Dynamical Systems

$\vec{z}(s) = \vec{M}(s; s_0, \vec{z}_0)$ dynamical variable $\vec{z}$

Flow, transport map $\vec{M}$

By referring to a reference trajectory, transport maps in accelerators become origin preserving: $\vec{M}(s; s_0, \vec{0}) = \vec{0}$

Flows build a group under concatenation:

$$\vec{M}(s; s_1, \cdot) \circ \vec{M}(s_1; s_0, \vec{z}_0) = \vec{M}(s; s_1, \vec{M}(s_1; s_0, \vec{z}_0)) = \vec{M}(s; s_0, \vec{z}_0)$$

1) Identity element: $\vec{M}(s; s_0, \vec{z}) = \vec{z}$

2) Inverse element of $\vec{M}(s; s_0, \vec{z}) = \vec{M}^{-1}(s; s_0, \vec{z})$ is $\vec{M}(s_0; s, \vec{z})$

In physics, the flow is often given as a solution of a first order ODE $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z}, s)$

(Note that an nth order ODE can be rewritten as an n-dimensional first order ODE.)

Uniqueness

Note that not all ODEs $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z}, s)$

have a unique solution $\vec{z}(s)$

through a given point $\vec{z}(0) = \vec{z}_0$

Picard-Lindeloef:

A unique solution through $(\vec{z}_0, s_0)$ exists for $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z}, s)$ if $\vec{f}(\vec{z}, s)$ is Lipschitz continuous and bounded,

i.e. it is continuous, bounded, and there is a number $\boldsymbol{N}$ such that

$$|\vec{f}(\vec{z}_1, s) - \vec{f}(\vec{z}_2, s)| < N |\vec{z}_1 - \vec{z}_2|$$

Example: $H = \frac{1}{2}p^2 + V(q)\,, \quad V(q) = -8\sqrt{|q|}^3 \quad \Rightarrow \quad \dot{q} = p\,, \quad \dot{p} = 12\sqrt{|q|}$

There are two solutions through the point (q,p,t)=(0,0,0)

1. $(q(t), p(t)) = (0,0)$
2. $(q(t), p(t)) = (t^4, 4t^3) \quad \Rightarrow \quad (\dot{q}, \dot{p}) = (4t^3, 12t^2)$

In our following treatments we do require uniqueness of solutions.

Linear Systems

Linear ODEs in N dimesions $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z},s)$ have $\vec{f}(\lambda\vec{z},s) = \lambda\vec{f}(\vec{z},s)$

$$\frac{d}{ds}\vec{z} = \underline{L}(s)\vec{z}$$

There are N linearly independent solutions. For example $\vec{z}_n(s)$ going through $(\vec{z}_0, s_0)$
With $z_{0i} = 0$ for $i \neq n$ and $z_{0n} = 1$

$$\vec{z}_0 = (0,\ldots,1,\ldots,0)^T \quad \Rightarrow \quad \vec{z}_n(s)$$

One speaks of N fundamental solutions.

Superposition for linear ODEs:
If z1 is a solution and z2 is a solution, then
any linear combination Az1 +Bz2 is also a solution

$$\frac{d}{ds}\vec{z}_1 = \underline{L}(s)\vec{z}_1 \quad \& \quad \frac{d}{ds}\vec{z}_2 = \underline{L}(s)\vec{z}_2 \quad \Rightarrow \quad \frac{d}{ds}(A\vec{z}_1 + B\vec{z}_2) = \underline{L}(s)(A\vec{z}_1 + B\vec{z}_2)$$

Therefore any solution through $(\vec{z}_0, s_0)$ can be written as $\vec{z}(s) = \sum_{n=1}^{N} \vec{z}_n(s) z_{0n}$

$$\vec{z}(s) = \vec{M}(s; s_0, \vec{z}_0) = \underline{M}(s, s_0)\vec{z}_0 \qquad \underline{M}(s, s_0) = (\vec{z}_1(s), \ldots, \vec{z}_N(s))$$

Nonlinear Systems

Noninear ODEs in N dimesions $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z}, s)$

Have no fundamental solutions. Each solution has to be determined separately for each initial condition.

Examples: Plasma, Galaxies

$$H(\ldots, \vec{r}_j, \ldots, \ldots, \vec{p}_j, \ldots) = \sum_j \frac{\vec{p}_j^{\,2}}{2m_j} + \sum_{k \neq j} \frac{q_j q_k}{|\vec{r}_j - \vec{r}_k|}$$

Finding a general solution, flow, or transport map can be very hard.
This has not even been possible for the 3 body problem.

$$\vec{z}(s) = \vec{M}(s; s_0, \vec{z}_0)$$

Weakly Nonlinear Systems

Weakly nonlinear ODEs $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z}, s)$

Have a right hand side that can be

approximated well by a truncated Taylor expansion

$$\vec{f}(\vec{z}, s) \approx \underline{L}(s)\vec{z} + \sum_{j,k} \vec{f}_{jk} z_j z_k + \sum_{j,k,l} \vec{f}_{jkl} z_j z_k z_l + \ldots + \sum_{\vec{k},\text{order}\,O} \vec{f}_{\vec{k}} \vec{z}^{\vec{k}} + \ldots$$

$$\vec{z}^{\vec{k}} = \prod_{n=1}^{N} z_n^{k_n}, \quad \sum_{\vec{k},\text{order}\,O} \ldots = \sum_{n=1}^{N} \sum_{k_n} \ldots \quad \text{with} \quad \sum_{n=1}^{N} k_n = O$$

By solving the Taylor expanded ODE one tries to find a Taylor expansion of the transport map: $\vec{M}(s; s_0, \vec{z}_0) \approx \underline{M}(s, s_0)\vec{z}_0 + \ldots + \sum_{\vec{k},\text{order}\,O} \vec{M}_{\vec{k}} \vec{z}_0^{\vec{k}} + \ldots$

Note:

While this approach is usually chosen, it is not certain that a transport map of the Taylor expanded ODE is a Taylor expansion of the transport map of the original ODE. One therefore often speaks of “formally” finding the Taylor expansion of the transport map.

Aberrations and Sensitivities

$$\vec{M}(s;s_0,\vec{z}_0) \approx \underline{M}(s,s_0)\vec{z}_0 + \ldots + \sum_{\vec{k},\text{order } O} \vec{M}_{\vec{k}}\vec{z}_0^{\vec{k}} + \ldots$$

The Taylor coefficients are called aberrations of order O and are denoted by

$$(z_i, z_1^{k_1} \ldots z_6^{k_6}) \equiv M_{\vec{k},i}\ , \quad \text{order} \quad O = \sum_{n=1}^{6} k_n$$

Parameter dependences lead to sensitivities:

$$\tfrac{d}{ds}\vec{z} = \vec{f}(\vec{z},s,\varepsilon) \quad \Rightarrow \quad \vec{z}(s) = \vec{M}(s,\varepsilon;s_0,\vec{z}_0)$$

$$\vec{M}(s,\varepsilon;s_0,\vec{z}_0) \approx \underline{M}(s,s_0)\vec{z}_0 + \underline{M}^1(s,s_0)\vec{z}_0\varepsilon + \ldots + \sum_{\vec{k},n,\text{order } O} \vec{M}_{\vec{k}}^{`n}\vec{z}_0^{\vec{k}}\varepsilon^n$$

$$(z_i, z_1^{k_1} \ldots z_6^{k_6}\varepsilon^n) \equiv M_{\vec{k},i}^{n}\ , \quad \text{order} \quad O = n + \sum_{j=1}^{6} k_j$$

How can all these Taylor coefficients be computed?

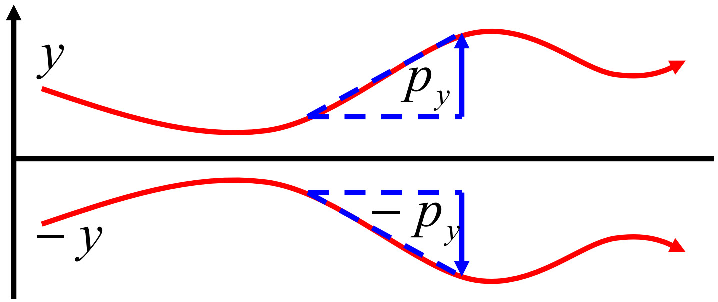
Horizontal Midplane Symmetry

This is the most important symmetry in nearly all accelerators.

$$\vec{r}^{\oplus} = (x, -y, z)$$

$$\vec{p}^{\oplus} = (p_x, -p_y, p_z)$$

$$\tfrac{d}{dt}\vec{p} = \vec{F}(\vec{r}, \vec{p}) \quad \Rightarrow \quad \tfrac{d}{dt}\vec{p}^{\oplus} = \vec{F}(\vec{r}^{\oplus}, \vec{p}^{\oplus})$$



$$\vec{z} \;= (x, a, y, b, \tau, \delta) \qquad \vec{z}(s) \;= \vec{M}(s, \vec{z}_0)$$

$$\vec{z}^{\oplus} = (x, a, -y, -b, \tau, \delta) \qquad \vec{z}^{\oplus}(s) = \vec{M}(s, \vec{z}_0^{\oplus})$$

$$M_i(s, \vec{z}_0^{\oplus}) = \;\; M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{1,2,5,6\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = -M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{3,4\}$$

$$(x, x_0^{k_1} \ldots \delta_0^{k_6}) = 0 \quad \text{for} \quad k_3 + k_4 \text{ is odd}$$ similarly for a, τ and δ

$$(y, x_0^{k_1} \ldots \delta_0^{k_6}) = 0 \quad \text{for} \quad k_3 + k_4 \text{ is even}$$ similarly for b

Double Midplane Symmetry

In addition to midplane symmetry, some elements are symmetric around the vertical plane, e.g. quadrupoles, glass lenses

$$\vec{z} = (x, a, y, b, \tau, \delta) \qquad \vec{z}\,(s) \; = \vec{M}(s, \vec{z}_0)$$

$$\vec{z}^{\oplus} = (x, a, -y, -b, \tau, \delta) \qquad \vec{z}^{\oplus}(s) = \vec{M}(s, \vec{z}_0^{\oplus})$$

$$\vec{z}^{\otimes} = (-x, -a, y, b, \tau, \delta) \qquad \vec{z}^{\otimes}(s) = \vec{M}(s, \vec{z}_0^{\otimes})$$

$$M_i(s, \vec{z}_0^{\oplus}) = \;\; M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{1,2,5,6\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = -M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{3,4\}$$

$$M_i(s, \vec{z}_0^{\otimes}) = -M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{1,2\}$$

$$M_i(s, \vec{z}_0^{\otimes}) = \;\; M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{3,4,5,6\}$$

$$(x, x_0^{k_1} \ldots \delta_0^{k_6}) = 0 \quad \text{for} \quad k_1 + k_2 \text{ is even or } k_3 + k_4 \text{ is odd}$$ similarly for a, τ and δ

$$(y, x_0^{k_1} \ldots \delta_0^{k_6}) = 0 \quad \text{for} \quad k_1 + k_2 \text{ is odd or } k_3 + k_4 \text{ is even}$$ similarly for b

Rotational Symmetry

Some optical elements are completely rotationally symmetric in the x-y plane, e.g. solenoid magnets, many glass lenses

$$w = x + iy\ , \quad \alpha = a + ib$$

$$\vec{z} = (w, \overline{w}, \alpha, \overline{\alpha}, \tau, \delta)$$

$$\vec{z}^{\oplus} = (e^{i\varphi} w, e^{-i\varphi} \overline{w}, e^{i\varphi} \alpha, e^{-i\varphi} \overline{\alpha}, \tau, \delta)$$

$$\vec{z}(s) = \vec{M}(s, \vec{z}_0)$$

$$\vec{z}^{\oplus}(s) = \vec{M}(s, \vec{z}_0^{\oplus})$$

$$M_i(s, \vec{z}_0^{\oplus}) = e^{i\varphi} M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{1,3\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = e^{-i\varphi} M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{2,4\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{5,6\}$$

$(w, w^{k_1} \dots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq 1$ similarly for α

$(\overline{w}, w^{k_1} \dots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq -1$ similarly for α*

$(\tau, w^{k_1} \dots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq 0$ similarly for δ

$(w, |w_0|^2 w_0), (\overline{\alpha}, |w_0|^2 \overline{w}_0), (\alpha, w_0^2 \overline{\alpha}_0), (\tau, |w_0|^2)$ can all be non-zero

C_n Symmetry

Some optical elements have C_n symmetric in the x-y plane,
e.g. C_2 for quadrupole, C_3 for sextupoles, etc.

$$w = x + iy\,, \quad \alpha = a + ib$$

$$\vec{z} = (w, \overline{w}, \alpha, \overline{\alpha}, \tau, \delta)$$

$$\vec{z}^{\oplus} = (e^{i\frac{2\pi}{n}} w, e^{-i\frac{2\pi}{n}} \overline{w}, e^{i\frac{2\pi}{n}} \alpha, e^{-i\frac{2\pi}{n}} \overline{\alpha}, \tau, \delta)$$

$$\vec{z}(s) = \vec{M}(s, \vec{z}_0)$$

$$\vec{z}^{\oplus}(s) = \vec{M}(s, \vec{z}_0^{\oplus})$$

$$M_i(s, \vec{z}_0^{\oplus}) = e^{i\frac{2\pi}{n}} M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{1,3\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = e^{-i\frac{2\pi}{n}} M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{2,4\}$$

$$M_i(s, \vec{z}_0^{\oplus}) = M_i(s, \vec{z}_0) \quad \text{for} \quad i \in \{5,6\}$$

$$(w, w^{k_1} \ldots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq jn + 1$$ similarly for α

$$(\overline{w}, w^{k_1} \ldots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq jn - 1$$ similarly for α*

$$(\tau, w^{k_1} \ldots \delta^{k_6}) = 0 \quad \text{for} \quad k_1 - k_2 + k_3 - k_4 \neq jn$$ similarly for δ

$$(w, \overline{w}_0), (\overline{\alpha}, |w_0|^2 w_0), (\alpha, \overline{w}_0^2 \alpha_0), (\tau, |w_0|^2)$$ can all be non-zero for C_2

Symplecticity

$$[\vec{\partial}\vec{M}^T]^T \underline{J}[\vec{\partial}\vec{M}^T] = \underline{J}$$

Symplecticity leads to the requirement that sums over certain products of aberrations must be either 0 or 1.

Separation into linear an nonlinear part of the map: $\vec{M}(\vec{z}) = \underline{M}_1(\vec{z} + \vec{N}(\vec{z}))$

$$\underline{M}(\vec{z}) = [\vec{\partial}\vec{M}^T]^T = [(\underline{1} + \vec{\partial}\vec{N}^T)\underline{M}_1^T]^T = \underline{M}_1(\underline{1} + \underline{N}(\vec{z}))$$

$$(1+\underline{N})^T \underline{M}_1^T \underline{J}\underline{M}_1(1+\underline{N}) = \underline{J} \quad \Rightarrow \quad \underline{M}_1^T \underline{J}\underline{M}_1 = \underline{J}\,, \quad \underline{N}^T \underline{J} + \underline{J}\underline{N} = -\underline{N}^T \underline{J}\underline{N}$$

For the leading order n-1 (the first order that appears in N): $\underline{N}^T \underline{J} + \underline{J}\underline{N} = 0 + O^n$

N is a Hamiltonian matrix up to order n and can thus be written up to order n as: $\vec{N}(\vec{z}) = \underline{J}\vec{\partial} f(\vec{z}) + O^{n+1}$

$$\begin{aligned} w(s_f) &= (x,x_0)\partial_a f + (x,y_0)\partial_b f + i[(y,x_0)\partial_a f + (y,y_0)\partial_b f] \\ &= (w,x_0)\partial_a f + (w,y_0)\partial_b f \\ &= \tfrac{1}{2}[(w,w_0)+(w,\overline{w}_0)][\partial_\alpha f + \partial_{\overline{\alpha}} f] - \tfrac{1}{2}[(w,w_0)-(w,\overline{w}_0)][\partial_\alpha f - \partial_{\overline{\alpha}} f] \\ &= (w,w_0)\partial_{\overline{\alpha}} f + (w,\overline{w}_0)\partial_\alpha f = (w,w_0)\partial_{\overline{\alpha}} f \end{aligned}$$

Special Aberrations

Dispersion (for δ as parameter of 4-dimensional motion) $\vec{z} = \underline{M}(s)\vec{z}_0 + \vec{D}(s)\delta$

Chromatic aberrations $(x,\ldots\delta^n)\,,\quad n \neq 0$

Geometric aberrations $(x, x^{k_1} a^{k_2} y^{k_3} b^{k_4} \ldots)\,,\quad \sum_{i=1}^{4} k_i \neq 0$

Purely Geometric aberrations $(x,\ldots\delta^n)\,,\quad n = 0$

Opening aberrations $(x, x^{k_1} \ldots y^{k_2} \ldots)\,,\quad k_1 + k_2 = 0$

Field aberrations $(x, x^{k_1} \ldots y^{k_2} \ldots)\,,\quad k_1 + k_2 \neq 0$

Spherical imaging systems: $(w,\alpha) = 0$

Spherical aberration for rotational symmetry $(w, \alpha |\alpha|^2)$

Coma line $(w, w|\alpha|^2)$

Coma circle $(w, \overline{w}\alpha^2)$

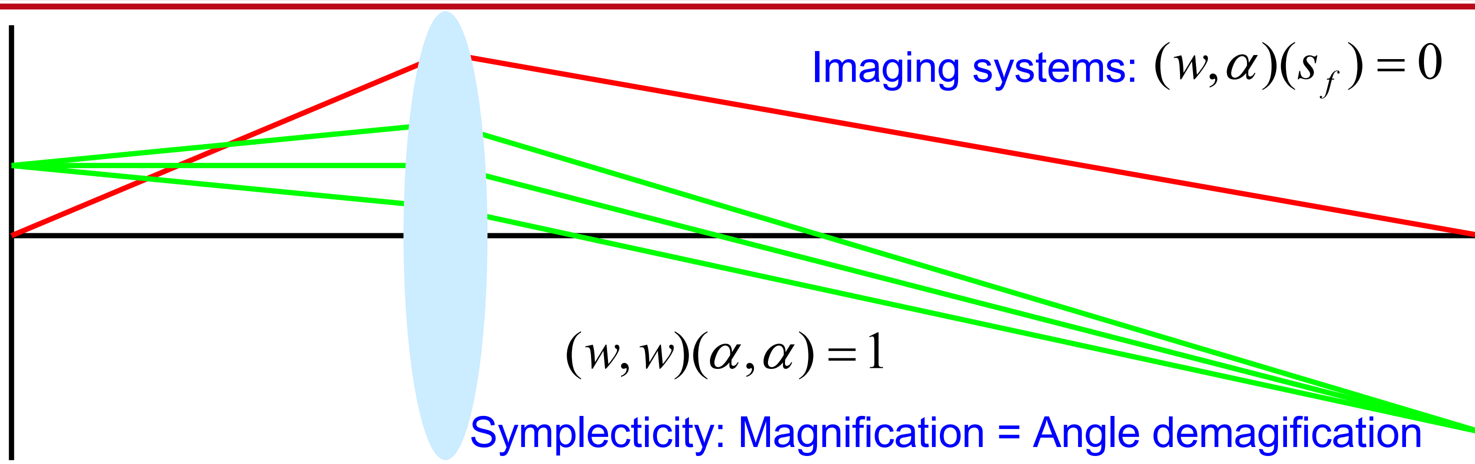
Astigmatism $(w, w^2\overline{\alpha})$

Curvature of Image $(w, |w|^2 \alpha)$

Distortion $(w, w|w|^2)$

Aberrations for rotational symmetry

Imaging systems: $(w,\alpha)(s_f) = 0$

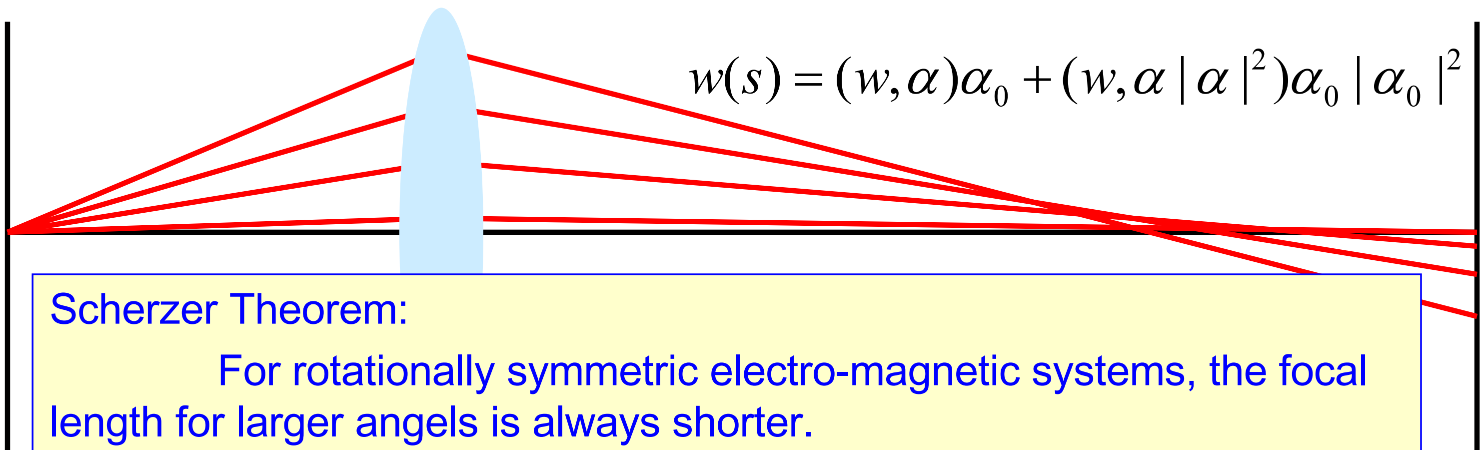


$$(w,w)(\alpha,\alpha) = 1$$

Symplecticity: Magnification = Angle demagification

Spherical aberration for rotational symmetry $(w,\alpha|\alpha|^2)$

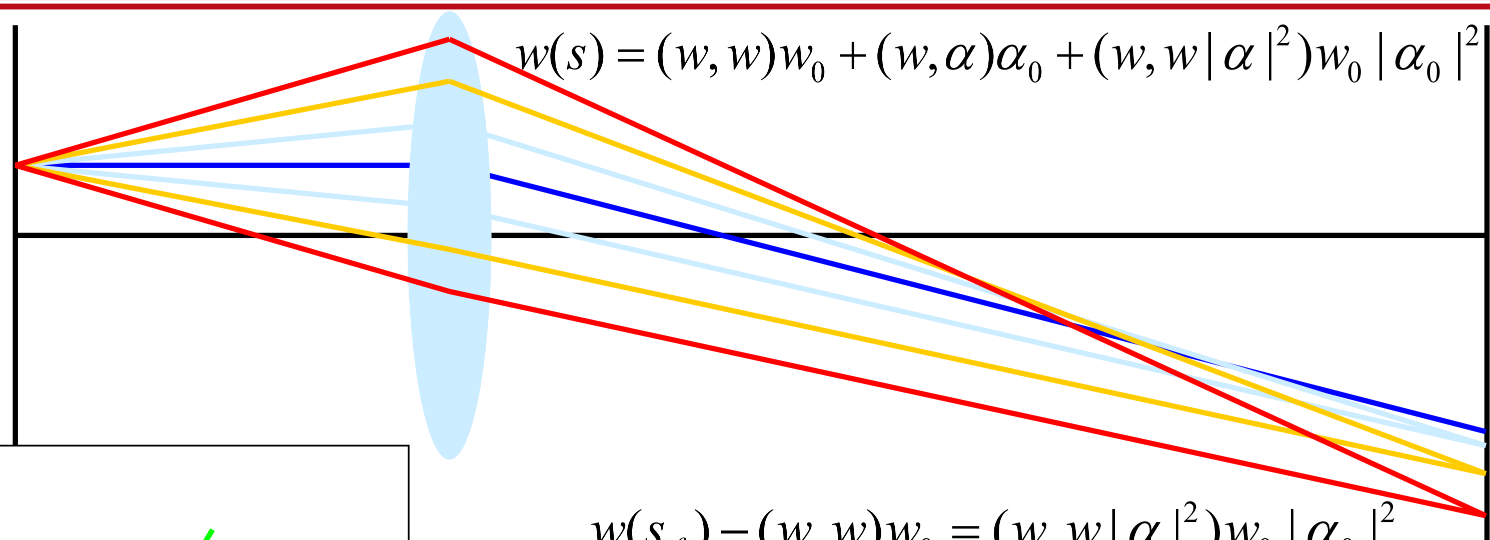
$$w(s) = (w,\alpha)\alpha_0 + (w,\alpha|\alpha|^2)\alpha_0|\alpha_0|^2$$



Scherzer Theorem:

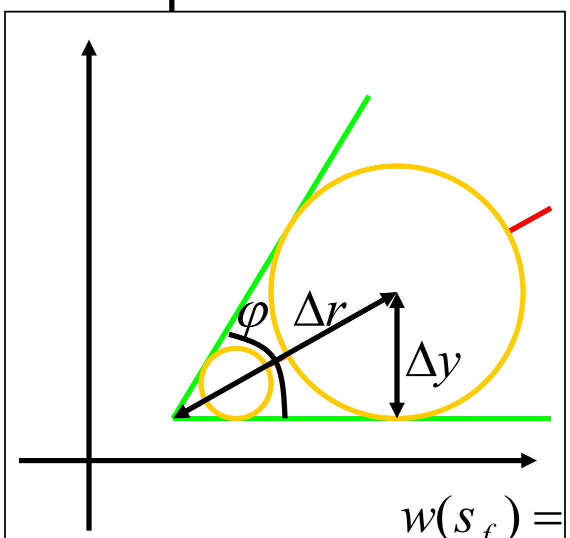
For rotationally symmetric electro-magnetic systems, the focal length for larger angels is always shorter.

Koma Line and Koma Circle



$$w(s) = (w,w)w_0 + (w,\alpha)\alpha_0 + (w,w|\alpha|^2)w_0|\alpha_0|^2$$

$$w(s_f) - (w,w)w_0 = (w,w|\alpha|^2)w_0|\alpha_0|^2 + (w,\overline{w}\alpha^2)\overline{w}_0\alpha_0^2$$



$$\varphi = 2\arcsin(\tfrac{\Delta y}{\Delta r}) = 2\arcsin(\tfrac{(w,\overline{w}\alpha^2)}{(w,w|\alpha|^2)})$$

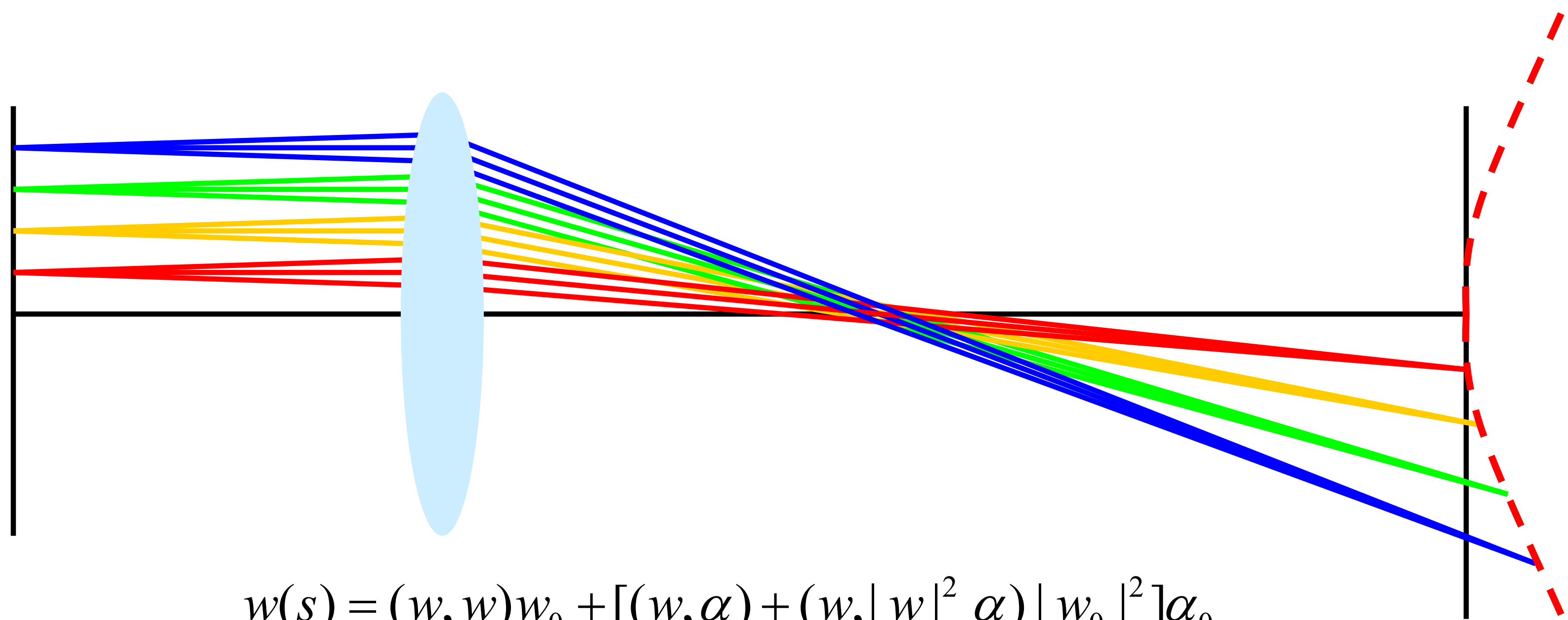
Symplecticity yields:

$$(w,w|\alpha|^2) = 2(w,\overline{w}\alpha^2) \quad \Rightarrow \quad \varphi = 60°$$

Since:

$$w(s_f) = (w,w_0)\partial_{\overline{\alpha}}[\ldots + \mathrm{Re}\{Kw\alpha\overline{\alpha}^2\}] = (w,w_0)[\ldots + Kw\alpha\overline{\alpha} + \tfrac{1}{2}\overline{K}\overline{w}\alpha^2]$$

Curvature of Image

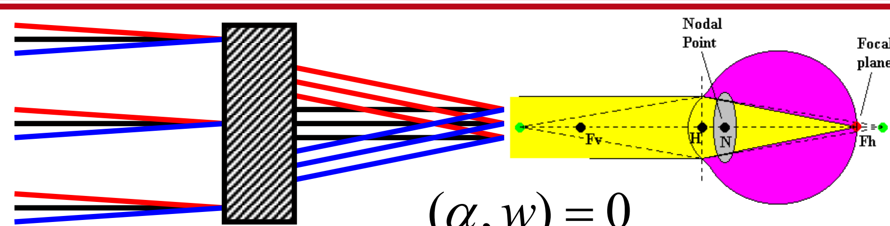


$$w(s) = (w,w)w_0 + [(w,\alpha) + (w,|w|^2\alpha)|w_0|^2]\alpha_0$$

The focus occurs at $(w,\alpha)(s_f) + (w,|w|^2\alpha)(s_f)|w_0|^2 = 0$

Other special systems

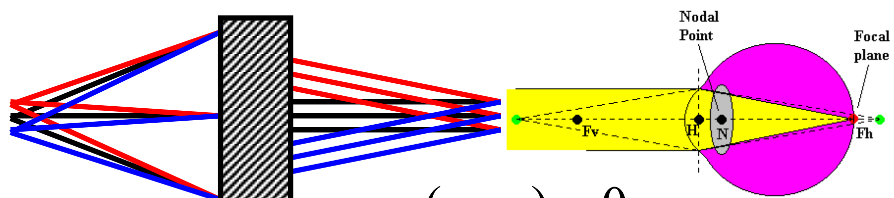
Telescope:
parallel to parallel system



$$(\alpha, w) = 0$$

Nonlinearly corrected telescope: $(\alpha, w^n) = 0$

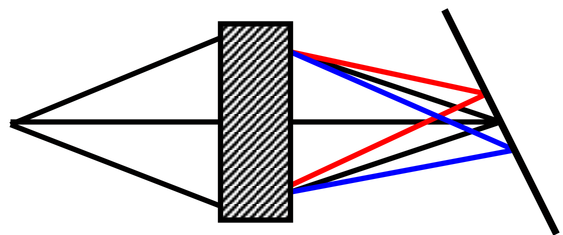
Microscope:
point to parallel system



$$(\alpha, \alpha) = 0$$

Nonlinearly corrected microscope: $(\alpha, \alpha^n) = 0$

Spectrograph:
point to parallel system



(x, δ) large
$(x, a) = 0$
(x, x) small

Nonlinearly corrected spectrograph: $(x, a^n b^m) = 0$

Tilt of focal plane: $(x, a\delta) \neq 0$ the focus is at $(x, a)(s_f) + (x, a\delta)(s_f)\delta = 0$

Variation of constants

$$\vec{z}\,' = \vec{f}(\vec{z},s)$$

$$\vec{z}\,' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z},s)$$

Field errors, nonlinear fields, etc can lead to $\Delta\vec{f}(\vec{z},s)$

$$\vec{z}_H\,' = \underline{L}(s)\vec{z}_H \quad \Rightarrow \quad \vec{z}_H(s) = \underline{M}(s)\vec{z}_{H0} \quad \text{with} \quad \underline{M}\,'(s)\vec{a} = \underline{L}(s)\underline{M}(s)\vec{a}$$

$$\vec{z}(s) = \underline{M}(s)\vec{a}(s) \quad \Rightarrow \quad \vec{z}\,'(s) = \underline{M}\,'(s)\vec{a} + \underline{M}(s)\vec{a}\,'(s) = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z},s)$$

$$\vec{a}(s) = \vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vec{z}(s) = \underline{M}(s)\left\{\vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}\right\}$$

$$= \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}$$

Perturbations are propagated from s to s’

Iteration of Aberrations

$$\vec{z}(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vec{z}_1(s) = \vec{z}_H(s)$$

$$\vec{z}_2(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_1(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vdots$$

$$\vec{z}_n(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_{n-1}(\hat{s}),\hat{s})\,d\hat{s}$$

Taylor expansions: $\Delta\vec{f}(\vec{z},s) = \Delta\vec{f}_2(\vec{z},s) + \Delta\vec{f}_3(\vec{z},s) + \dots$, $\Delta f_O = \sum_{\vec{k},\text{order } O} \vec{f}_{\vec{k}}\,\vec{z}^{\,\vec{k}}$

$$\vec{z}_1(s) = \underline{M}(s)\vec{z}_0$$

$$\vec{z}_2(s) = \underline{M}(s)\vec{z}_0 + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}_2(\vec{z}_1(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vec{z}_3(s) = \underline{M}(s)\vec{z}_0 + \int_0^s \underline{M}(s,\hat{s})\{[\Delta\vec{f}_2(\vec{z}_2(\hat{s}),\hat{s})]_3 + \Delta\vec{f}_3(\vec{z}_1(\hat{s}),\hat{s})\}\,d\hat{s}$$

$$\vdots$$

Poisson Bracket

The Poisson Bracket is defined as

$$[f(\vec{z}),g(\vec{z})]=\sum_i \partial_{q_j} f\,\partial_{p_j} g-\partial_{p_j} f\,\partial_{q_j} g=\vec{\partial}^T f\,\underline{J}\,\vec{\partial} g$$

The Poisson Bracket can be viewed as a product on the vector space of phase space functions. It is:

1) Linear: $[f,ag]=[af,g]=a[f,g]\,,\quad a\in IR$
2) Distributive: $[f,g+h]=[f,g]+[f,h]$

This turns the vector space into an algebra.

The multiplication is furthermore:

1) Anti-commutative: $[f,g]=-[g,f]$
2) Has a Jacobi-identity: $[f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0$

   as can be proven by the product rule: $[f,gh]=g[f,h]+[f,g]h$

This turns the algebra into a Lie algebra.

Example: $\vec{a}\times\vec{b}$ turns IR^3 into a Lie algebra.

Map computation by Lie Algebra

The Poisson-Bracket operator of f, $:g:$ is defined as $:g:h=[g,h]$

$$:H:g=[H,g]=-[g,H]=-\vec{\partial}^T g \underline{J}\,\vec{\partial}H=-\vec{\partial}^T g \tfrac{d}{ds}\vec{z}=-\tfrac{d}{ds}g(\vec{z})$$

$$\tfrac{d}{ds}\vec{z}=\vec{f}(\vec{z},s) \quad \Rightarrow \quad -:H:z_j=\tfrac{d}{ds}z_j=f_j(\vec{z},s)\,, \quad -:H:f_j=\tfrac{d}{ds}f_j-\tfrac{\partial}{\partial s}f_j$$

In the main field region where $\tfrac{d}{ds}\vec{z}=\vec{f}(\vec{z}) \quad \Rightarrow \quad -:H:f_j=\tfrac{d}{ds}f_j=\tfrac{d^2}{ds^2}z_j$

$$\text{If } g(\vec{z})=\tfrac{d^n}{ds^n}z_j \text{ then } -:H:g=\tfrac{d}{ds}g=\tfrac{d^{n+1}}{ds^{n+1}}z_j \quad \Rightarrow \quad (-:H:)^n z_j=\tfrac{d^n}{ds^n}z_j$$

Propagator: $e^{-\Delta s:H:}\vec{z}=\sum_{n=0}^{\infty}\frac{(-\Delta s:H:)^n}{n!}\vec{z}=\sum_{n=0}^{\infty}\frac{\Delta s^n}{n!}\frac{d^n}{ds^n}\vec{z}=\vec{M}(s+\Delta s;s,\vec{z})$

$$\vec{M}_2\circ\vec{M}_1(\vec{z}_0)=\vec{M}_2(\Delta s_2,\vec{z}(\Delta s_1))=\sum_{n=1}^{\infty}\frac{(-\Delta s_1:H_1:)^n}{n!}\vec{M}_2(\Delta s_2,\vec{z}_0)$$

$$=e^{-\Delta s_1:H_1(\vec{z}_0):}e^{-\Delta s_2:H_2(\vec{z}_0):}\vec{z}_0$$

$$\vec{M}(s;\vec{z}_0)=\vec{M}_n\circ\ldots\circ\vec{M}_2\circ\vec{M}_1(\vec{z}_0)=e^{-\Delta s_1:H_1:}\ldots e^{-\Delta s_n:H_n:}\vec{z}_0$$

Poisson Bracket Invariance

The Poisson Bracket is invariant under a symplectic transfer map

$$[f(\vec{M}(\vec{z})), g(\vec{M}(\vec{z}))] = \vec{\partial}^T f\Big|_{\vec{M}} \underline{M}\,\underline{J}\,\underline{M}^T \vec{\partial} g\Big|_{\vec{M}} = [f(\vec{z}), g(\vec{z})]\Big|_{\vec{M}(\vec{z})}$$

For nonlinear expansions, one writes the transport map as a linear matrix

and a nonlinear Lie exponent,

$$\vec{M}_1(\vec{z}) = \underline{M}_1 e^{:H_1(\vec{z}):^n}\vec{z} = \underline{M}_1 \sum_{n=0}^{\infty} \frac{:H_1^n:}{n!}\vec{z}$$

since a linear Lie exponent requires infinitely many terms in the power sum, but the nonlinear exponent terminates when a finite order expansion is sought.

$$(\underline{M}_2 e^{:H_2^n(\vec{z}):}\vec{z}) \circ (\underline{M}_1 e^{:H_1^n(\vec{z}):}\vec{z}) = \underline{M}_2 e^{:H_2^n(\underline{M}_1 e^{:H_1^n(\vec{z}):}\vec{z}):} \underline{M}_1 e^{:H_1^n(\vec{z}):}\vec{z}$$

$$= \underline{M}_2 \underline{M}_1 e^{:H_2^n(\underline{M}_1 e^{:H_1^n(\vec{z}):}\vec{z}):} e^{:H_1^n(\vec{z}):}\vec{z} = \underline{M}_2 \underline{M}_1 e^{:H_1^n(\vec{z}):} e^{:H_2^n(\underline{M}_1\vec{z}):}\vec{z}$$

When these equations are used to compute and manipulate transfer maps, one speaks of the Lie algebraic method.

Computing Taylor Expansions

$$f(x) \approx \sum_{n=1}^{\text{order } O} \tfrac{x^n}{n!} \partial^n f\Big|_0$$

But taking this approach for complicated functions would be very cumbersome:

1\.

$$f(x) = \frac{1}{1+\sin x} - 1\,, \quad f(0) = 0\,, \quad \partial f\Big|_0 = \frac{-\cos x}{(1+\sin x)^2}\Bigg|_0 = -1\,,$$

$$\partial^2 f(x) = \frac{\sin x(1+\sin x) + 2\cos^2 x}{(1+\sin x)^3}\Bigg|_0 = 2\,, \quad f(x) \approx -x + x^2 + O^3$$

2\.

$$f(x) = \frac{1}{1+\sin x} - 1\,,$$

$$f(x) \approx \frac{1}{1 + x - \frac{1}{6}x^3 + O^4} - 1$$

$$\approx -(x - \tfrac{1}{6}x^3) + (x - \tfrac{1}{6}x^3)^2 - (x - \tfrac{1}{6}x^3)^3 + O^4$$

$$\approx -x + x^2 - \tfrac{5}{6}x^3 + O^4$$

This approach is formalized in the field of automatic differentiation using a Differential Algebra.

Computations with TPSA(n)

Computation of a function in *IR* is done by a finite number of elementary operations (+,-,x) and elementary function evaluations (sin, cos, exp, 1/x, …).

$$f(x) = \frac{1}{1+\sin x} - 1$$

1. $x \in IR$
2. $\sin$
3. $1+$
4. $1/$
5. -1

If $g_n(x)$ is the truncated power series of order n of g(x) and $h_n(x)$ is that of h(x)

we can look for elementary operations ("+","-","x") so that

g_n"+"h_n is the TPS(n) of g+h

g_n"-"h_n is the TPS(n) of g-h

g_n"x"h_n is the TPS(n) of gxh

Similarly we can look for elementary functions ("sin","cos","exp","1/x",…) so that

"sin"(g_n) is the TPS(n) of sin(g), "exp"(g_n) is the TPS(n) of exp(g), etc.

Evaluating all elementary operations and elementary functions in f(x) in terms of "+","…" starting with the TPS(n) of x, leads to the TPS(n) of f(x).

Automatic Differentiation with TPSA(n)

Example: computing the TPS(3) of $f(x) = \frac{1}{1+\sin x} - 1$

1. TPS(3) of x is x
2. TPS(3) of "sin" x is $x - \frac{1}{6}x^3$
3. $1"+"x - \frac{1}{3}x^3 = 1 + x - \frac{1}{6}x^3$
4. $i(x) = \frac{1}{1+x}, "i"(x) = 1 - x + x^2 - x^3, "i"(x - \frac{1}{6}x^3) = 1 - x + x^2 - \frac{5}{6}x^3$
5. $1 - x + x^2 - \frac{5}{6}x^3 "-" 1 = -x + x^2 - \frac{5}{6}x^3$

This automatically (i.e. with a computer) leads to derivatives of f(x):

$$f(0) = 0,\ f'(0) = -1,\ f''(0) = 2,\ f'''(0) = -4$$

Truncated power series can be added "+" and multiplied "x" and there is a neutral element of multiplication (i.e.1). Therefore the vector space of TPS(n) forms an algebra. It is called the Truncated Power Series Algebra TPSA(n).

The Differential Algebra ${}_1D_1$

An addition and multiplication with a scalar leads to a vector space over IR^2:

$$\{a_0,a_1\},\ \{b_0,b_1\}\in IR^2,\ t\in IR$$

$$\{a_0,a_1\}+\{b_0,b_1\}=\{a_0+b_0,a_1+b_1\}$$

$$t\{a_0,a_1\}=\{ta_0,ta_1\}$$

The introduction of a multiplication $\{a_0,a_1\}\{b_0,b_1\}=\{a_0b_0,a_0b_1+a_1b_0\}$ leads to an algebra if it is:

1) Distribut. $\{a_0,a_1\}(\{b_0,b_1\}+\{c_0,c_1\})=\{a_0,a_1\}\{b_0,b_1\}+\{a_0,a_1\}\{c_0,c_1\}$

2) Has a neutral element: $\{a_0,a_1\}\{1,0\}=\{a_0,a_1\}$

and additionally to a ring if it is

3) Commutative: $\{a_0,a_1\}\{b_0,b_1\}=\{b_0,b_1\}\{a_0,a_1\}$

4) Associative: $\{a_0,a_1\}(\{b_0,b_1\}\{c_0,c_1\})=(\{a_0,a_1\}\{b_0,b_1\})\{c_0,c_1\}$

All these properties are clearly given, since first order power expansion have this multiplication: $(a_0+a_1x)(b_0+b_1x)=a_0b_0+(a_0b_1+a_1b_0)x+O^2$

The Differential Algebra ${}_1D_1$

By the introduced addition and multiplication we created an algebra, since the multiplication is commutative and associative we also created a ring, but not a field. Complex numbers are a field since there is a multiplicative inverse for all numbers except 0.

$$\{a_0,a_1\}\{b_0,b_1\}=\{a_0b_0,a_0b_1+a_1b_0\} \quad \Rightarrow \quad \{a_0,a_1\}\{\tfrac{1}{a_0},-\tfrac{a_1}{a_0^2}\}=\{1,0\}$$

We further introduce a differentiation: $\partial\{a_0,a_1\}=\{a_1,0\}$

It is a differentiation since it satisfies a product rule:

$$\partial(\{a_0,a_1\}\{b_0,b_1\})=\{a_0b_1+a_1b_0,0\}=(\partial\{a_0,a_1\})\{b_0,b_1\}+\{a_0,a_1\}(\partial\{b_0,b_1\})$$

By adding a differentiation we have created a Differential Algebra (DA).

Differentiation of Polynomials: $f(x)=2+x^2 \quad \Rightarrow \quad f'(x)=2x$

$$f(\{2,1\})=\{2,0\}+\{4,4\}=\{6,4\}=\{f(2),f'(2)\}$$

Since $\{f,f'\}+\{g,g'\}=\{(f+g),(f+g)'\}$, $\{f,f'\}\{g,g'\}=\{(fg),(fg)'\}$

Every polynomial: $P(\{f,f'\})=\{P(f),[P(f)]'\}$ and $P(\{x,1\})=\{P(x),P'(x)\}$

Elementary Functions in ${}_1D_1$

$$e(a_0 + a_1 x) = e(a_0) + e'(a_0) a_1 x + O^2$$

leads to

$$e(\{a_0, a_1\}) = \{e(a_0), a_1 e'(a_0)\}$$

$$\sin(\{a_0, a_1\}) = \{\sin a_0, a_1 \cos a_0\}$$

$$\cos(\{a_0, a_1\}) = \{\cos a_0, -a_1 \sin a_0\}$$

Since $\{f, f'\} + \{g, g'\} = \{(f+g), (f+g)'\}$, $\{f, f'\}\{g, g'\} = \{(fg), (fg)'\}$

and $e(\{f, f'\}) = \{e(f), [e(f)]'\}$

Therefore $F(\{f, f'\}) = \{F(f), [F(f)]'\}$ and $F(\{x, 1\}) = \{F(x), F'(x)\}$

So that automatic differentiation works not only for Polynomials but for any function that is constructed from a finite number of operations and elementary functions.

Computer programs that have differential algebra elements as data types can evaluate any function or algorithm in this data type and obtain derivatives of the function or derivatives of the algorithm.

The Differential Algebra ${}_nD_v$

The concept of ${}_1D_1$ can be extended to truncated power series of order n and to v variables. This leads to the differential algebra ${}_nD_v$. For each coefficient in the nth order expansion there is one dimension in the vectors of ${}_nD_v$.

Power expansions for v variables have extremely many expansion coefficients:

A polynomial of order n in v variables has $\dim({}_nD_v) = \frac{(n+v)!}{n!v!}$ coefficients

since

$$\underbrace{\dim({}_nD_v) - \dim({}_{n-1}D_v)}_{z_1^{k_1}\ldots z_v^{k_v},\sum_{j=1}^{v} k_j = n} = \underbrace{\dim({}_nD_{v-1})}_{z_1^{k_1}\ldots z_{v-1}^{k_{v-1}},\sum_{j=1}^{v} k_j \le n} \quad , \quad \frac{(n+v)!}{n!v!} - \frac{(n-1+v)!}{(n-1)!v!} = \frac{(n+v-1)!}{n!(v-1)!}$$

and iteration of ${}_nD_v$ starts with the correct conditions: $\dim({}_nD_1) = n+1 = \frac{(n+1)!}{n!}$

$$\dim({}_0D_v) = 1 = \frac{v!}{v!}$$

Example: $\dim({}_{10}D_6) = 8008$

Computer programs that have differential algebra elements as data types produce the nth order power expansion of v-dimensional functions or algorithms automatically.

Equivalence classes in ${}_nD_v$

A TPS(n) of a function f(x) defines the equivalence class of all functions that have the same TPS(n).

$$\text{Def}: \quad f =_n g \quad \text{if} \quad \vec{\partial}^{\vec{k}} f(0) = \vec{\partial}^{\vec{k}} g(0) \quad \forall\, \vec{k} \text{ with order} \le n$$

$=_n$ is an equivalence relation since it has

1) the identity property $f =_n f \quad \forall\, f$
2) the symmetry property $f =_n g \quad if \quad g =_n f$
3) the transitivity property $f =_n h \quad if \quad f =_n g \; and \; g =_n h$

Equivalence classes: $\text{Def}: \quad [f]_n = \{g \,|\, g =_n f\}$

Arithmetic of equivalence class:

Those operations generate a differential algebra.

$$[f]_n + [g]_n \equiv [f+g]_n$$
$$[f]_n[g]_n \equiv [f\,g]_n$$
$$t[f]_n \equiv [t\,f]_n$$
$$\partial_j[f]_n \equiv [\partial_j f]_{n-1}$$
$$e([f]_n) \equiv [e(f)]_n$$

Concatenation of maps

$$f(x),\, g(x) \quad \text{and} \quad [f]_n, [g_n] \in_n D_1$$

$$[f(g(x))]_0 = [f(g(0))]_0$$

$$[f(g(x))]_1 = [f(g(0)) + g'(0) f'(g(0)) x]_1$$

The composition of two TPS(n) can only be computed if the first one is origin preserving, then $[f]_n \circ [g]_n \equiv [f(g(x))]_n$

If two maps that are know to order n and the first one is origin preserving, then the composition of the maps is known to order n.

$$[\vec{M}_1]_n, [\vec{M}_2]_n \in_n D_v^v$$

$$[\vec{M}_2]_n \circ [\vec{M}_1]_n = [\vec{M}_2(\vec{M}_1(\vec{z}))]_n$$

Therefore the reference trajectory is always chosen as origin for the maps accelerator elements.

Inversion of Maps

The nth order inverse of an origin preserving function can be computed within the differential algebra (DA):

$$\vec{M}(\vec{z}) = \underline{M}_1\vec{z} + \vec{N}(\vec{z})$$

$$\vec{M} \circ \vec{M}^{-1}(\vec{z}) = \underline{M}_1\vec{M}^{-1} + \vec{N} \circ \vec{M}^{-1} = \vec{z}$$

$$\vec{M}^{-1} = \underline{M}_1^{-1}(\vec{z} - \vec{N} \circ \vec{M}^{-1})$$

$$[\vec{M}^{-1}]_n = \underline{M}_1^{-1}[\vec{z} - \vec{N} \circ \vec{M}^{-1}]_n = \underline{M}_1^{-1}(\vec{z} - [\vec{N}]_n \circ [\vec{M}^{-1}]_{n-1})$$

Iterative computation of the inverse:

$$[\vec{M}^{-1}]_1 = [\underline{M}_1^{-1}\vec{z}]_1$$

$$[\vec{M}^{-1}]_2 = \underline{M}_1^{-1}(\vec{z} - [\vec{N}]_2 \circ [\underline{M}_1^{-1}\vec{z}]_1)$$

$$[\vec{M}^{-1}]_3 = \underline{M}_1^{-1}(\vec{z} - [\vec{N}]_3 \circ (\underline{M}_1^{-1}(\vec{z} - [\vec{N}]_2 \circ [\underline{M}_1^{-1}\vec{z}]_1)))$$

Generating Functions

The motion of particles can be represented by Generating Functions

Each flow or transport map: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

With a Jacobi Matrix : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = \left(\vec{\partial}_0 \vec{M}^T\right)^T$

That is Symplectic: $\underline{M}\,\underline{J}\,\underline{M}^T = \underline{J}$

Can be represented by a Generating Function:

$$F_1(\vec{q}, \vec{q}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_1 \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_1$$

$$F_2(\vec{p}, \vec{q}_0, s) \quad \text{with} \quad \vec{q} = \vec{\partial}_p F_2 \quad , \quad \vec{p}_0 = \vec{\partial}_{q_0} F_2$$

$$F_3(\vec{q}, \vec{p}_0, s) \quad \text{with} \quad \vec{p} = -\vec{\partial}_q F_3 \quad , \quad \vec{q}_0 = -\vec{\partial}_{p_0} F_3$$

$$F_4(\vec{p}, \vec{p}_0, s) \quad \text{with} \quad \vec{q} = \vec{\partial}_q F_4 \quad , \quad \vec{q}_0 = -\vec{\partial}_{p_0} F_4$$

6-dimensional motion needs only one function ! But to obtain the transport map this has to be inverted.

Computation of Generating Functions

For any map for which the TPS(n) is know, a TPS(n+1) of a generating function that produces this map can be computed. For example, looking for

$$F_1(\vec q,\vec q_0,s) \quad \text{with} \quad \vec p = -\vec\partial_q F_1(\vec q,\vec q_0,s) \quad , \quad \vec p_0 = \vec\partial_{q_0} F_1(\vec q,\vec q_0,s)$$

$\vec z = \vec M(\vec z_0)$ is given as TPS(n)

$$\begin{pmatrix}\vec q\\ \vec q_0\end{pmatrix} = \begin{pmatrix}\vec M_q(\vec z_0)\\ \vec q_0\end{pmatrix} = \vec l(\vec z_0)\,, \quad \begin{pmatrix}\vec p_0\\ \vec p\end{pmatrix} = \begin{pmatrix}\vec p_0\\ \vec M_p(\vec z_0)\end{pmatrix} = \vec h(\vec z_0) = \underline{J}\left[\vec\partial F_1(\vec q,\vec q_0)\right]_{\vec l(\vec z_0)}$$

$$\vec\partial F_1 = -\underline{J}\vec h\circ\vec l^{\,-1} \quad \Rightarrow F_1 = -\underline{J}\int_0^{(\vec q,\vec q_0)} \vec h\circ\vec l^{\,-1}(\vec Q)\,d\vec Q$$

$$[\vec M]_n \quad \Rightarrow [\vec l\,]_n, [\vec h]_n \quad \Rightarrow [\vec l^{\,-1}]_n, [\vec h]_n\circ[\vec l^{\,-1}]_n$$

$$[F_1]_{n+1} = -\underline{J}\int_0^{(\vec q,\vec q_0)} [\vec h]_n\circ[\vec l^{\,-1}]_n\,d\vec Q$$

Particle coordinates (q0,p0) are propagated by such generating functions when zeros of the following equations are found numerically:

$$\vec p + \vec\partial_q F_1(\vec q,\vec q_0,s) = \vec 0 \quad \text{and} \quad \vec p_0 - \vec\partial_{q_0} F_1(\vec q,\vec q_0,s) = \vec 0$$