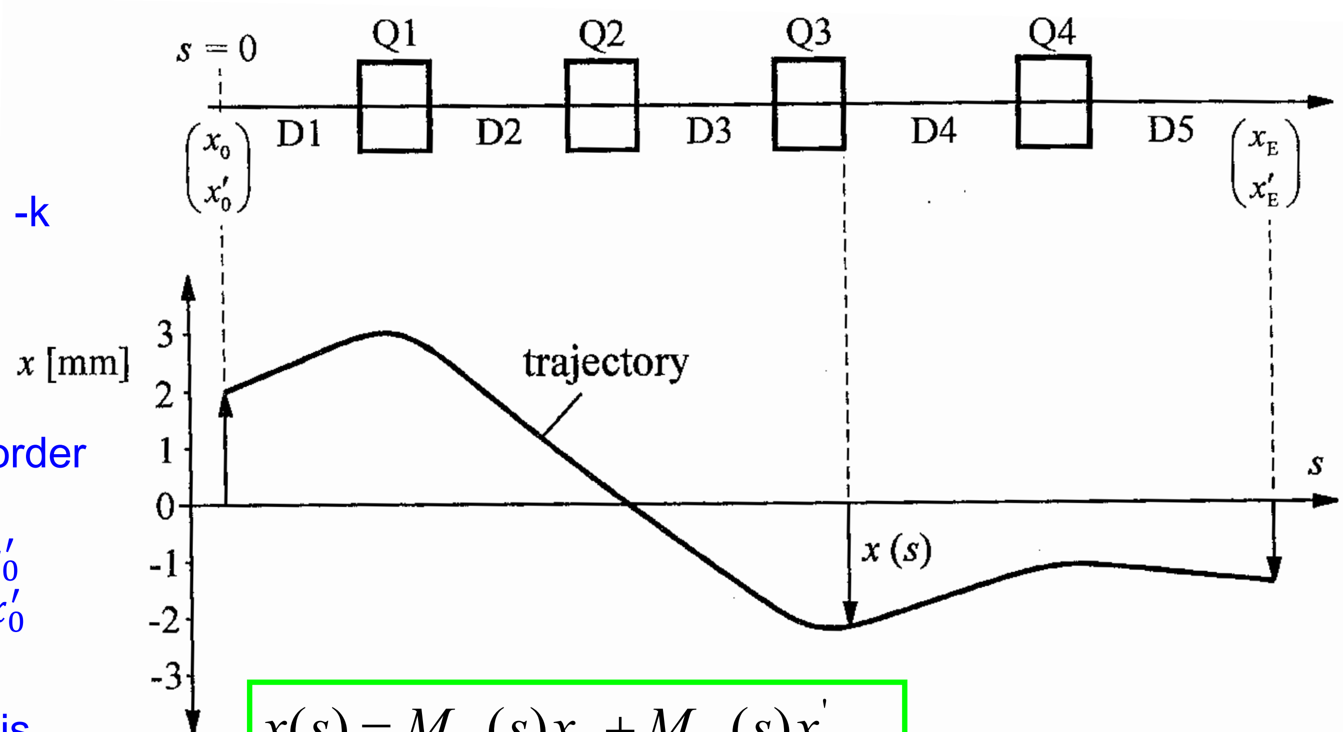
In y: quadrupole defocusing -k

In x: K = k + $\frac{1}{\rho^2}$

General solution of the 2nd order linear equation of motion:

$$x(s) = M_{11}(s)x_0 + M_{12}x_0'$$
$$x'(s) = M_{21}(s)x_0 + M_{22}x_0'$$

The vector $\vec{z} = (x, x', y, y')^T$ is transported by a matrix multiplication $\vec{z}(s) = \underline{M}(s)\vec{z}_0$



$$x(s) = M_{11}(s)x_0 + M_{12}(s)x_0'$$
$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s) + \phi_0)$$

Twiss parameters

$$x'' = -k\,x$$

$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s)+\phi_0)$$

$$x'(s) = \sqrt{\tfrac{2J}{\beta}}[\beta\psi'\cos(\psi(s)+\phi_0) - \alpha\sin(\psi(s)+\phi_0)] \quad \text{with} \quad \alpha = -\tfrac{1}{2}\beta'$$

$$x''(s) = \sqrt{\tfrac{2J}{\beta}}[(\beta\psi''-2\alpha\psi')\cos(\psi(s)+\phi_0) - (\alpha' + \tfrac{\alpha^2}{\beta} + \beta\psi'^2)\sin(\psi(s)+\phi_0)]$$

$$= \sqrt{\tfrac{2J}{\beta}}[-k\beta\sin(\psi(s)+\phi_0)]$$

$$\beta\psi'' - 2\alpha\psi' = \beta\psi'' + \beta'\psi' = (\beta\psi')' = 0 \quad \Rightarrow \quad \psi' = \tfrac{I}{\beta}$$

$$\alpha' + \gamma = k\beta \quad with \quad \gamma = \tfrac{I^2+\alpha^2}{\beta}$$

Universal choice: I=1!

$$\beta' = -2\alpha$$

$$\alpha' = k\beta - \gamma$$

$$\psi = \int_0^s \tfrac{I}{\beta(s')}ds'$$

$\alpha, \beta, \gamma, \psi$ are called Twiss parameters.

What are the initial conditions?

The phase ellipse

Particles with a common J and different ϕ all lie on an ellipse in phase space:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{I}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

(Linear transform of a circle)

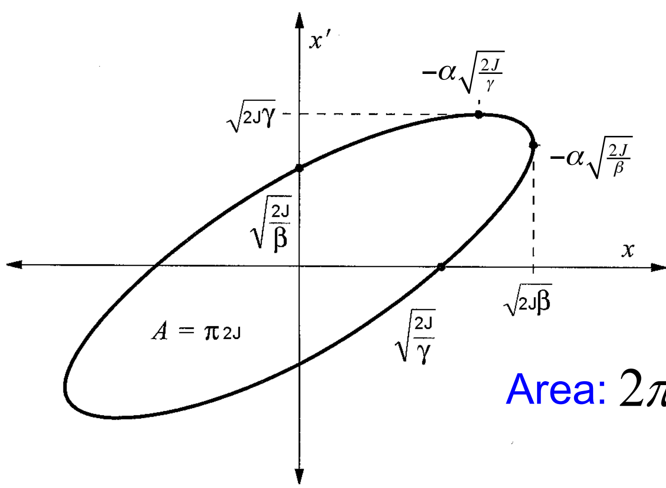
$$x_{\max} = \sqrt{2J\beta} \text{ at } x' = -\alpha\sqrt{\tfrac{2J}{\beta}}$$

$$(x, x')\begin{pmatrix} \frac{I}{\sqrt{\beta}} & \frac{\alpha}{\sqrt{\beta}} \\ 0 & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} \frac{I}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = (x, x')\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = 2J$$

(Quadratic form)

$$\beta\gamma - \alpha^2 = I^2$$

Area: $2\pi J / I$



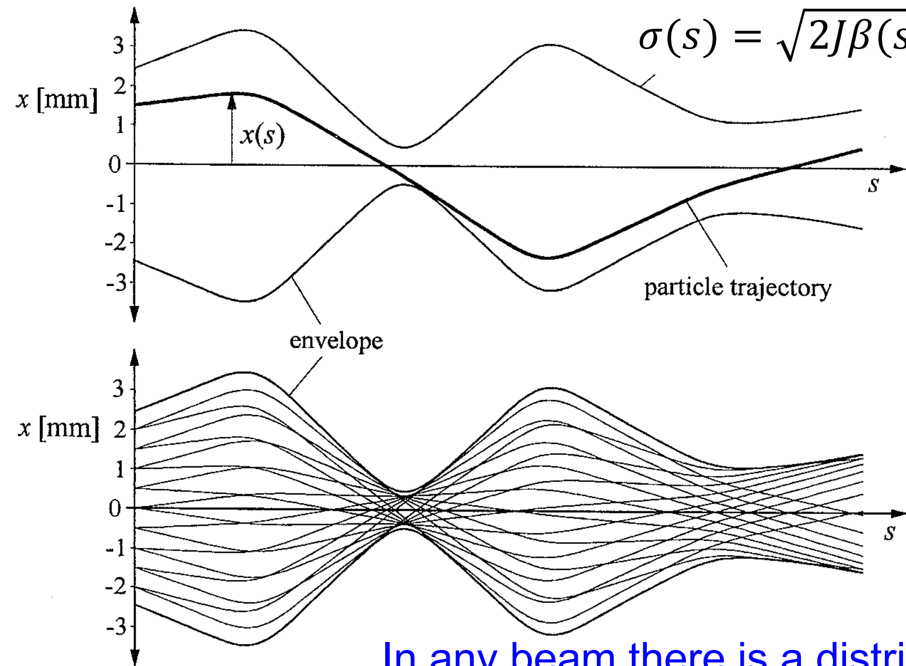
I=1 is therefore a useful choice!

What β is for x, γ is for x'

$$x'_{\max} = \sqrt{2J\gamma} \text{ at } x = -\alpha\sqrt{\tfrac{2J}{\gamma}}$$

Area: $2\pi J$ → $$\int_0^{2\pi}\int_0^{J} dJ\, d\phi = 2\pi J = \iint dx\, dx'$$

The beam envelope



$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s)+\phi_0)$$

$$\sigma' = -\alpha\sqrt{\frac{2J}{\beta}}$$

$$\sigma'' = -(k\beta-\gamma)\sqrt{\frac{2J}{\beta}} - \alpha^2\sqrt{\frac{2J}{\beta^3}}$$

$$\sigma'' = -(k\beta^2-1)\sqrt{\frac{2J}{\beta^3}}$$

In any beam there is a distribution of initial parameters. If the particles with the largest J are distributed in ϕ over all angles, then the envelope of the beam is described by $\sigma = \sqrt{2J_{max}\beta(s)}$.

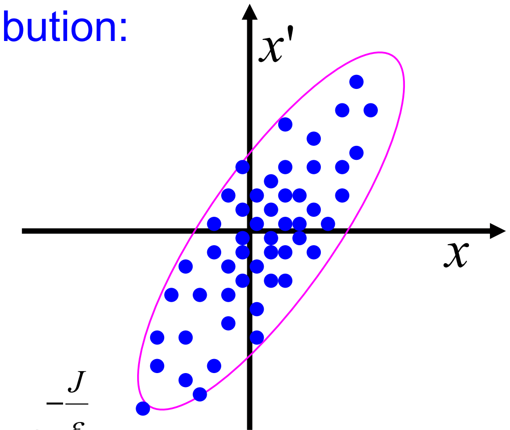
The initial conditions of β and α are chosen so that this is approximately the case.

The envelope equation: $\sigma'' = -k\sigma + \frac{(2J)^2}{\sigma^3}$

The phase space distribution

Often one can fit a Gauss distribution to the particle distribution:

$$\rho(x,x') = \frac{1}{2\pi\varepsilon} e^{-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{2\varepsilon}}$$



The equi-density lines are then ellipses. And one chooses the starting conditions for β and α according to these ellipses!

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix} \qquad \rho(J,\phi_0) = \frac{1}{2\pi\varepsilon} e^{-\frac{J}{\varepsilon}}$$

$$\langle 1 \rangle = \frac{1}{2\pi\varepsilon} \int_0^{2\pi} \int_0^{\infty} e^{-J/\varepsilon} dJ d\phi_0 = 1$$

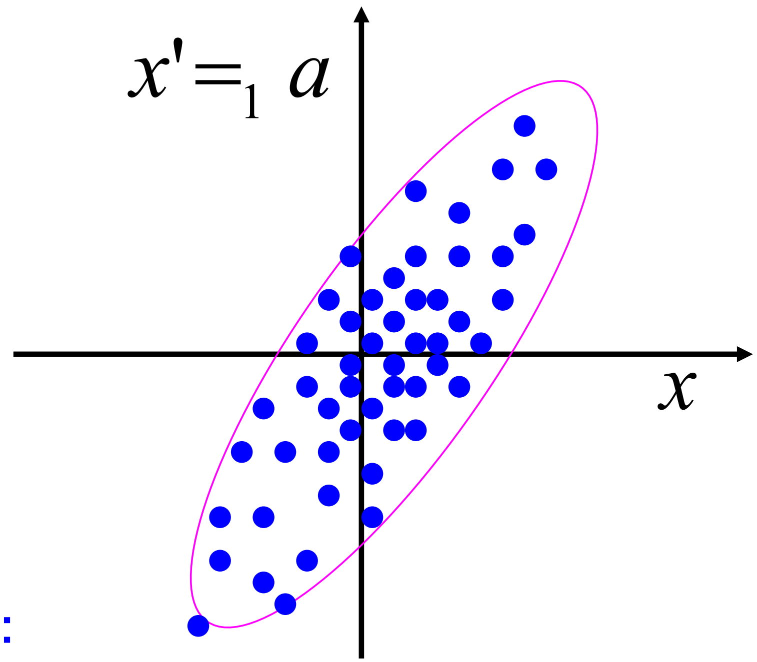
Initial beam distribution ⟶ initial α, β, γ

$$\langle x^2 \rangle = \frac{1}{2\pi\varepsilon} \iint 2J\beta \sin\phi_0^2 e^{-J/\varepsilon} dJ d\phi_0 = \varepsilon\beta \longrightarrow \langle x'^2 \rangle = \varepsilon\gamma$$

$$\langle x x' \rangle = -\frac{1}{2\pi\varepsilon} \iint 2J\alpha \sin\varphi_0^2 e^{-J/\varepsilon} dJ d\varphi_0 = -\varepsilon\alpha$$

$$\varepsilon = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}$$ is called the emittance.

The normalized emittance



$with\ a = p_x/p_0$
and the beam's reference momentum p_0.

Remarks:

(1) The phase space area that a beam fills in (x , a) phase space shrinks during acceleration by the factor p_0/p. This area is the emittance ε.

(2) The phase space area that a beam fills in (x , p_x) phase space is conserved. This area (divided by mc) is the normalized emittance ε_n.

$$\varepsilon = \varepsilon_n / \beta_r \gamma_r$$

Invariant of motion

$$x(s) = \sqrt{2J\beta(s)}\sin(\psi(s)+\phi_0)$$

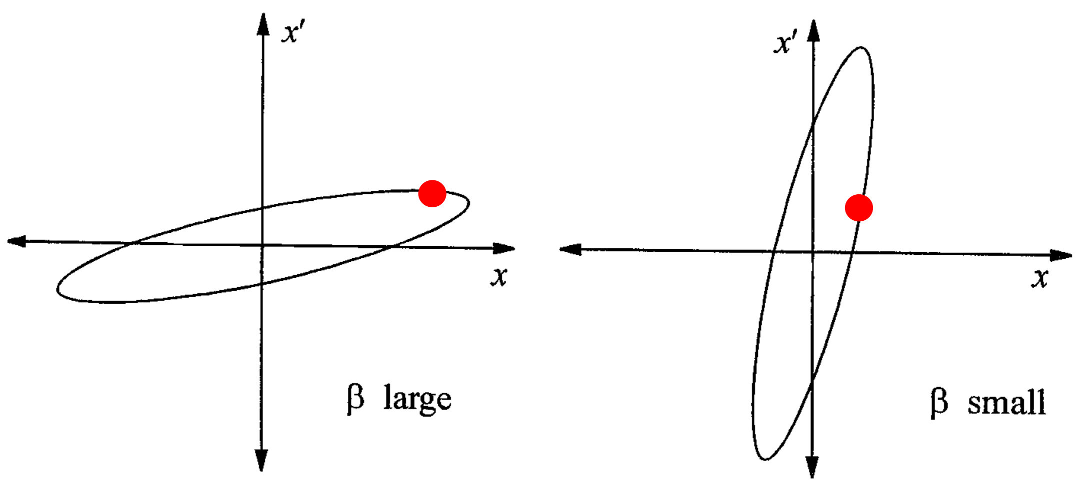
Where J and ϕ are given by the starting conditions x_0 and x'_0.

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = 2J$$

Leads to the invariant of motion:

$$f(x,x',s) = \gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 \quad \Rightarrow \quad \tfrac{d}{ds} f = 0$$

It is called the Courant-Snyder invariant.



Twiss differential equation ➔ usually too hard

$$\gamma = \frac{1+\alpha^2}{\beta}$$
$$\beta' = -2\alpha$$
$$\alpha' = k\beta - \gamma$$

$$\beta'' = 2\gamma = 2\frac{1+\frac{1}{4}\beta'^2}{\beta} = \frac{d\beta'}{d\beta}\frac{d\beta}{ds}$$
$$\frac{\beta'}{1+\frac{1}{4}\beta'^2}d\beta' = 2\frac{d\beta}{\beta}$$
$$\log(1+\tfrac{1}{4}\beta'^2) = \log(\beta/\beta_0)$$
$$\beta' = 2\sqrt{\beta/\beta_0 - 1}$$
$$\frac{d\beta}{2\sqrt{\beta/\beta_0 - 1}} = ds$$
$$\beta_0\sqrt{\beta/\beta_0 - 1} = s - s_0$$
$$\beta(s) = \beta_0\left(1+\left(\frac{s-s_0}{\beta_0}\right)^2\right)$$

Propagation of Twiss parameters

$$(x_0, x_0') \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = 2J$$

$$(x, x') \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = 2J = (x_0, x_0') \underline{M}^T \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \underline{M} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

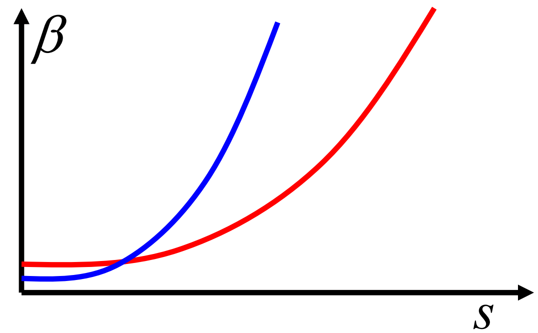
$$\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} = \underline{M}^{-T} \begin{pmatrix} \gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0 \end{pmatrix} \underline{M}^{-1}$$

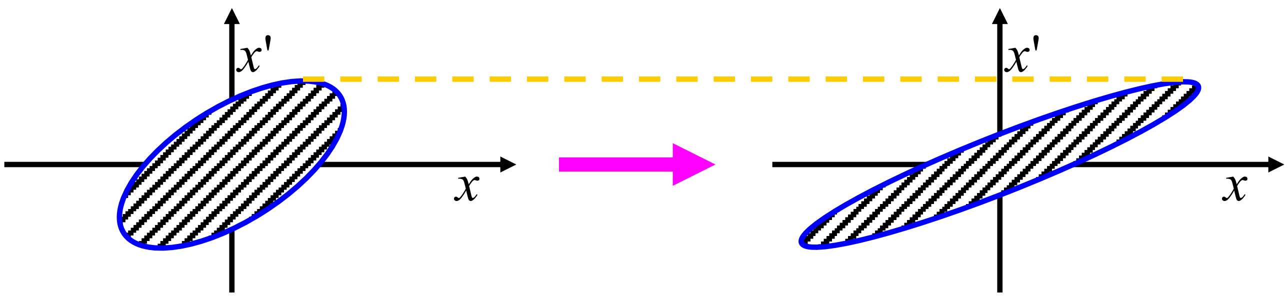
$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \underline{M} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \underline{M}^T$$

Twiss parameters in a drift

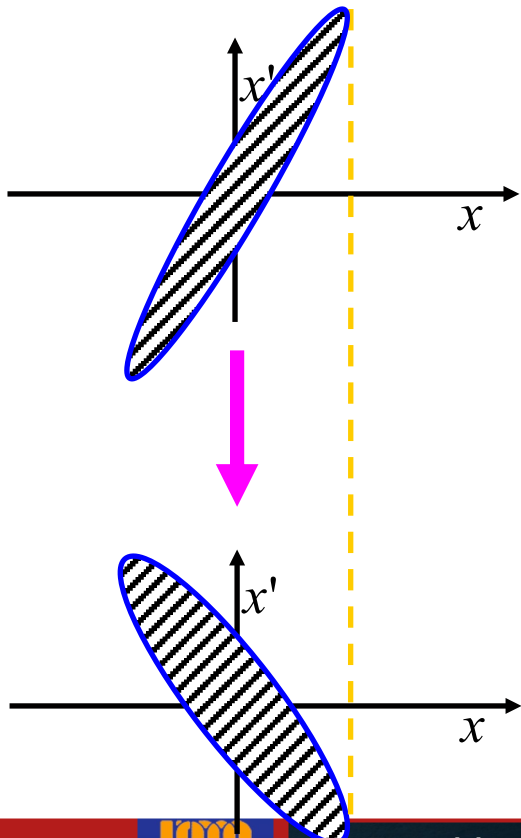
$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} \beta_0 - 2\alpha_0 s + \gamma_0 s^2 & \gamma_0 s - \alpha_0 \\ \gamma_0 s - \alpha_0 & \gamma_0 \end{pmatrix}$$

$$\beta = \beta_0^*[1 + (\tfrac{s}{\beta_0^*})^2] \quad \text{for} \quad \alpha_0^* = 0$$



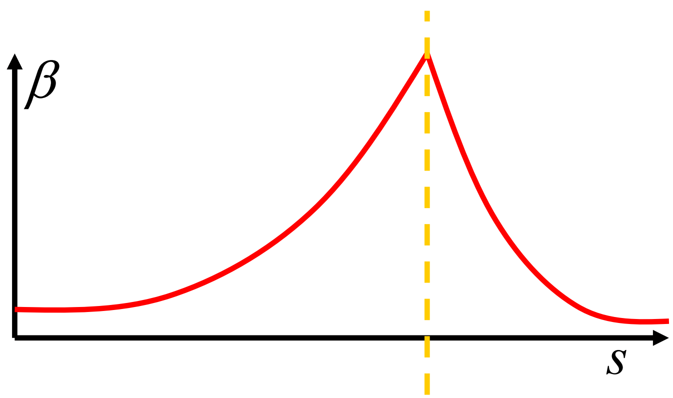


Twiss parameters in a thin quadrupole



$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\alpha = \alpha_0 + k\beta_0}$$



From Twiss parameter to Transfer Marix

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta_0} & 0 \\ -\frac{\alpha_0}{\sqrt{\beta_0}} & \frac{1}{\sqrt{\beta_0}} \end{pmatrix} \begin{pmatrix} \sin(\phi_0) \\ \cos(\phi_0) \end{pmatrix}$$

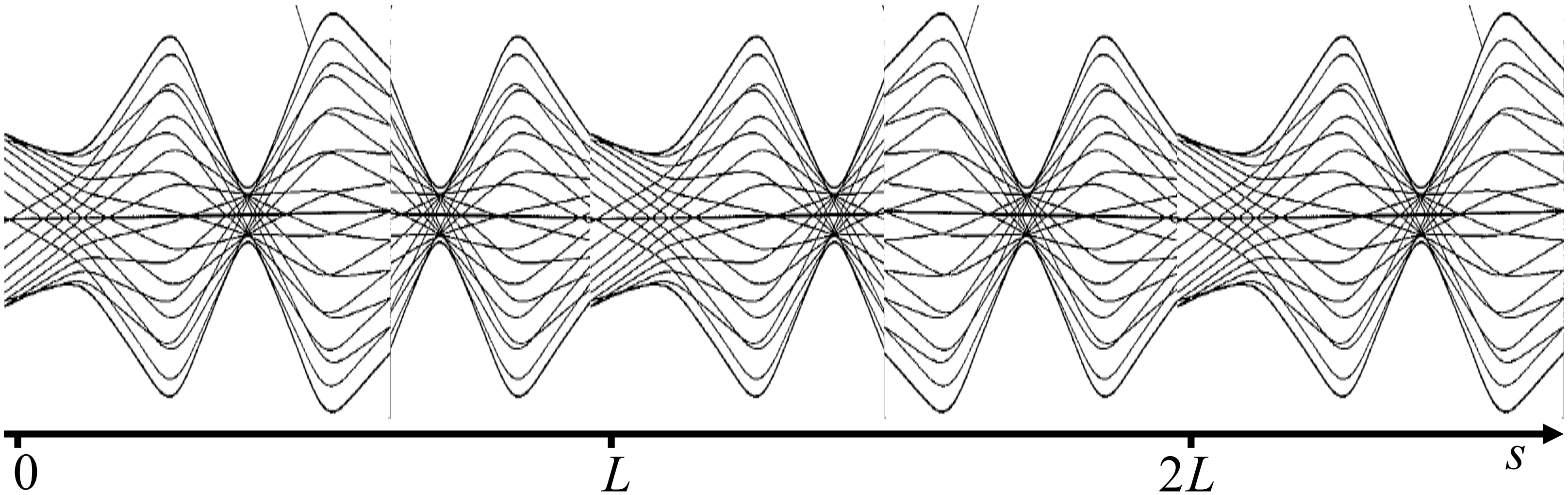
$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

$$= \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\psi(s) & \sin\psi(s) \\ -\sin\psi(s) & \cos\psi(s) \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix}$$

$$\underline{M}(s) = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\psi(s) & \sin\psi(s) \\ -\sin\psi(s) & \cos\psi(s) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}[\cos\psi + \alpha_0 \sin\psi] & \sqrt{\beta_0 \beta}\sin\psi \\ \sqrt{\frac{1}{\beta_0 \beta}}[(\alpha_0 - \alpha)\cos\psi - (1+\alpha_0\alpha)\sin\psi] & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi - \alpha\sin\psi] \end{pmatrix}$$

Periodic solutions in a periodic accelerator



$$\vec{z}(s) = \underline{M}(s,0)\vec{z}(0)$$

$$\vec{z}(L) = \underline{M}(L,0)\vec{z}(0)$$

$$\vec{z}(s+L) = \underline{M}_0(s)\vec{z}(s) \quad , \quad \underline{M}_0 = \underline{M}(s+L,s)$$

$$\vec{z}(s+nL) = \underline{M}_0^n(s)\vec{z}(s)$$

Periodic beta functions

If the particle distribution in a ring or any other periodic structure is stable, it is periodic from turn to turn.

$$\rho(x,x',s+L)=\rho(x,x',s)$$

To be matched to such a beam, the Twiss parameters α, β, γ must be the same after every turn.

$$\underline{M}(s,0)=\begin{pmatrix}\sqrt{\frac{\beta}{\beta_0}}[\cos\psi+\alpha_0\sin\psi] & \sqrt{\beta_0\beta}\sin\psi \\ \sqrt{\frac{1}{\beta_0\beta}}[(\alpha_0-\alpha)\cos\psi-(1+\alpha_0\alpha)\sin\psi] & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi-\alpha\sin\psi]\end{pmatrix}$$

$$\underline{M}_p(s)=\begin{pmatrix}\cos\mu+\alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu-\alpha\sin\mu\end{pmatrix}=\underline{1}\cos\mu+\begin{pmatrix}\alpha & \beta \\ -\gamma & -\alpha\end{pmatrix}\sin\mu$$

$$\mu=\psi(s+L)-\psi(s)$$

One turn matrix to periodic Twiss parameters

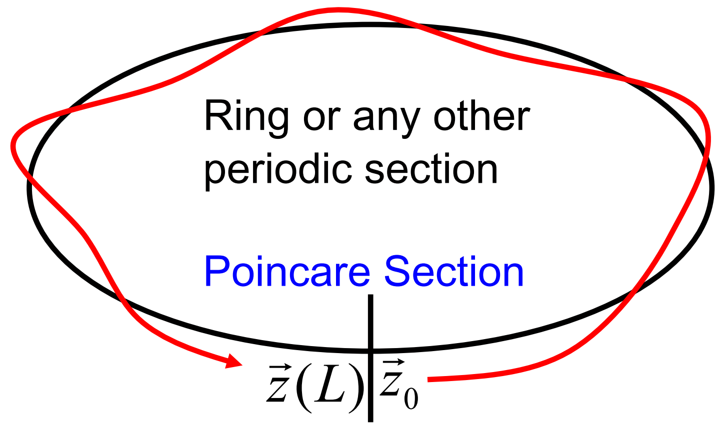
The periodic Twiss parameters are the solution of a nonlinear differential equation with periodic boundary conditions:

$$\beta' = -2\alpha \qquad \text{with} \qquad \beta(L) = \beta(0)$$

$$\alpha' = k\beta - \tfrac{1+\alpha^2}{\beta} \qquad \text{with} \qquad \alpha(L) = \alpha(0)$$

$$\mu = \int_0^L \tfrac{1}{\beta(\hat{s})}\, d\hat{s}$$

Note: $\beta(s) > 0$



$$\underline{M}_0(s) = \underline{1}\cos\mu + \underline{\beta}\sin\mu\,;\ \underline{\beta} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$\cos\mu = \tfrac{1}{2}\mathrm{Tr}[\underline{M}_0(s)]$$

$$\beta = \underline{M}_{0,12}\,\tfrac{1}{\sin\mu}$$

$$\alpha = (\underline{M}_{0,11} - \underline{M}_{0,22})\,\tfrac{1}{2\sin\mu}$$

$$\gamma = \tfrac{1+\alpha^2}{\beta}$$

Stable beam motion and thus a periodic beta function can only exist when |Tr[$\underline{M}$]| < 2.

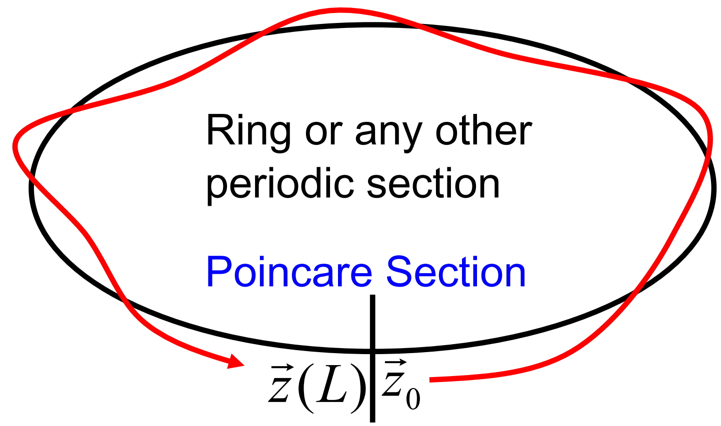
The tune of a particle accelerator

The betatron phase advance per turn devided by 2π is called the TUNE.

$$\mu = 2\pi\nu = \psi(s+L) - \psi(s)$$

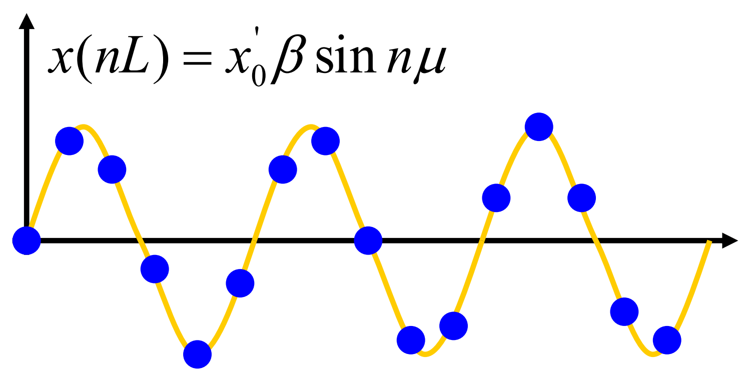
It is a property of the ring and does not depend on the azimuth s.

$$\underline{M}_0(s) = \underline{1}\cos\mu + \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} \sin\mu$$



$$2\cos\underline{\mu(s)} = \mathrm{Tr}[\underline{M}_0(s)] = \mathrm{Tr}[\underline{M}(s,0)\underline{M}_0(0)\underline{M}^{-1}(s,0)]$$

$$= \mathrm{Tr}[\underline{M}_0(0)] = 2\cos\underline{\mu(0)}$$



$$\underline{M}_0^n = \underline{1}\cos n\mu + \underline{\beta}\sin n\mu$$