Variation of constants

$$\vec{z}\,' = \vec{f}(\vec{z}, s)$$

$$\vec{z}\,' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s)$$

Field errors, nonlinear fields, etc can lead to $\Delta\vec{f}(\vec{z}, s)$

$$\vec{z}\,'_H = \underline{L}(s)\vec{z}_H \quad \Rightarrow \quad \vec{z}_H(s) = \underline{M}(s)\vec{z}_{H0} \quad \text{with} \quad \underline{M}'(s)\vec{a} = \underline{L}(s)\underline{M}(s)\vec{a}$$

$$\vec{z}(s) = \underline{M}(s)\vec{a}(s) \quad \Rightarrow \quad \vec{z}\,'(s) = \underline{M}'(s)\vec{a} + \underline{M}(s)\vec{a}\,'(s) = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s)$$

$$\vec{a}(s) = \vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\, d\hat{s}$$

$$\vec{z}(s) = \underline{M}(s)\left\{\vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\, d\hat{s}\right\}$$

$$= \vec{z}_H(s) + \int_0^s \underline{M}(s - \hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\, d\hat{s}$$

Perturbations are propagated from s to s’

Orbit distortions for a one-pass accelerator

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \Delta f$$

The extra force can for example come from an erroneous dipole field or from a correction coil: $\Delta f = \frac{q}{p}\Delta B_y = \Delta\kappa$

Variation of constants: $\vec{z} = \underline{M}\vec{z}_0 + \Delta\vec{z}$ with $\Delta\vec{z} = \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix} 0 \\ \Delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$

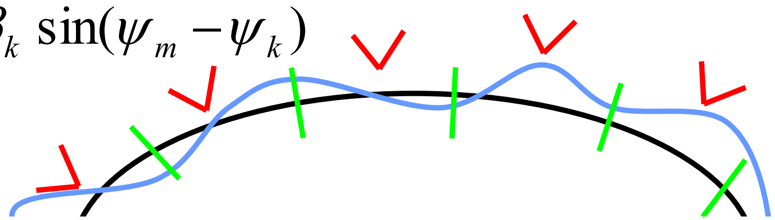
$$\Delta x(s) = \sum_k \Delta\vartheta_k \sqrt{\beta(s)\beta_k}\,\sin(\psi(s) - \psi_k)$$

Orbit correction for a one-pass accelerator

When the closed orbit $x_{\mathrm{co}}^{\mathrm{old}}(s_m)$ is measured at beam position monitors (BPMs, index m) and is influenced by corrector magnets (index k), then the monitor readings before and after changing the kick angles created in the correctors by $\Delta\vartheta_k$ are related by

$$x_{\mathrm{co}}^{\mathrm{new}}(s_m) = x_{\mathrm{co}}^{\mathrm{old}}(s_m) + \sum_k \Delta\vartheta_k \sqrt{\beta_m \beta_k}\,\sin(\psi_m - \psi_k)$$

$$= x_{\mathrm{co}}^{\mathrm{old}}(s_m) + \sum_k O_{mk} \Delta\vartheta_k$$



$$\vec{x}_{\mathrm{co}}^{\mathrm{new}} = \vec{x}_{\mathrm{co}}^{\mathrm{old}} + \underline{O}\Delta\vec{\vartheta}$$

$$\Delta\vec{\vartheta} = -\underline{O}^{-1}\vec{x}_{\mathrm{co}}^{\mathrm{old}} \quad \Rightarrow \quad \vec{x}_{\mathrm{co}}^{\mathrm{new}} = 0$$

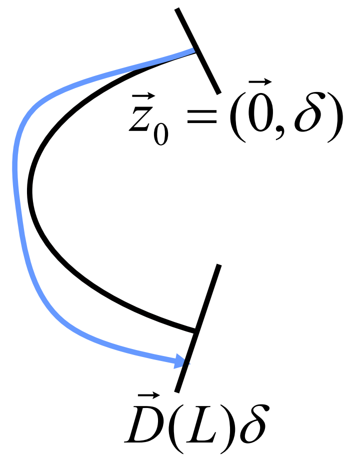
It is often better not to try to correct the closed orbit at the the BPMs to zero in this way since

1. computation of the inverse can be numerically unstable, so that small errors in the old closed orbit measurement lead to a large error in the corrector coil settings.
2. A zero orbit at all BPMs can be a bad orbit in-between BPMs

Dispersion of one-pass accelerators

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \kappa\delta$$



$$\vec{z} = \underline{M}\vec{z}_0 + \int_0^s \underline{M}(s-\hat{s}) \begin{pmatrix} 0 \\ \delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$$

$$\Rightarrow \quad \vec{D}(s) = \int_0^s \underline{M}(s-\hat{s}) \begin{pmatrix} 0 \\ \kappa(\hat{s}) \end{pmatrix} ds'$$

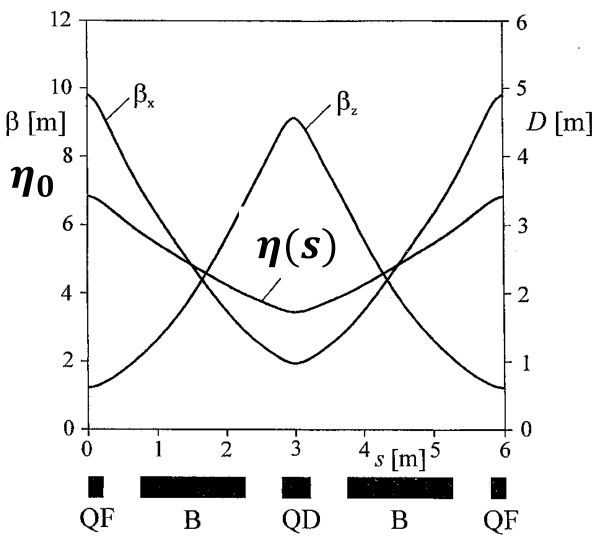
$$\Delta\kappa = \delta\kappa$$

$$D(s) = \sqrt{\beta(s)} \int_0^s \kappa(\hat{s}) \sqrt{\beta(\hat{s})} \sin(\psi(s) - \psi(\hat{s})) d\hat{s}$$

Alternatively, one can multiply the 6x6 matrices and take $\boldsymbol{D(s) = M_{16}(s)}$

Fodo Cells and periodic dispersion

Alternating gradients allow focusing in both transverse plains. Therefore, focusing and defocusing quadrupoles are usually alternated and interleaved with bending magnets.



The dispersion that starts with 0 is called $\boldsymbol{D}(\boldsymbol{s})$, the dispersion that is periodic in a section is called $\boldsymbol{\eta}(\boldsymbol{s})$.

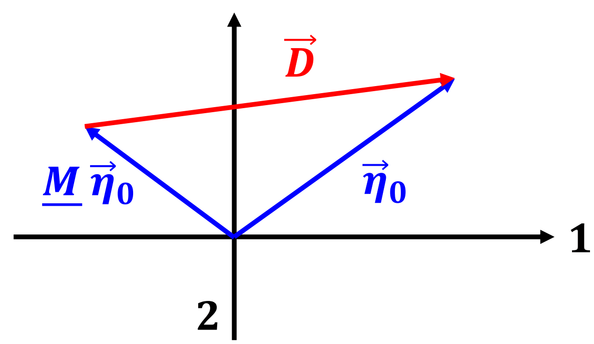
$$\underline{M}_0 = \underline{M}_{FoDo}^N$$

The periodic beta function and dispersion for each FODO is also periodic for an accelerator section that consists of many FODO cells. Often large sections of an accelerator consist of FODOs.

$$\vec{\eta}(s) = \begin{pmatrix} \eta(s) \\ \eta'(s) \end{pmatrix}, \vec{D}(s) = \begin{pmatrix} D(s) \\ D'(s) \end{pmatrix}$$

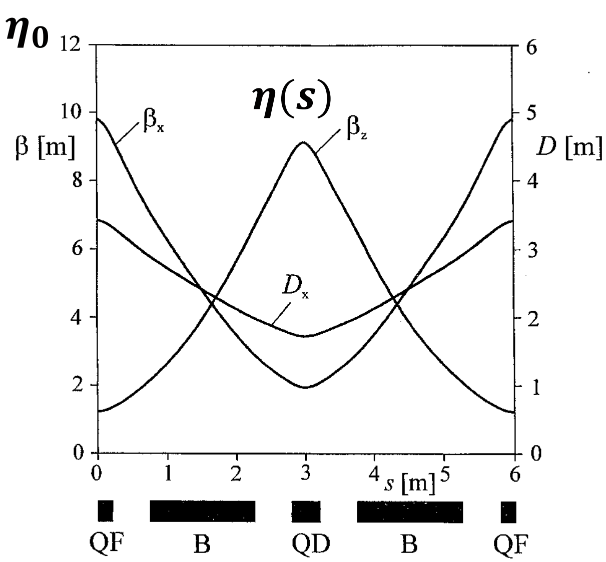
$$\vec{\eta}(s) = \underline{M}(s)\,\vec{\eta}_0 + \vec{D}(s)$$

After the FoDo: $\vec{\eta}_0 = \underline{M}(s)\,\vec{\eta}_0 + \vec{D} \;\Rightarrow\; \vec{\eta}_0 = (\underline{1} - \underline{M})^{-1}\,\vec{D}$

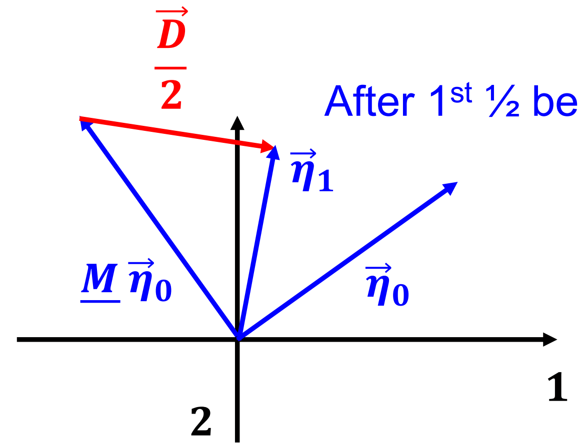


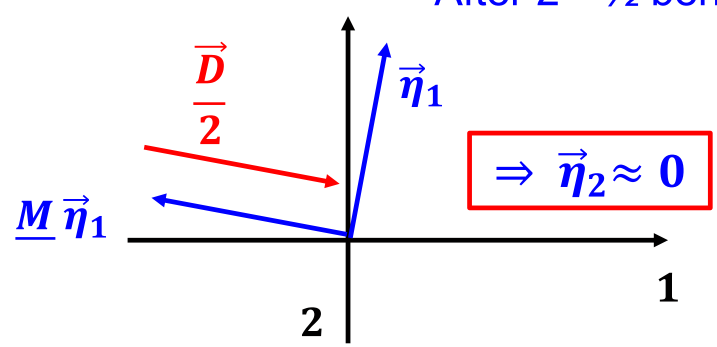
Dispersion suppression by missing bends

After an arc of periodic FoDo cells one would often like to suppress the dispersion to 0 while not changing the betas much from the periodic cells. This can be done by having one FoDo with bends reduced by a factor α followed by one reduced by (1- α).



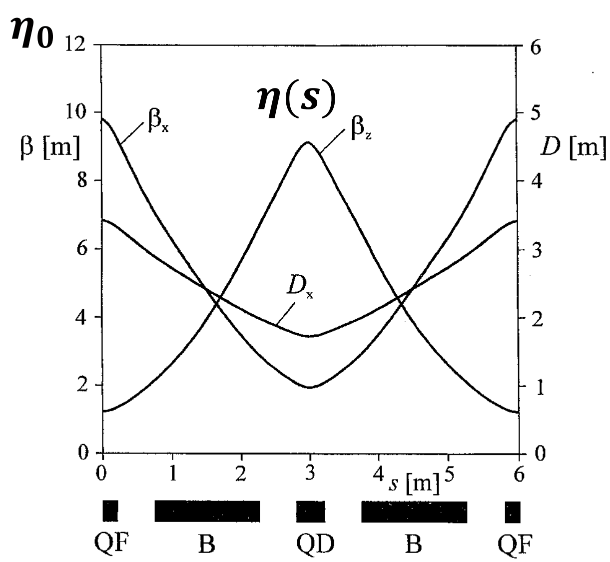
Example: **90 degrees** FoDo cell and $\boldsymbol{\alpha} = \frac{1}{2}$:



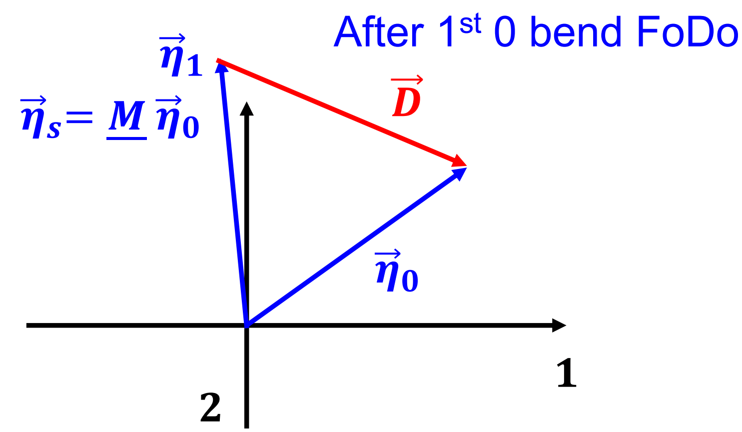


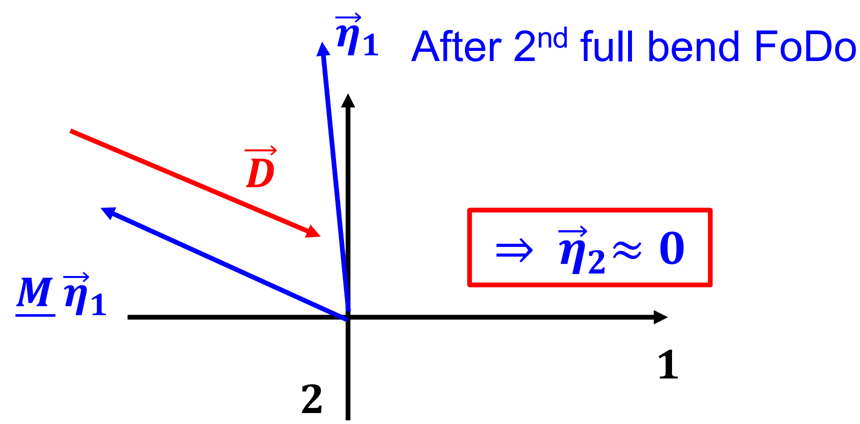
Dispersion suppression by missing bends

After an arc of periodic FoDo cells one would often like to suppress the dispersion to 0 while not changing the betas much from the periodic cells. This can be done by having one FoDo with bends reduced by a factor α followed by one reduced by (1- α).



Example: **60 degrees** FoDo cell and $\boldsymbol{\alpha} = \mathbf{0}$:



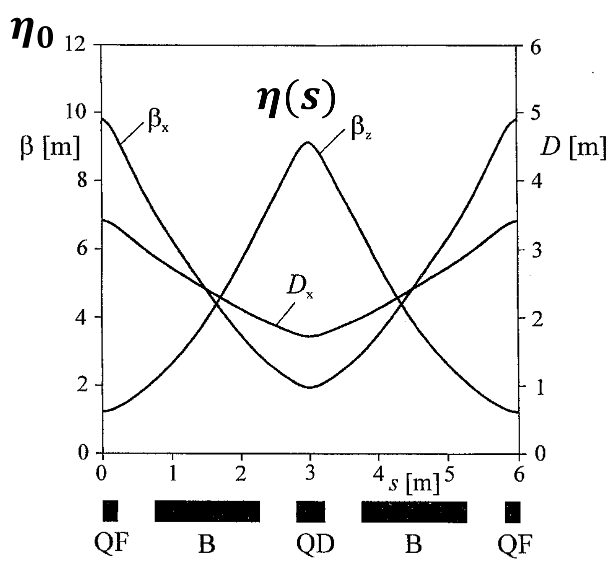


For every FoDo phase advance there is an α to make η 0.

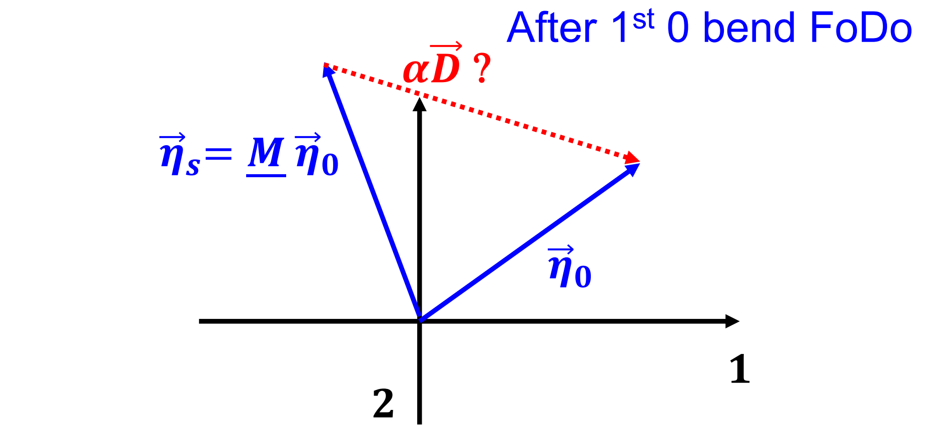
Cornell University Brookhaven National Laboratory

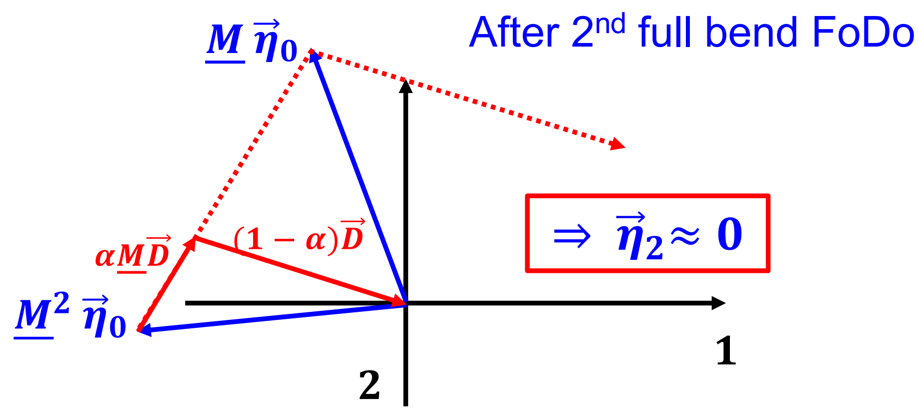
Dispersion suppression by missing bends

After an arc of periodic FoDo cells one would often like to suppress the dispersion to 0 while not changing the betas much from the periodic cells. This can be done by having one FoDo with bends reduced by a factor α followed by one reduced by (1- α).



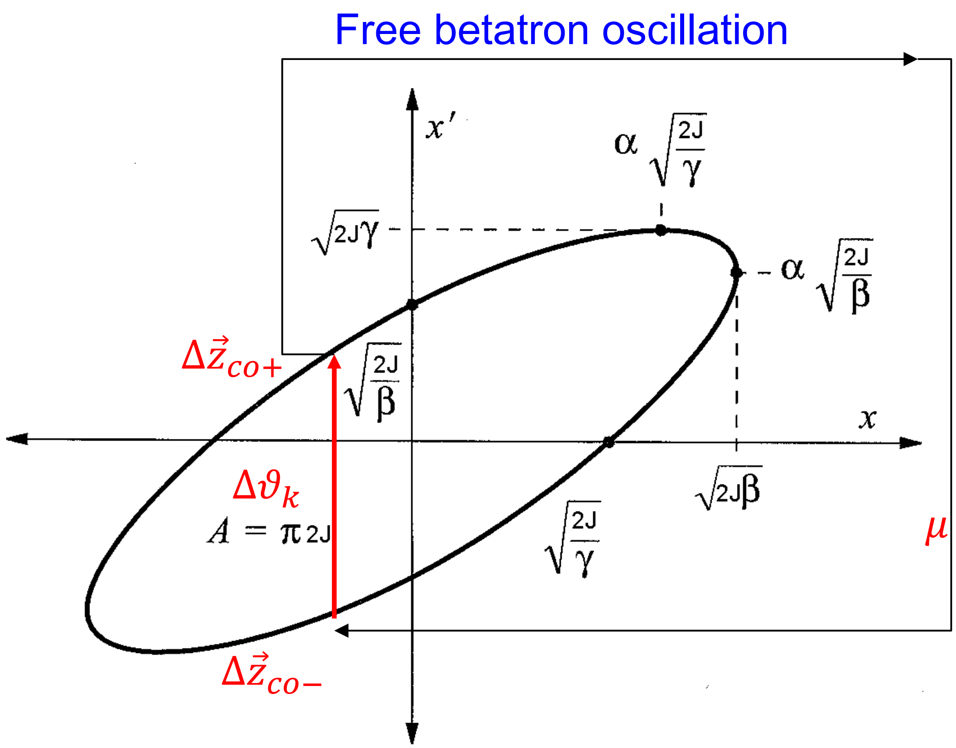
For any other FoDo phase advance: is there an $\boldsymbol{\alpha}$?





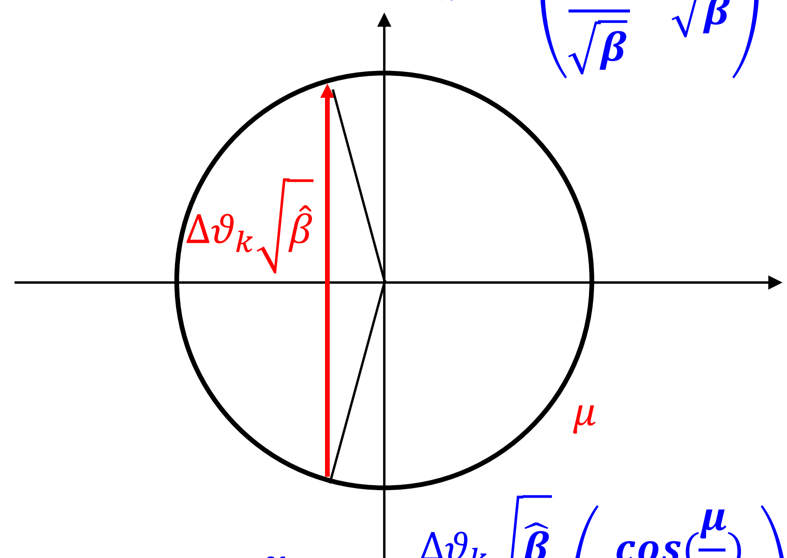
For every FoDo phase advance there is an α to make η 0.

Closed orbit for one kick



$$x_n(s) = \frac{\Delta\vartheta_k\sqrt{\beta\hat{\beta}}}{2\sin(\frac{\mu}{2})}\cos(\frac{\mu}{2} - |\psi - \psi_k|)$$

$$\begin{pmatrix}x_n\\a_n\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{\beta}} & 0\\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta}\end{pmatrix}\begin{pmatrix}x\\x'\end{pmatrix}$$



$$\begin{pmatrix}x_{n\pm}\\a_{n\pm}\end{pmatrix} = \frac{\Delta\vartheta_k\sqrt{\hat{\beta}}}{2\sin(\frac{\mu}{2})}\begin{pmatrix}\cos(\frac{\mu}{2})\\ \pm\sin(\frac{\mu}{2})\end{pmatrix}$$

$$\begin{pmatrix}x_{n\pm}\\a_{n\pm}\end{pmatrix} = \frac{\Delta\vartheta_k\sqrt{\hat{\beta}}}{2\sin(\frac{\mu}{2})}\cos\begin{pmatrix}\frac{\mu}{2} - |\psi - \psi_k|\\ \ldots\end{pmatrix}$$

Closed orbit correction in periodic accelerators

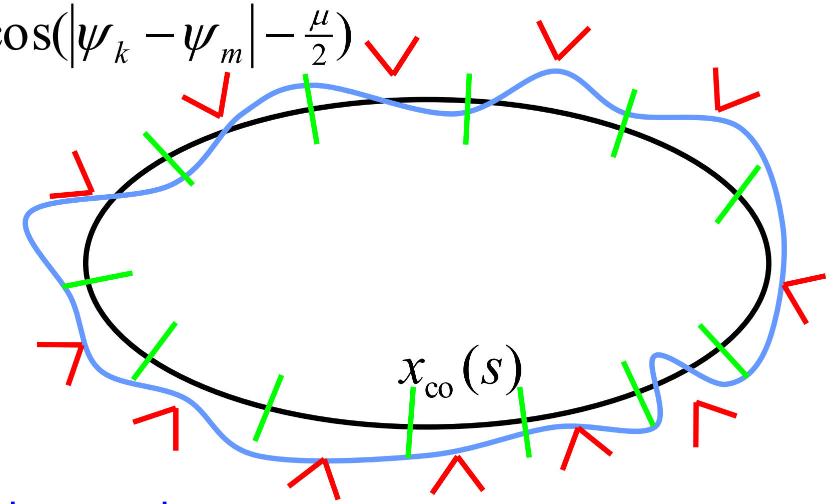
When the closed orbit $x_{co}^{old}(s_m)$ is measured at beam position monitors (BPMs, index m) and is influenced by corrector magnets (index k), then the monitor readings before and after changing the kick angles created in the correctors by $\Delta\vartheta_k$ are related by

$$x_{co}^{new}(s_m) = x_{co}^{old}(s_m) + \sum_k \Delta\vartheta_k \frac{\sqrt{\beta_m\beta_k}}{2\sin\frac{\mu}{2}} \cos(|\psi_k - \psi_m| - \tfrac{\mu}{2})$$

$$= x_{co}^{old}(s_m) + \sum_k O_{mk}\Delta\vartheta_k$$

$$\vec{x}_{co}^{new} = \vec{x}_{co}^{old} + \underline{O}\Delta\vec{\vartheta}$$

$$\Delta\vec{\vartheta} = -\underline{O}^{-1}\vec{x}_{co}^{old} \quad \Rightarrow \quad \vec{x}_{co}^{new} = 0$$

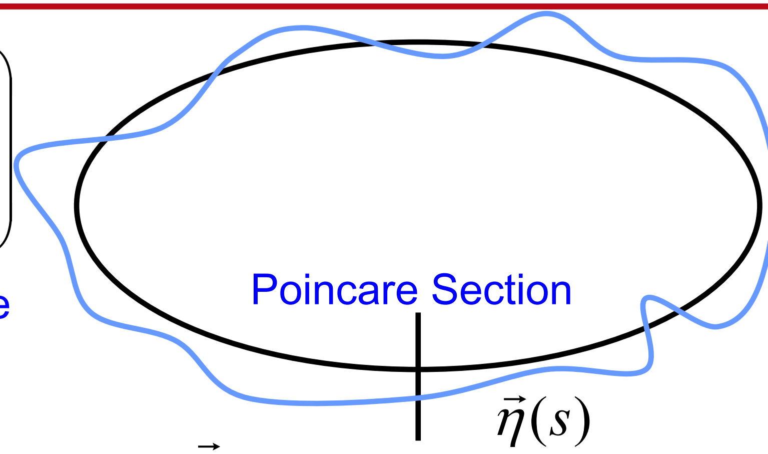


It is often better not to try to correct the closed orbit at the the BPMs to zero in this way since

1. computation of the inverse can be numerically unstable, so that small errors in the old closed orbit measurement lead to a large error in the corrector coil settings.
2. A zero orbit at all BPMs can be a bad orbit inbetween BPMs

Periodic dispersion

$$\begin{pmatrix} \underline{M}_{0x}\vec{z}_0+\vec{D}(L)\delta \\ M_{56}\delta \\ \delta \end{pmatrix} = \begin{pmatrix} \underline{M}_{0x} & \vec{0} & \vec{D}(L) \\ \vec{T}^T & 1 & M_{56} \\ \vec{0}^T & 0 & 1 \end{pmatrix}\begin{pmatrix} \vec{z}_0 \\ 0 \\ \delta \end{pmatrix}$$



The periodic orbit for particles with relative energy deviation δ is

$$\vec{\eta}(0)=\underline{M}_0\vec{\eta}(0)+\vec{D}(L) \qquad \vec{\eta}(L)=\underline{M}_0\vec{\eta}(0)+\vec{D}(L) \quad \text{with} \quad \vec{\eta}(L)=\vec{\eta}(0)$$

$$\Downarrow$$

$$\vec{\eta}(0)=[\underline{1}-\underline{M}_0(0)]^{-1}\vec{D}(L)$$

Particles with energy deviation δ oscillates around this periodic orbit.

$$\vec{z}=\vec{z}_\beta+\delta\vec{\eta}$$

$$\vec{z}_\beta(L)+\delta\vec{\eta}(L)=\vec{z}(L)=\underline{M}_0\vec{z}(0)+\vec{D}(L)\delta=\underline{M}_0[\vec{z}_\beta(0)+\delta\vec{\eta}(0)]+\vec{D}(L)\delta$$

$$=\underline{M}_0\vec{z}_\beta(0)+\delta\vec{\eta}(L)$$