Higher-order multipoles

$$\psi = \Psi_n \operatorname{Im}\{(x-iy)^n\} = \Psi_n \cdot (\ldots - i\,n\,x^{n-1}y) \quad \Rightarrow \quad \vec{B}(y=0) = \Psi_n\, n \begin{pmatrix} 0 \\ x^{n-1} \end{pmatrix}$$

Multipole strength: $k_n = \frac{q}{p} \partial_x^n B_y \big|_{x,y=0} = \frac{q}{p} \Psi_{n+1}\,(n+1)!$ units: $\frac{1}{\mathrm{m}^{n+1}}$

p/q is also called Bρ and used to describe the energy of multiply charge ions

Names: dipole, quadrupole, sextupole, octupole, decapole, duodecapole, …

Higher order multipoles come from

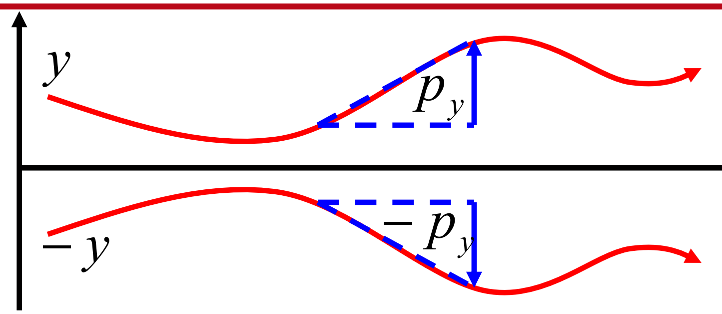
- Field errors in magnets
- Magnetized materials
- From multipole magnets that compensate such erroneous fields
- To compensate nonlinear effects of other magnets
- To stabilize the motion of many particle systems
- To stabilize the nonlinear motion of individual particles

Midplane-symmetric motion

$$\vec{r}^{\oplus} = (x, -y, z)$$

$$\vec{p}^{\oplus} = (p_x, -p_y, p_z)$$

$$\tfrac{d}{dt}\vec{p} = \vec{F}(\vec{r}, \vec{p}) \quad \Rightarrow \quad \tfrac{d}{dt}\vec{p}^{\oplus} = \vec{F}(\vec{r}^{\oplus}, \vec{p}^{\oplus})$$



$$\begin{aligned} v_y B_z - v_z B_y &= -v_y B_z(x,-y,z) - v_z B_y(x,-y,z) & & B_x(x,-y,z) = -B_x(x,y,z) \\ v_z B_x - v_x B_z &= -v_z B_x(x,-y,z) + v_x B_z(x,-y,z) \Rightarrow & & B_y(x,-y,z) = \;\; B_y(x,y,z) \\ v_x B_y - v_y B_x &= \;\; v_x B_y(x,-y,z) + v_y B_x(x,-y,z) & & B_z(x,-y,z) = -B_z(x,y,z) \end{aligned}$$

$$\boxed{\psi(x,-y,z) = -\psi(x,y,z)}$$

$$\Psi_n \operatorname{Im}\left\{ e^{in\vartheta_n}(x+iy)^n \right\} = -\Psi_n \operatorname{Im}\left\{ e^{in\vartheta_n}(x-iy)^n \right\}$$

$$\Rightarrow \quad \Psi_n \operatorname{Im}\left[e^{in\vartheta_n} 2\operatorname{Re}\left\{ (x+iy)^n \right\} \right] = 0 \quad \Rightarrow \quad \boxed{\vartheta_n = 0}$$

The discussed multipoles

produce midplane symmetric motion. When the field is rotated by π/2, i.e $\vartheta_n = \pi/2n$, one speaks of a skew multipole.

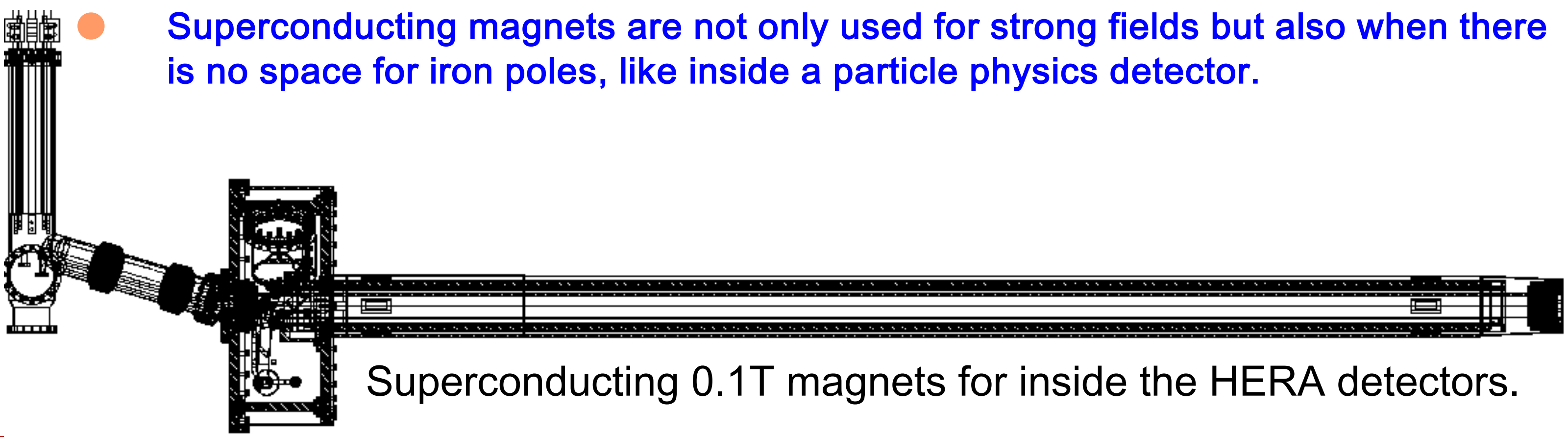
Superconducting magnets

Above 2T the field from the bare coils dominate over the magnetization of the iron.
But Cu wires cannot create much filed without iron poles:
5T at 5cm distance from a 3cm wire would require a current density of

$$j = \frac{I}{d^2} = \frac{1}{d^2}\frac{2\pi\, r\, B}{\mu_0} = 1389\,\frac{\mathrm{A}}{\mathrm{mm}^2}$$

Cu can only support about 100A/mm^2.

- Superconducting cables routinely allow current densities of 1500A/mm^2 at 4.6 K and 6T. Materials used are usually Nb aloys, e.g. NbTi, Nb_3Ti or Nb_3Sn.
- Superconducting magnets are not only used for strong fields but also when there is no space for iron poles, like inside a particle physics detector.

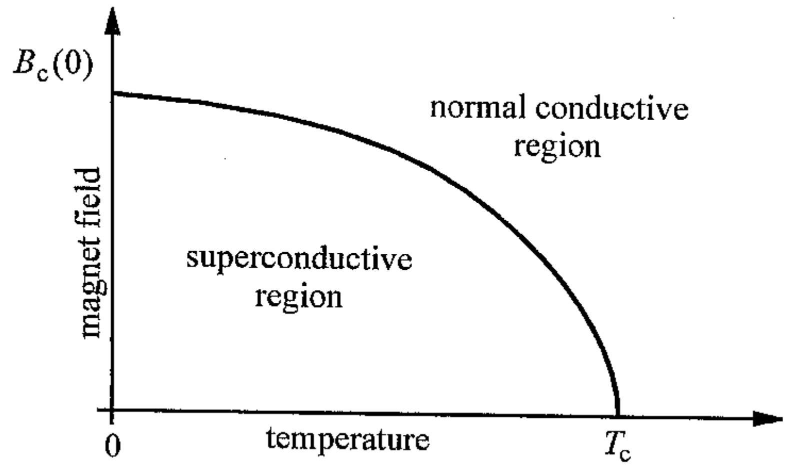


Superconducting 0.1T magnets for inside the HERA detectors.

Superconducting cables

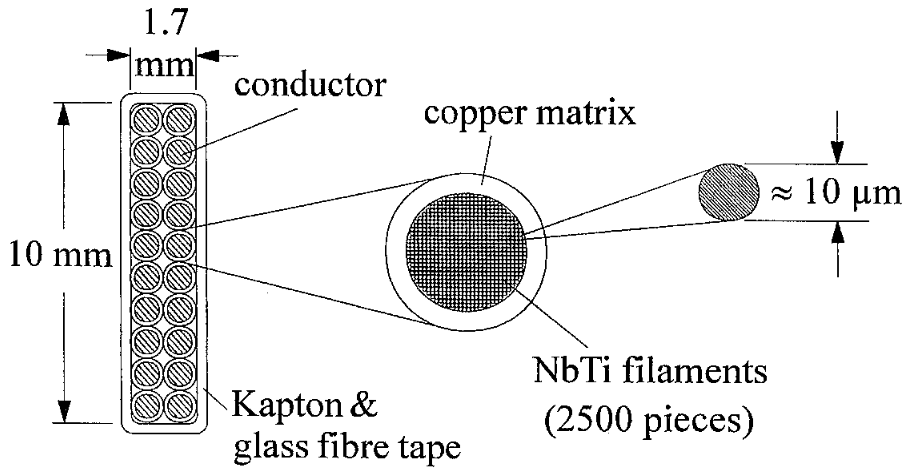
Problems:

- Superconductivity brakes down for too large fields
- Due to the Meissner-Ochsenfeld effect superconductivity current only flows on a thin surface layer.



Remedy:

- Superconducting cable consists of many very thin filaments (about 10μm).



Complex scalar magnetic potential of a wire

Straight wire at the origin: $\vec{\nabla}\times\vec{B}=\mu_0\,\vec{j} \quad\Rightarrow\quad \vec{B}(r)=\frac{\mu_0 I}{2\pi r}\vec{e}_\varphi=\frac{\mu_0 I}{2\pi r}\begin{pmatrix}-y\\ x\end{pmatrix}$

Wire at $\vec{a}$:

$$\vec{B}(x,y)=\frac{\mu_0 I}{2\pi(\vec{r}-\vec{a})^2}\begin{pmatrix}-[y-a_y]\\ x-a_x\end{pmatrix}$$

This can be represented by complex multipole coefficients Ψ_ν

$$\vec{B}(x,y)=-\vec{\nabla}\Psi \quad\Rightarrow\quad B_x+iB_y=-(\partial_x+i\partial_y)\psi=-2\partial_{\overline{w}}\psi$$

$$B_x+iB_y=\tfrac{\mu_0 I}{2\pi}\frac{-i(w_a-w)}{(w_a-w)(\overline{w}_a-\overline{w})}=i\,\tfrac{\mu_0 I}{2\pi}\frac{-\frac{w_a}{a^2}}{1-\frac{\overline{w}w_a}{a^2}}$$

$$=i\,\tfrac{\mu_0 I}{2\pi}\partial_{\overline{w}}\ln(1-\tfrac{\overline{w}w_a}{a^2})=-2\partial_{\overline{w}}\,\mathrm{Im}\left\{\tfrac{\mu_0 I}{2\pi}\ln(1-\tfrac{\overline{w}w_a}{a^2})\right\}$$

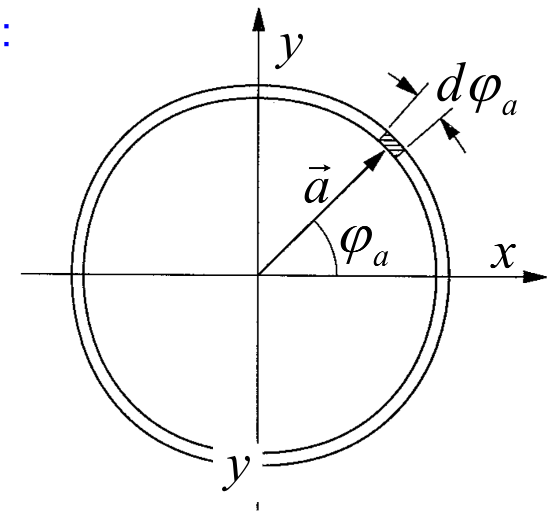
$$\psi=\mathrm{Im}\left\{\tfrac{\mu_0 I}{2\pi}\ln(1-\tfrac{\overline{w}w_a}{a^2})\right\}=-\mathrm{Im}\left\{\tfrac{\mu_0 I}{2\pi}\sum_{\nu=1}^{\infty}\tfrac{1}{\nu}\left(\tfrac{w_a}{a^2}\right)^\nu\overline{w}^\nu\right\} \quad\Rightarrow\quad \Psi_\nu=\tfrac{\mu_0 I}{2\pi}\tfrac{1}{\nu}\tfrac{1}{a^\nu}e^{i\nu\varphi_a}$$

Air-coil multipoles

Creating a multipole be created by an arrangement of wires:

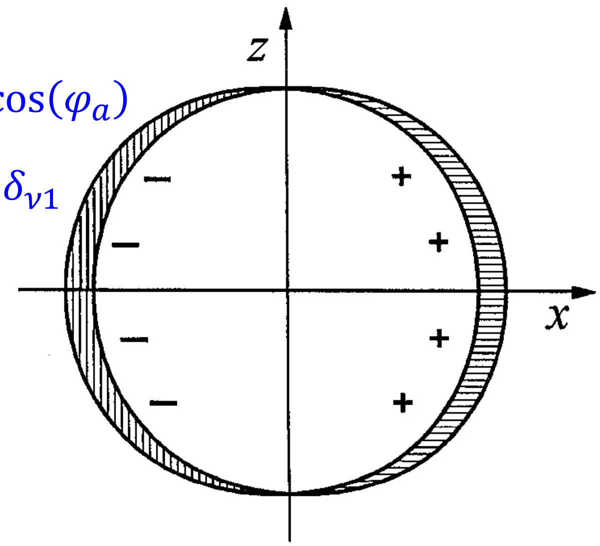
$$\Psi_\nu = \int_0^{2\pi} \frac{\mu_0}{2\pi} \frac{1}{\nu} \frac{1}{a^\nu} e^{i\nu\varphi_a} \frac{dI}{d\varphi_a} d\varphi_a$$

The Multipole coefficient Ψ_ν is the **Fourier coefficient** of the angular charge distribution $\frac{\mu_0}{\nu a^\nu}\frac{dI}{d\varphi_a}$.

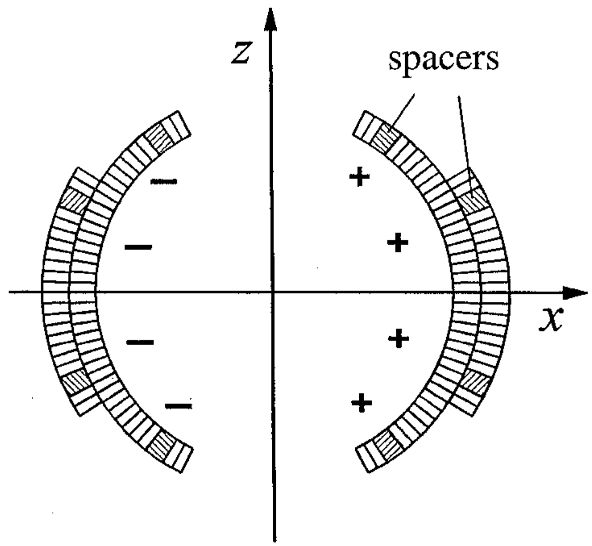


$$\frac{dI}{d\varphi_a} \propto \cos(\varphi_a)$$

➔ $\Psi_\nu \propto \delta_{\nu 1}$

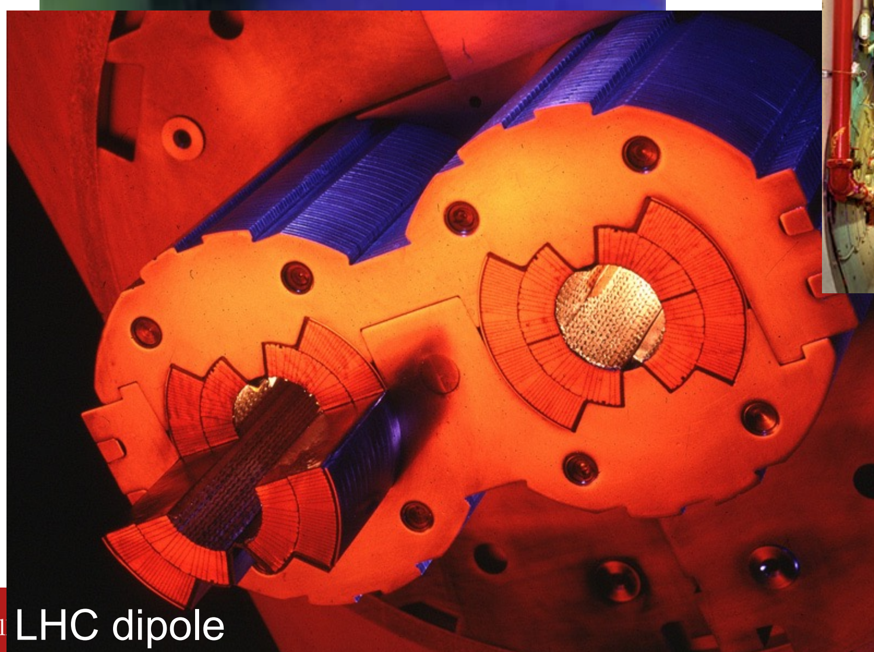


Ideal multipole



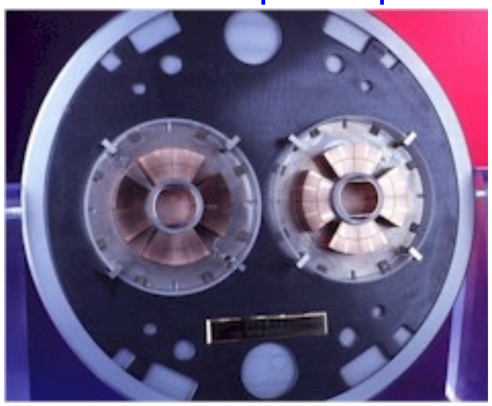
Approximate multipole

Real Air-coil multipoles

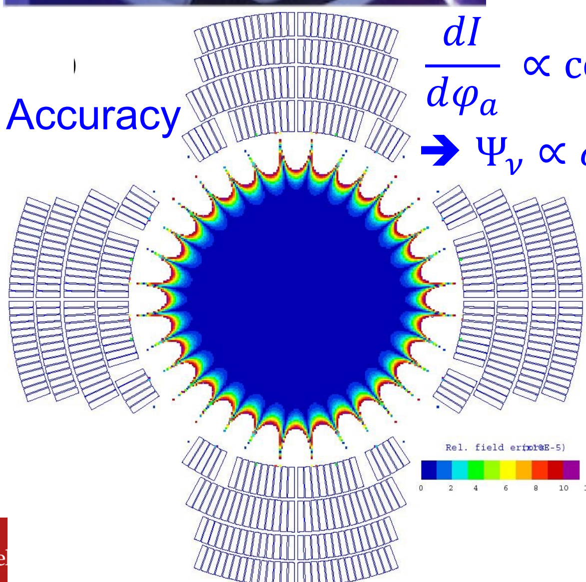


Special super-conducting Air-coil magnets

LHC double quadrupole

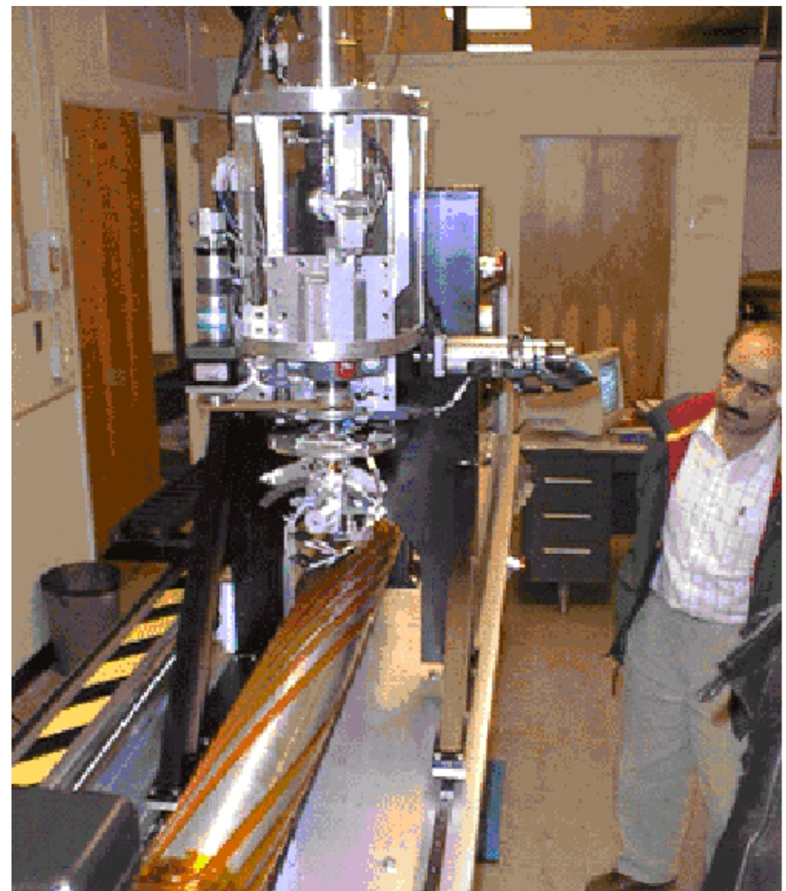


Accuracy



$$\frac{dI}{d\varphi_a} \propto \cos(2\varphi_a)$$

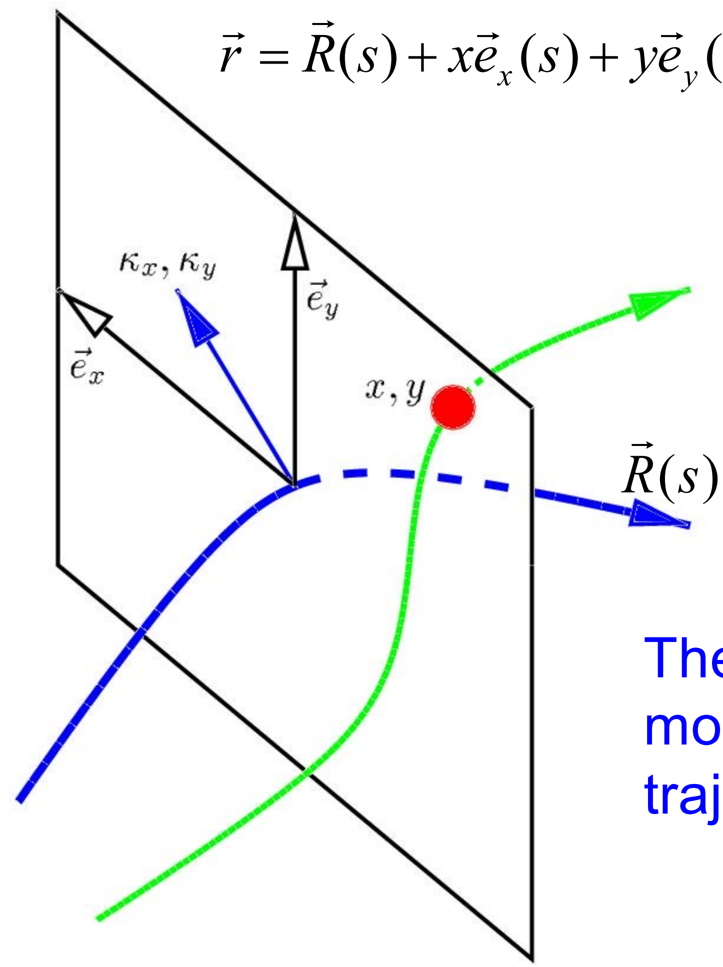
$$\Rightarrow \Psi_\nu \propto \delta_{\nu 2}$$



RHIC Siberian Snake dipole

The comoving coordinate system

$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$



$$\left|d\vec{R}\right| = ds$$

$$\boxed{\vec{e}_s \equiv \tfrac{d}{ds}\vec{R}(s)}$$

The time dependence of a particle's motion is often not as interesting as the trajectory along the accelerator length "s".

The 4-dimensional equation of motion

$$\frac{d^2}{dt^2}\vec{r} = \vec{f}_r(\vec{r}, \tfrac{d}{dt}\vec{r}, t)$$

3 dimensional ODE of 2nd order can be changed to a
6 dimensional ODE of 1st order:

$$\left.\begin{aligned} \frac{d}{dt}\vec{r} &= \frac{1}{m\gamma}\vec{p} = \frac{c}{\sqrt{p^2-(mc)^2}}\vec{p} \\ \frac{d}{dt}\vec{p} &= \vec{F}(\vec{r},\vec{p},t) \end{aligned}\right\} \quad \frac{d}{dt}\vec{Z} = \vec{f}_Z(\vec{Z},t)\,, \quad \vec{Z} = (\vec{r},\vec{p})$$

If the force does not depend on time, as in a typical beam line magnet, the energy is conserved so that one can reduce the dimension to 5. The equation of motion is then autonomous.

Furthermore, the time dependence is often not as interesting as the trajectory along the accelerator length "s". Using "s" as the independent variable reduces the dimensions to 4. The equation of motion is then no longer autonomous.

$$\frac{d}{ds}\vec{z} = \vec{f}_z(\vec{z},s)\,, \quad \vec{z} = (x, y, p_x, p_y)$$

6D equation of motion

Usually one prefers to compute the trajectory as a function of "s" along the accelerator even when the energy is not conserved, as when accelerating cavities are in the accelerator.

Then the energy "E" and the time "t" at which a particle arrives at the cavities are important. And the equations become 6 dimensional again:

$$\tfrac{d}{ds}\vec{z} = \vec{f}_z(\vec{z}, s)\,, \quad \vec{z} = (x, y, p_x, p_y, -t, E)$$

But: $\vec{z} = (\vec{r}, \vec{p})$ is an especially suitable variable, since it is a phase space vector so that its equation of motion comes from a Hamiltonian, or by variation principle from a Lagrangian.

$$\delta\int\left[p_x\dot{x} + p_y\dot{y} + p_s\dot{s} - H(\vec{r}, \vec{p}, t)\right]dt = 0 \quad \Rightarrow$$ Hamiltonian motion

$$\delta\int\left[p_x x' + p_y y' - H\,t' + p_s(x, y, p_x, p_y, t, H)\right]ds = 0 \quad \Rightarrow$$ Hamiltonian motion

The new canonical coordinates are: $\vec{z} = (x, y, p_x, p_y, -t, E)$ with $\mathrm{E} = \mathrm{H}$

The new Hamiltonian is: $K = -p_s(\vec{z}, s)$

6D phase space motion

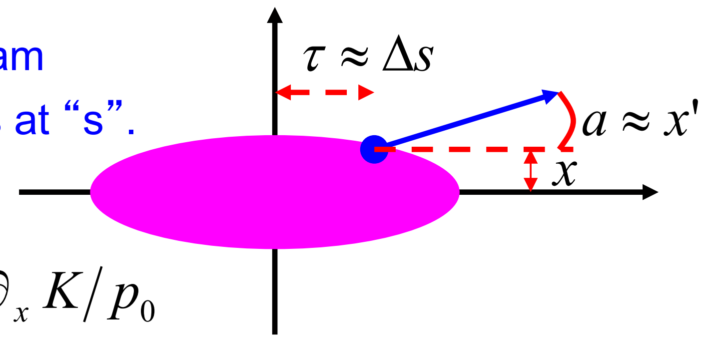
Using a reference momentum p_0 and a reference time t_0:

$$\vec{z} = (x, a, y, b, \tau, \delta)$$

$$a = \frac{p_x}{p_0}, \quad b = \frac{p_y}{p_0}, \quad \delta = \frac{E - E_0}{E_0}, \quad \tau = (t_0 - t)\frac{c^2}{v_0} = (t_0 - t)\frac{E_0}{p_0}$$

Usually p_0 is the design momentum of the beam

And t_0 is the time at which the bunch center is at “s”.



$$\left.\begin{array}{l} x' = \partial_{p_x} K \\ p_x' = -\partial_x K \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x' = \partial_a K/p_0, \quad a' = -\partial_x K/p_0 \\ y' = \partial_b K/p_0, \quad b' = -\partial_y K/p_0 \end{array}\right.$$

$$-t' = \partial_E K \quad \Rightarrow \quad \tau' = \tfrac{c^2}{v_0}\partial_\delta K/E_0 = \partial_\delta K/p_0$$

$$E' = -\partial_{-t} K \quad \Rightarrow \quad \delta' = -\tfrac{1}{E_0}\partial_\tau K \tfrac{c^2}{v_0} = -\partial_\tau K/p_0$$

New Hamiltonian:

$$\tilde{H} = K/p_0$$

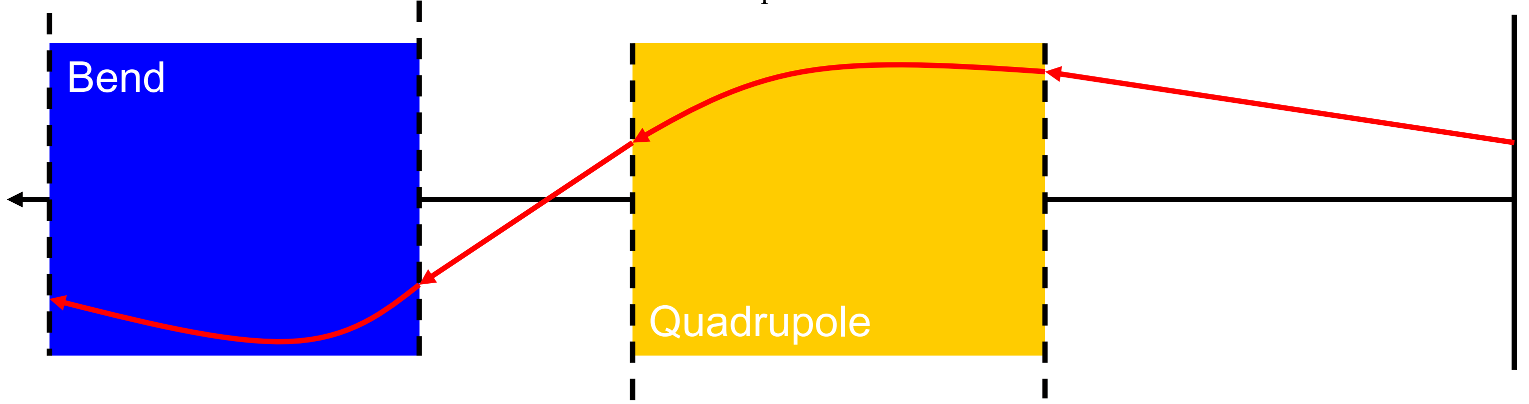
The matrix solution of linear equations of motions

Linear equation of motion: $\vec{z}' = \underline{F}(s)\vec{z}$ $\rightarrow$ $\vec{z}(s) = \underline{M}(s)\,\vec{z}_0$

Matrix solution of the starting condition $\vec{z}(0) = \vec{z}_0$

$$\vec{z} = \underline{M}_{\text{bend}}(L_4)\underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{drift}}(L_3)\underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$



$$\vec{z} = \underline{M}_{\text{drift}}(L_1)\vec{z}_0 \qquad \vec{z}_0$$

$$\vec{z} = \underline{M}_{\text{quad}}(L_2)\underline{M}_{\text{drift}}(L_1)\vec{z}_0$$

Simplest example: motion through an empty drift

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\ddot{x} = 0 \quad \Rightarrow \quad x'' = 0 \quad \Rightarrow \quad a = x', a' = 0$$

Linear solution:

$$x(s) = x_0 + x_0' s$$

$$\begin{pmatrix} x \\ a \\ y \\ b \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} x_0 + s a_0 \\ a \\ y_0 + s b_0 \\ b_0 \\ \tau_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \underline{0} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \vec{z}_0$$

Significance of the Hamiltonian

The equations of motion can be determined by one function:

$$\tfrac{d}{ds}x = \partial_{p_x} H(\vec{z},s)\,,\quad \tfrac{d}{ds}p_x = -\partial_x H(\vec{z},s)\,,\quad \ldots$$

$$\tfrac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial} H(\vec{z},s) = \vec{F}(\vec{z},s) \quad \text{with} \quad \underline{J} = \mathrm{diag}(\underline{J}_2)\,,\quad \underline{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The force has a Hamiltonian Jacobi Matrix:

A linear force: $\vec{F}(\vec{z},s) = \underline{F}(s)\cdot\vec{z}$

The Jacobi Matrix of a linear force: $\underline{F}(s)$

The general Jacobi Matrix : $F_{ij} = \partial_{z_j} F_i$ or $\underline{F} = \left(\vec{\partial}\,\vec{F}^T\right)^T$

Hamiltonian Matrices: $\underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = 0$

Prove : $F_{ij} = \partial_{z_j} F_i = \partial_{z_j} J_{ik} \partial_{z_k} H = J_{ik}\,\partial_k \partial_j H \quad \Rightarrow \quad \underline{F} = \underline{J}\,\underline{D}\underline{H}$

$$\underline{F}\,\underline{J} + \underline{J}\,\underline{F}^T = \underline{J}\,\underline{D}\underline{J}H + \underline{J}\,\underline{D}^T\,\underline{J}^T H = 0$$

Hamiltonian → Symplectic Flow

The flow of a Hamiltonian equation of motion has a symplectic Jacobi Matrix

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

A linear flow: $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0$

The Jacobi Matrix of a linear flow: $\underline{M}(s)$

The general Jacobi Matrix : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = \left(\vec{\partial}_0 \vec{M}^T\right)^T$

The Symplectic Group SP(2N) : $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

$$\tfrac{d}{ds}\vec{z} = \tfrac{d}{ds}\vec{M}(s, \vec{z}_0) = \underline{J}\,\vec{\nabla} H = \vec{F}$$

$$\tfrac{d}{ds} M_{ij} = \partial_{z_{0j}} F_i(\vec{z}, s) = \partial_{z_{0j}} M_k \partial_{z_k} F_i(\vec{z}, s)$$

$$\tfrac{d}{ds}\underline{M}(s, \vec{z}_0) = \underline{F}(\vec{z}, s)\underline{M}(s, \vec{z}_0)$$

$$K = \underline{M}\,\underline{J}\,\underline{M}^T$$

$$\tfrac{d}{ds}\underline{K} = \tfrac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T + \underline{M}\,\underline{J}\,\tfrac{d}{ds}\underline{M}^T = \underline{F}\,\underline{M}\,\underline{J}\,\underline{M}^T + \underline{M}\,\underline{J}\,\underline{M}^T\underline{F}^T = \underline{F}\,\underline{K} + \underline{K}\,\underline{F}^T$$

$\underline{K} = \underline{J}$ is a solution. Since this is a linear ODE , $\underline{K} = \underline{J}$ is the unique solution.

Symplectic Flow → Hamiltonian

For every symplectic transport map there is a Hamilton function

The flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

Force vector: $\vec{h}(\vec{z}, s) = -\underline{J}\left[\frac{d}{ds}\vec{M}(s, \vec{z}_0)\right]_{\vec{z}_0 = \vec{M}^{-1}(\vec{z}, s)}$

Since then: $\frac{d}{ds}\vec{z} = \underline{J}\vec{h}(\vec{z}, s)$

There is a Hamilton function H with: $\vec{h} = \vec{\partial} H$

<u>If and only if:</u> $\partial_{z_j} h_i = \partial_{z_i} h_j \quad \Rightarrow \quad \underline{h} = \underline{h}^T$

$$\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J} \quad \Rightarrow \quad \begin{cases} \frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T = -\underline{M}\,\underline{J}\frac{d}{ds}\underline{M}^T \\ \underline{M}^{-1} = -\underline{J}\,\underline{M}^T\underline{J} \end{cases}$$

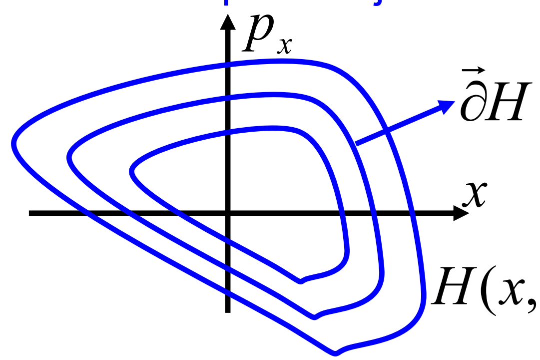
$$\vec{h} \circ \vec{M} = -\underline{J}\frac{d}{ds}\vec{M}$$

$$\underline{h}(\vec{M})\underline{M} = -\underline{J}\frac{d}{ds}\underline{M}$$

$$\underline{h}(\vec{M}) = -\underline{J}\frac{d}{ds}\underline{M}\,\underline{M}^{-1} = \underline{J}\frac{d}{ds}\underline{M}\,\underline{J}\,\underline{M}^T\underline{J} = -\underline{J}\,\underline{M}\,\underline{J}\frac{d}{ds}\underline{M}^T\underline{J} = \underline{M}^{-T}\frac{d}{ds}\underline{M}^T\underline{J} = \underline{h}^T$$

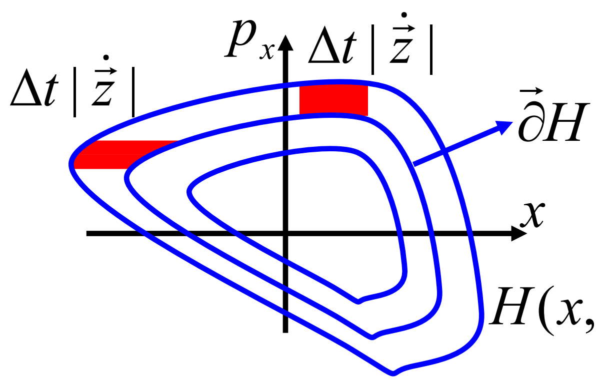
Phase space densities in 2D

- Phase space trajectories move on surfaces of constant energy



$$\tfrac{d}{ds}\vec{z} = \underline{J}\,\vec{\partial} H \quad \Rightarrow \quad \tfrac{d}{ds}\vec{z} \perp \vec{\partial} H$$

- A phase space volume does not change when it is transported by Hamiltonian motion.

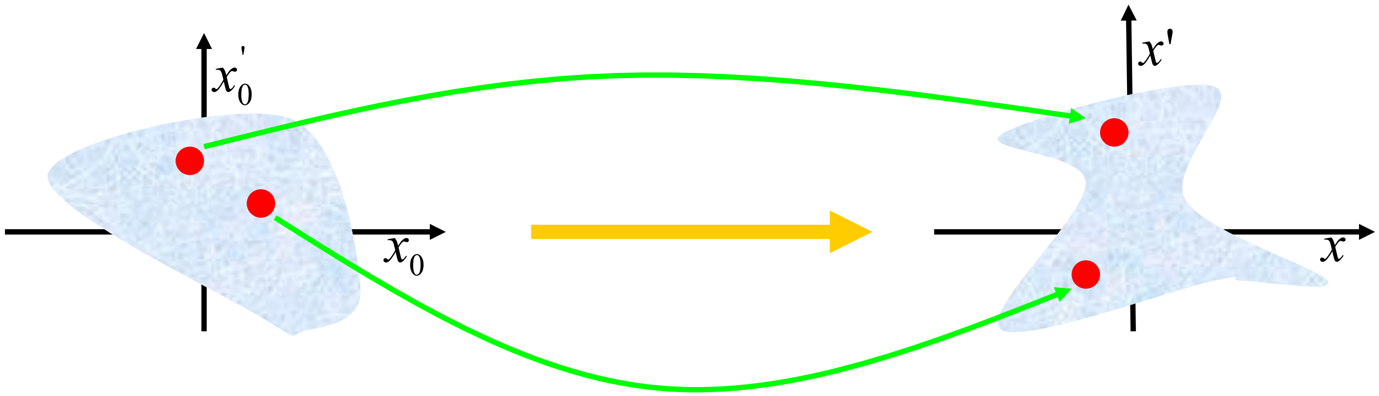


The distance d of lines with equal energy is proportional to $1/|\vec{\partial} H| \propto |\dot{\vec{z}}|^{-1}$

$$\boxed{d * \Delta t\,|\dot{\vec{z}}| = \text{const}}$$

Liouville's Theorem

- A phase space volume does not change when it is transported by Hamiltonian motion. $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0$ with $\det[\underline{M}(s)] = +1$

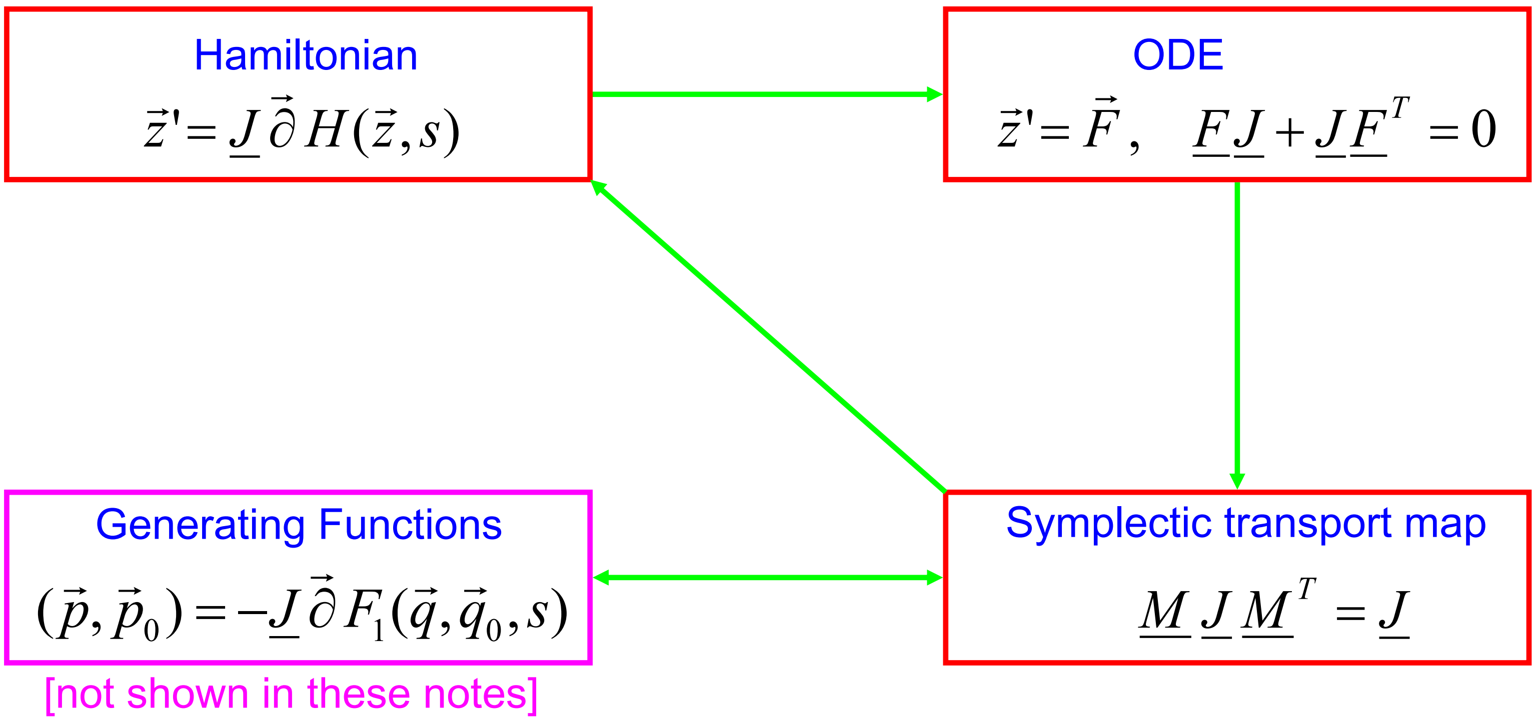


Volume = $V = \iint_V d^n\vec{z} = \iint_{V_0} \left| \frac{\partial \vec{z}}{\partial \vec{z}_0} \right| d^n\vec{z}_0 = \iint_{V_0} |\underline{M}| d^n\vec{z}_0 = \iint_{V_0} d^n\vec{z}_0 = V_0$

Hamiltonian Motion → $V = V_0$

But Hamiltonian requires symplecticity, which is much more than just $\det[\underline{M}(s)] = +1$

Symplectic representations



Eigenvalues of symplectic matrices

For matrices with real coefficients:

If there is an eigenvector and eigenvalue: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$

then the complex conjugates are also eigenvector and eigenvalue: $\underline{M}\vec{v}_i^{\,*} = \lambda_i^* \vec{v}_i^{\,*}$

For symplectic matrices:

If there are eigenvectors and eigenvalues: $\underline{M}\vec{v}_i = \lambda_i \vec{v}_i$ with $\underline{J} = \underline{M}^T \underline{J}\,\underline{M}$

then $\vec{v}_i^{\,T} \underline{J}\vec{v}_j = \vec{v}_i^{\,T} \underline{M}^T \underline{J}\,\underline{M}\vec{v}_j = \lambda_i \lambda_j \vec{v}_i^{\,T} \underline{J}\vec{v}_j \quad \Rightarrow \quad \vec{v}_i^{\,T} \underline{J}\vec{v}_j (\lambda_i \lambda_j - 1) = 0$

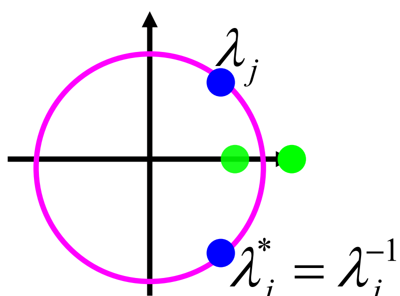
Therefore $\underline{J}\vec{v}_j$ is orthogonal to all eigenvectors with eigenvalues that are not $1/\lambda_j$. Since it cannot be orthogonal to all eigenvectors, there is at least one eigenvector with eigenvalue $1/\lambda_j$

Two dimensions: λ_j is eigenvalue

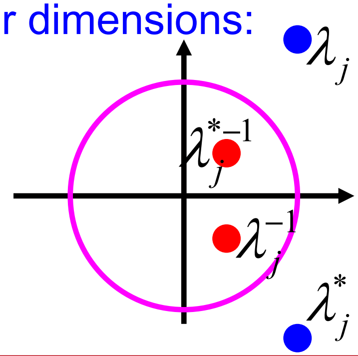
Then $1/\lambda_j$ and λ_j^* are eigenvalues

$\underline{\lambda_2 = 1/\lambda_1 = \lambda_1^*} \quad \Rightarrow \quad |\lambda_j| = 1$

$\lambda_2 = 1/\lambda_1 = \lambda_2^*$



Four dimensions:



Time of flight from symplecticity

$$\underline{M} = \begin{pmatrix} \underline{M}_4 & \vec{0} & \vec{D} \\ \vec{T}^T & 1 & M_{56} \\ \vec{0}^T & 0 & 1 \end{pmatrix}$$ is in SU(6) and therefore $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

$$\begin{pmatrix} \underline{M}_4 \underline{J}_4 & -\vec{D} & \vec{0} \\ \vec{T}^T \underline{J}_4 & -M_{56} & 1 \\ \vec{0}^T & -1 & 0 \end{pmatrix} \begin{pmatrix} \underline{M}_4^T & \vec{T} & \vec{0} \\ \vec{0}^T & 1 & 0 \\ \vec{D}^T & M_{56} & 1 \end{pmatrix} = \begin{pmatrix} \underline{J}_4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \underline{M}_4 \underline{J}_4 \underline{M}_4^T & \underline{M}_4 \underline{J}_4 \vec{T} - \vec{D} & \vec{0} \\ \vec{T}^T \underline{J}_4 \underline{M}_4^T + \vec{D}^T & 0 & 1 \\ \vec{0}^T & -1 & 0 \end{pmatrix} = \begin{pmatrix} \underline{J}_4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\vec{T} = -\underline{J}_4 \underline{M}_4^{-1} \vec{D}$$

It is sufficient to compute the 4D map $\underline{M}_4$, the Dispersion $\vec{D}$ and the time of flight term M_{56}

The drift

$$\begin{pmatrix} x' \\ a' \\ y' \\ b' \\ \tau' \\ \delta' \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Note that in nonlinear expansion $x' \neq a$ so that the drift does not have a linear transport map even though $x(s) = x_0 + x_0' s$ is completely linear.

$$\begin{pmatrix} x \\ a \\ y \\ b \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} x_0 + s a_0 \\ a \\ y_0 + s b_0 \\ b_0 \\ \tau_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix} 1 & s \\ 0 & 1 \end{matrix} & \underline{0} \\ \underline{0} & \underline{0} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \vec{z}_0$$

The quadrupole

$$x'' = -x\,k$$

$$y'' = \;\; y\,k$$

$$\underline{M}_4 = \begin{pmatrix} \cos(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}\,s) & & \\ -\sqrt{k}\sin(\sqrt{k}\,s) & \cos(\sqrt{k}\,s) & \multicolumn{2}{c}{\underline{0}} \\ & & \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ \multicolumn{2}{c}{\underline{0}} & \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix}$$

As for a drift, the energy does not change, i.e. $\delta = \delta_0$.
The time of flight only depends on energy, i.e. $\tau = \tau_0 + M56\,\delta$.

For k<0 one has to take into account that

$$\cos(\sqrt{k}\,s) = \cosh(\sqrt{|k|}\,s)\,, \quad \sin(\sqrt{k}\,s) = i\sinh(\sqrt{|k|}\,s)$$

$$\cosh(\sqrt{k}\,s) = \cos(\sqrt{|k|}\,s)\,, \quad \sinh(\sqrt{k}\,s) = i\sin(\sqrt{|k|}\,s)$$

Variation of constants

$$\vec{z}\,' = \vec{f}(\vec{z}, s)$$

$$\vec{z}\,' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s)$$

Field errors, nonlinear fields, etc can lead to $\Delta\vec{f}(\vec{z}, s)$

$$\vec{z}\,'_H = \underline{L}(s)\vec{z}_H \quad \Rightarrow \quad \vec{z}_H(s) = \underline{M}(s)\vec{z}_{H0} \quad \text{with} \quad \underline{M}'(s)\vec{a} = \underline{L}(s)\underline{M}(s)\vec{a}$$

$$\vec{z}(s) = \underline{M}(s)\vec{a}(s) \quad \Rightarrow \quad \vec{z}\,'(s) = \underline{M}'(s)\vec{a} + \underline{M}(s)\vec{a}\,'(s) = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s)$$

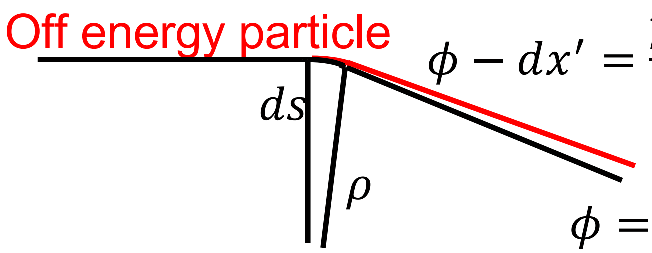
$$\vec{a}(s) = \vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\, d\hat{s}$$

$$\vec{z}(s) = \underline{M}(s)\left\{\vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\, d\hat{s}\right\}$$

$$= \vec{z}_H(s) + \int_0^s \underline{M}(s - \hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s})\, d\hat{s}$$

Perturbations are propagated from s to s’

The dipole equation of motion



$$\phi - dx' = \frac{p_0}{p} ds/\rho = \phi(1 - \frac{dp}{p})$$

$$\phi = ds/\rho$$

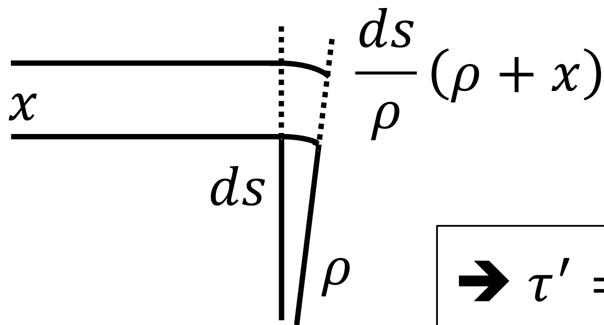
$$\Rightarrow x'' = \frac{1}{\rho}\frac{dp}{p} = \frac{1}{\rho}\frac{dp}{dE}\frac{E}{p}\delta = \frac{1}{\rho}\frac{1}{\beta^2}\delta$$

$$x'' + x\,\kappa^2 = \frac{\kappa}{\beta^2}\delta \quad \text{with} \quad \kappa = \frac{1}{\rho} \quad \text{or} \quad \begin{aligned} x' &= a \\ a' &= -\kappa^2 x + \frac{\kappa}{\beta^2}\delta \end{aligned}$$

$$x_H'' + x_H\,\kappa^2 = 0 \;\Rightarrow\; \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) \\ -\kappa\sin(\kappa s) & \cos(\kappa s) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \underline{M}(s) \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} - \underline{M}(s)\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \int_0^s \underline{M}(s-\zeta)\begin{pmatrix} 0 \\ \frac{\kappa}{\beta^2}\delta \end{pmatrix} d\zeta = \int_0^s \begin{pmatrix} \frac{1}{\kappa}\sin(\kappa s) \\ \cos(\kappa s) \end{pmatrix} d\zeta \frac{\kappa}{\beta^2}\delta = \begin{pmatrix} \frac{1}{\kappa}(1-\cos(\kappa s)) \\ \sin(\kappa s) \end{pmatrix} \frac{1}{\beta^2}\delta$$

The dipole transport matrix



$$\Rightarrow \tau' = -\frac{x}{\rho}\frac{1}{v}\frac{E_0}{p_0} = -\frac{x}{\rho}\frac{1}{\beta^2} = -\frac{\kappa}{\beta^2}x$$

$$\underline{M} = \begin{pmatrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) & \underline{0} & & 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ -\kappa\sin(\kappa s) & \cos(\kappa s) & & & 0 & \sin(\kappa s) \\ \underline{0} & & 1 & s & \underline{0} & \\ & & 0 & 1 & & \\ -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] & \underline{0} & & 1 & \kappa^{-1}[\sin(\kappa s)-s\kappa] \\ 0 & 0 & & & 0 & 1 \end{pmatrix}$$

(for $\beta = 1$)

$$M_{56} = -\kappa\int_0^s M_{16}\,ds$$

The combined function bend

$$x'' = -x\,(\underbrace{\kappa^2 + k}_{K}) + \delta\kappa$$

$$y'' = y\,k \quad , \quad \tau' = -\kappa\, x$$

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0}\,\vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{T} & \underline{0} & \underline{M}_\tau \end{pmatrix}$$

$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K}\,s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\,s) \\ -\sqrt{K}\sin(\sqrt{K}\,s) & \cos(\sqrt{K}\,s) \end{pmatrix}$$

$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k}\,s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\,s) \\ \sqrt{k}\sinh(\sqrt{k}\,s) & \cosh(\sqrt{k}\,s) \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \frac{\kappa}{K}[1-\cos(\sqrt{K}s)] \\ \frac{\kappa}{\sqrt{K}}\sin(\sqrt{K}s) \end{pmatrix}$$

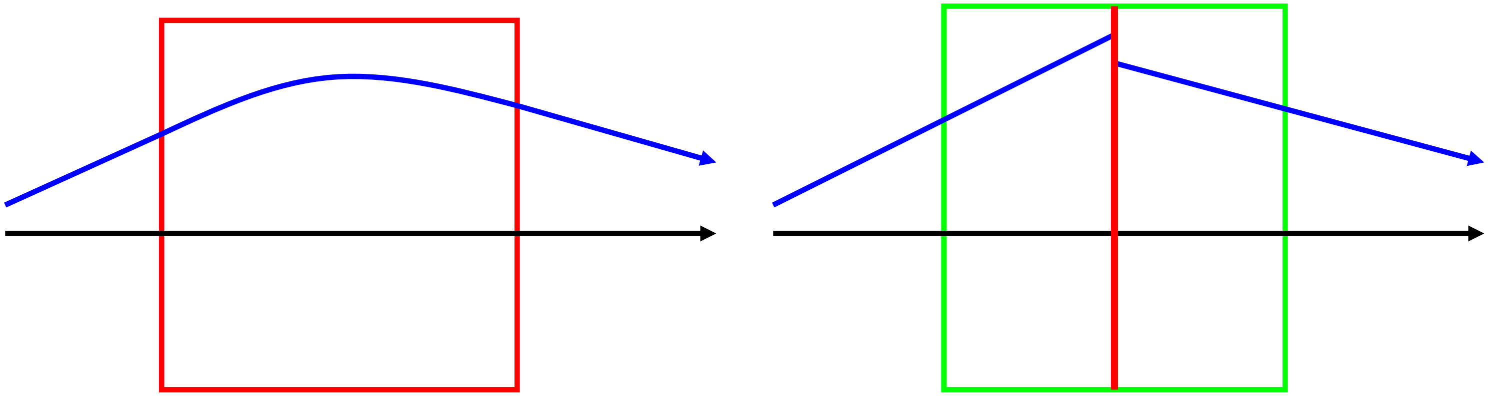
Options:

- For k>0:
  focusing in x, defocusing in y.
- For k<0, K<0:
  defocusing in x, focusing in y.
- For k<0, K>0:
  weak focusing in both planes.

$$\underline{M}_\tau = \begin{pmatrix} 1 & -\kappa\int_0^s M_{16}ds \\ 0 & 1 \end{pmatrix}$$

$\underline{T}$ from symplecticity

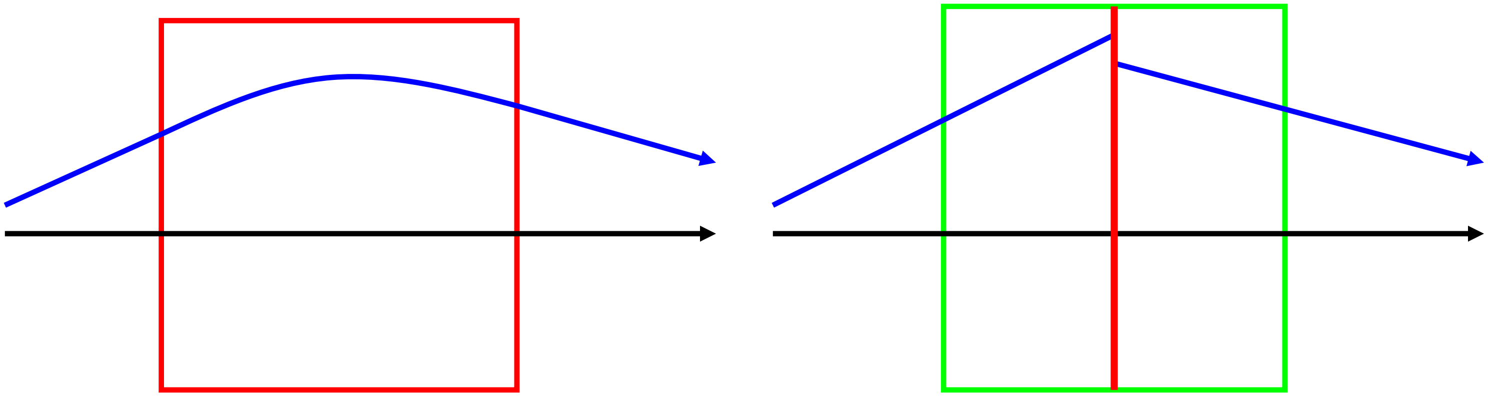
Thin lens approximation



$$\vec{z}(s) = \underline{M}(s)\,\vec{z}_0 = \underline{D}(\tfrac{s}{2})\underline{D}^{-1}(\tfrac{s}{2})\underline{M}(s)\underline{D}^{-1}(\tfrac{s}{2})\underline{D}(\tfrac{s}{2})\vec{z}_0$$

Drift: $\underline{M}^{\text{thin}}_{\text{drift}}(s) = \underline{D}^{-1}(\tfrac{s}{2})\underline{M}(s)\underline{D}^{-1}(\tfrac{s}{2}) = \underline{1}$

The thin lens quadrupole



$$\underline{M}^{\text{thin}}_{\text{quad,x}}(s) = \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\sqrt{k}s) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}s) \\ -\sqrt{k}\sin(\sqrt{k}s) & \cos(\sqrt{k}s) \end{pmatrix} \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & s \\ -ks & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{s}{2} \\ -ks & 1+\frac{ks^2}{2} \end{pmatrix}$$

Weak magnet limit: $\sqrt{k}s << 1$

$$\underline{M}^{\text{thin}}_{\text{quad,x}}(s) \approx \begin{pmatrix} 1 & 0 \\ -ks & 1 \end{pmatrix}$$

The thin lens dipole

$$\underline{M} = \begin{pmatrix} \cos(\kappa s) & \frac{1}{\kappa}\sin(\kappa s) & 0 & 0 & 0 & \kappa^{-1}[1-\cos(\kappa s)] \\ -\kappa\sin(\kappa s) & \cos(\kappa s) & 0 & 0 & 0 & \sin(\kappa s) \\ 0 & 0 & 1 & s & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s)-1] & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Weak magnet limit: $\kappa s << 1$

$$\underline{M}^{\text{thin}}_{\text{bend},x\tau}(s) = \underline{D}(-\tfrac{s}{2})\underline{M}_{\text{bend},x\tau}\underline{D}(-\tfrac{s}{2}) \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\kappa^2 s & 1 & 0 & \kappa s \\ -\kappa s & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The thin lens combined function bend

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0}\,\vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{T} & \underline{0} & \underline{1} \end{pmatrix}$$

Weak magnet limit: $\kappa s << 1$

$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K}\, s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}\, s) \\ -\sqrt{K}\sin(\sqrt{K}\, s) & \cos(\sqrt{K}\, s) \end{pmatrix}$$

$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k}\, s) & \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\, s) \\ \sqrt{k}\sinh(\sqrt{k}\, s) & \cosh(\sqrt{k}\, s) \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \frac{\kappa}{K}[1-\cos(\sqrt{K}s)] \\ \frac{\kappa}{\sqrt{K}}\sin(\sqrt{K}s) \end{pmatrix}$$

$\longrightarrow$

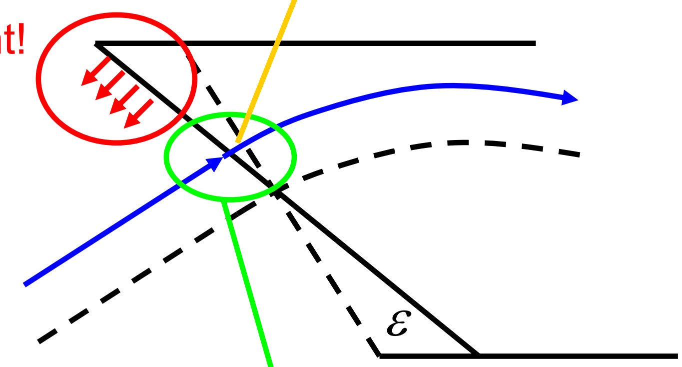
$$\underline{M}_x^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ -K\,s & 1 \end{pmatrix}$$

$$\underline{M}_y^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ k\,s & 1 \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} 0 \\ \kappa\,s \end{pmatrix}$$

Edge focusing

Top view :



Horizontal focusing with $\Delta x' = -x\dfrac{\tan(\varepsilon)}{\rho}$

$$B_x = \partial_y B_s\big|_{y=0}\, y\tan(\varepsilon) = \partial_s B_y\big|_{y=0}\, y\tan(\varepsilon)$$

$$y'' = \tfrac{q}{p}\,\partial_s B_y\big|_{y=0}\, y\tan(\varepsilon)$$

$$\Delta y' = \int y''\,ds = \tfrac{q}{p} B_y\, y\tan(\varepsilon) = y\frac{\tan(\varepsilon)}{\rho}$$

Quadrupole effect with

$$kl = \frac{\tan(\varepsilon)}{\rho}$$

$$\vec{z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\tan(\varepsilon)}{\rho} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\tan(\varepsilon)}{\rho} & 1 \end{pmatrix} \vec{z}_0$$