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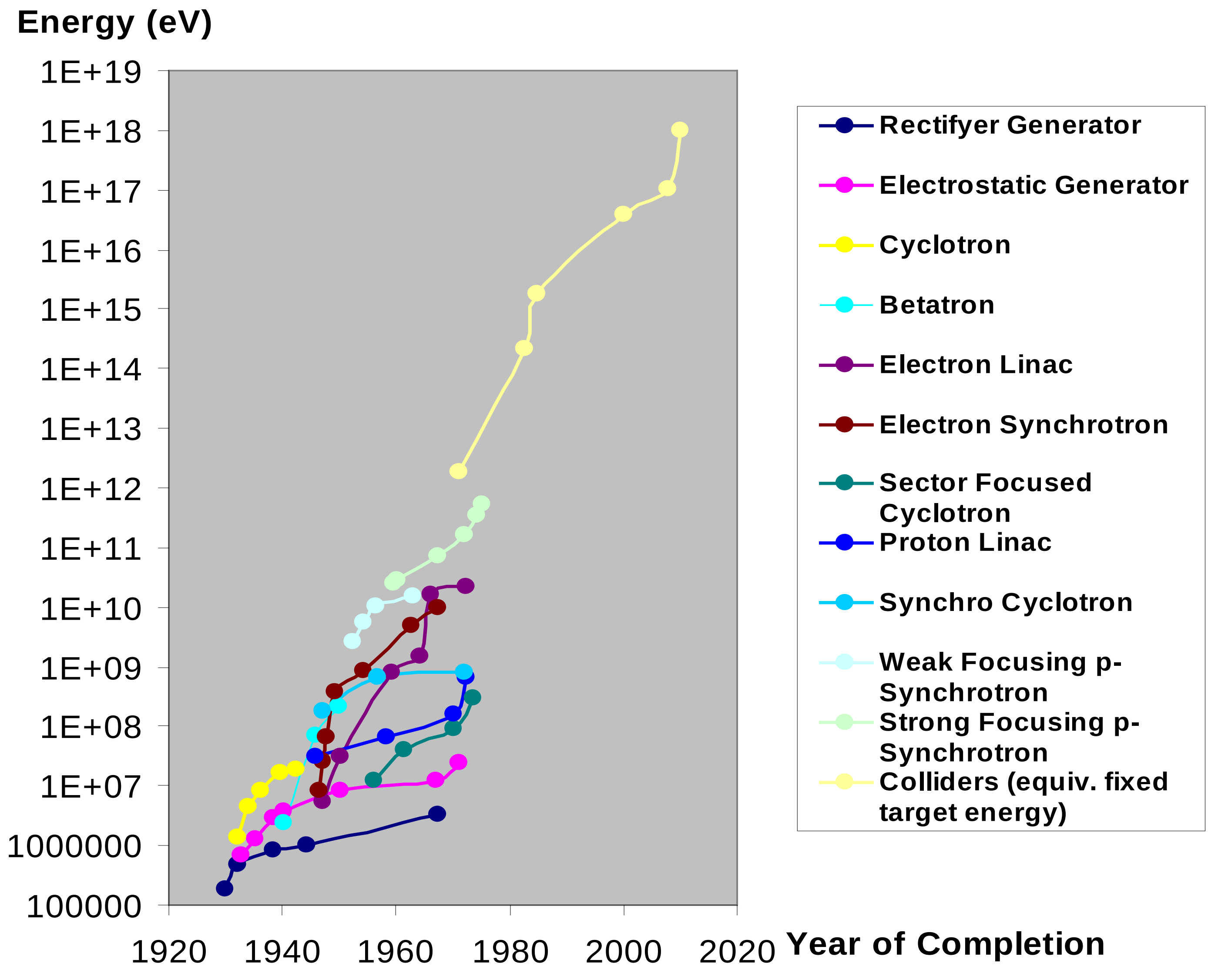
Physics of Accelerators

1) History: The Livingston chart
2) Circular accelerators at DESY
3) The macroscopic fields in accelerators
   - Solenoid magnets
   - Dipole magnets
   - Quadrupole magnets
   - Sextupole magnets
4) Particle motion in circular accelerators
   - Liouville's theorem
   - Parametrization of the one turn matrix
   - Invariants of motion
   - The emittance
   - The β-function
   - The tune
   - The closed orbit
   - The periodic dispersion
   - Chromaticity correction
   - Longitudinal dynamics
   - Nonlinear invariants of motion
5) Many particle effects
6) Literature

Livingston Chart



Energy in the center of momentum:

$$\frac{1}{c^2} E_{c.m.}^2 = (P_1^\mu + P_2^\mu)_{c.m.} (P_{1\mu} + P_{2\mu})_{c.m.} = (P_1^\mu + P_2^\mu)(P_{1\mu} + P_{2\mu})$$

$$= \frac{1}{c^2}(E_1 + E_2)^2 - (p_1 - p_2)^2 = 2\left(\frac{E_1 E_2}{c^2} + p_1 p_2\right) + m_1^2 + m_2^2$$

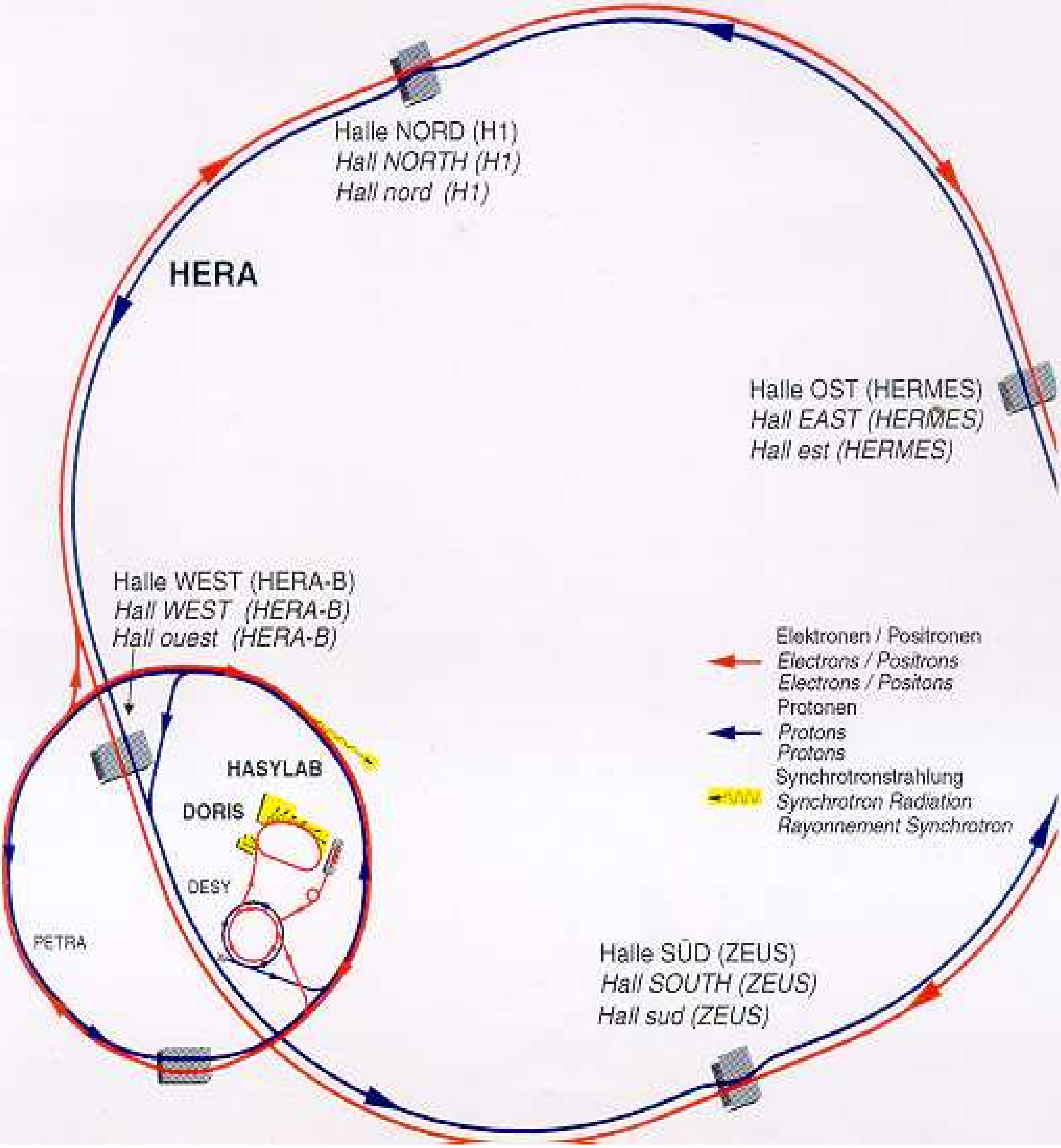
Collider experiment: $E_1 \gg m_1 \; , \; E_2 \gg m_2$

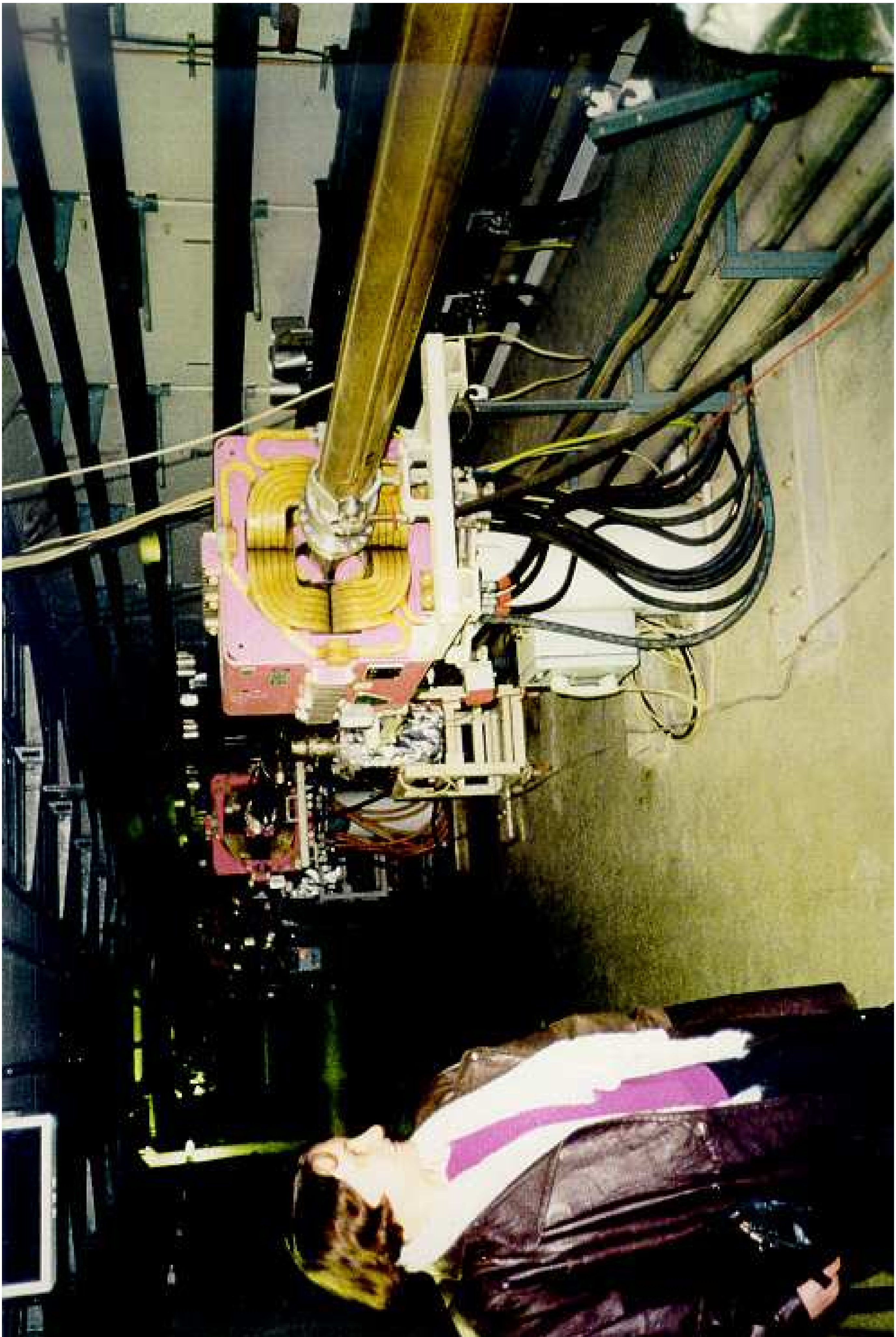
$$E_{c.m.} = 2\sqrt{E_1 E_2}$$

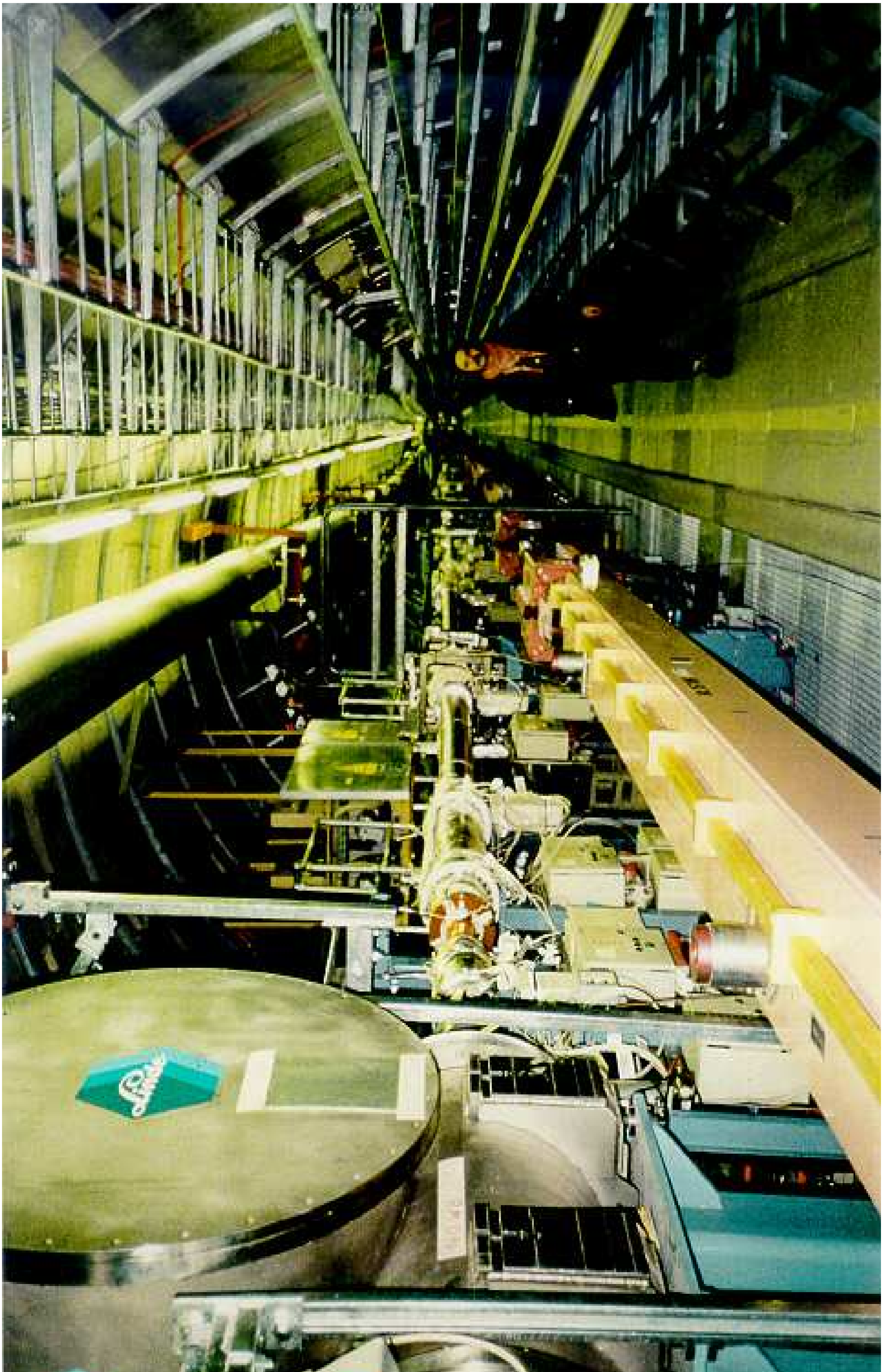
Target experiment: $E_1 \gg m_1, m_2 \; , \; p_2 = 0 \; , \; E_2 = m_2$

$$E_{c.m.} = \sqrt{2 E_1 m_2}$$

Circular accelerators at DESY





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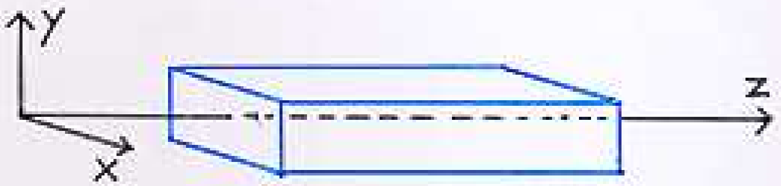
The macroscopic fields in accelerators

static magnetic fields: $\partial_t \vec{B} = 0$; $\vec{E} = 0$; charge free space : $\vec{j} = 0$

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{j} + \varepsilon_0 \partial_t \vec{E}) = 0 \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi(\vec{r})$$

$$\vec{\nabla} \vec{B} = 0 \quad \Rightarrow \quad \vec{\nabla}^2 \psi = 0$$



$(x=0, y=0)$ is the beams design curve

For finite fields on the design curve $\psi(\vec{r})$ can be power expanded in x and y

$$\psi = \sum_{n,m} b_{nm}(z)\, x^n y^m$$

Complex notation: $w = x + iy$, $\overline{w} = x - iy$

$$\partial_x = \partial_w + \partial_{\overline{w}} \quad , \quad \partial_y = i\partial_w - i\partial_{\overline{w}} = i(\partial_w - \partial_{\overline{w}})$$

$$\vec{\nabla}^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 = (\partial_w + \partial_{\overline{w}})^2 - (\partial_w - \partial_{\overline{w}})^2 + \partial_z^2 = 4\partial_w \partial_{\overline{w}} + \partial_z^2$$

$$\psi = \operatorname{Im}\{ \sum_{\nu\lambda=0} a_{\nu\lambda} (w\overline{w})^\lambda \overline{w}^\nu \}$$

$$\vec{\nabla}^2 \psi = \operatorname{Im}\{ \sum_{\nu\lambda=1} 4 a_{\nu\lambda} (\lambda+\nu)\lambda (w\overline{w})^{\lambda-1} \overline{w}^\nu + \sum_{\nu\lambda=0} a''_{\nu\lambda} (w\overline{w})^\lambda \overline{w}^\nu \}$$

$$= \operatorname{Im}\{ \sum_{\nu\lambda=0} [4(\lambda+1+\nu)(\lambda+1) a_{\nu\lambda+1} + a''_{\nu\lambda}] (w\overline{w})^\lambda \overline{w}^\nu \} = 0$$

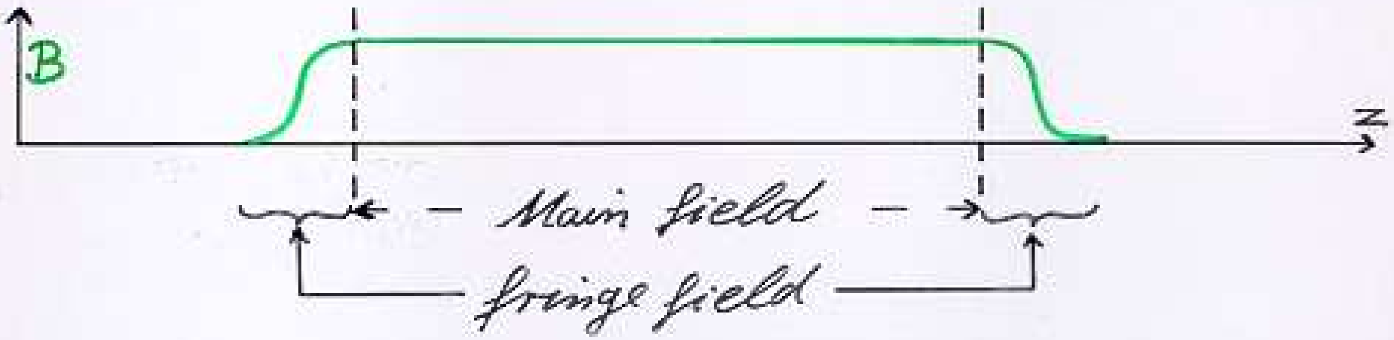
$\Rightarrow$ Iteration equation $\quad a_{\nu\lambda+1} = \dfrac{-1}{4(\lambda+1+\nu)(\lambda+1)} a''_{\nu\lambda}$, $a_{\nu 0} = \Psi_\nu(z)$

$\Psi_\nu(z)$ are called the z-dependent multipole coefficients

$$\psi = \mathrm{Im}\left\{ \sum_{\nu\lambda=0} \frac{(-1)^\lambda \nu!}{(\lambda+\nu)!\,\lambda!} \left(\frac{w\bar{w}}{4}\right)^\lambda \bar{w}^\nu \, \Psi_{\nu(z)}^{[2\lambda]} \right\}$$

$$= \sum_{\nu\lambda=0} \frac{(-1)^\lambda \nu!}{(\lambda+\nu)!\,\lambda!} \left(\frac{r}{2}\right)^{2\lambda} r^\nu \, \mathrm{Im}\left\{ \Psi_{\nu(z)}^{[2\lambda]} e^{-i\nu\varphi} \right\}$$

The index ν describes C_ν Symmetry around $\vec{e}_z$ due to a sign change after $\Delta\varphi = \frac{\pi}{\nu}$.



Only the fringe field region has terms with $\lambda \neq 0$, and $\partial_z^2 \psi \neq 0$.

Main fields in accelerator physics $(\partial_z^2 \psi = 0, \lambda = 0)$

$\Psi_0 = i|\Psi_0|,\ \Psi_\nu = |\Psi_\nu| e^{i\nu\vartheta_0}_{\{\nu\neq 0\}} \Rightarrow \psi = \sum_{\nu=1} r^\nu |\Psi_\nu| \,\mathrm{Im}\{e^{-i\nu(\varphi-\vartheta_0)}\} + |\Psi_0|$

Main field potential: $\Psi(\vec{r}) = \Psi_0(z) - \sum_{\nu=1} r^\nu \Psi_\nu \sin(\nu(\varphi - \vartheta_\nu))$

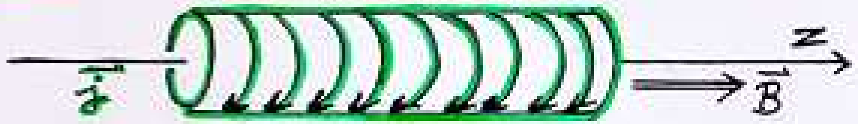
The isolated multipole $\boxed{\Psi(\vec{r}) = -r^\nu \Psi_\nu \sin(\nu\varphi)}$
with the rotation ϑ_ν of the coordinate system set to 0.

The potentials of different multipole components Ψ_ν have:

1. different rotation symmetry C_ν (periodic with $\Delta\varphi = \frac{2\pi}{\nu}$).
2. different radial dependence r^ν.

Macroscopic $\vec{B}$-Fields in Accelerators

$\nu = 0$ $\quad \Psi(\vec{r}) = \Psi_0(z) - \frac{w\bar{w}}{4}\Psi_0''(z) \pm \dots \quad , \quad \vec{B} = \begin{pmatrix} \frac{x}{2}\Psi_0'' \\ \frac{y}{2}\Psi_0'' \\ -\Psi_0' \end{pmatrix} \longrightarrow \vec{\nabla}\cdot\vec{B} = 0$



Solenoid magnets are used in detectors for particle identification

$m\gamma\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = q\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}\times\begin{pmatrix}-\frac{x}{2}B_z'\\ -\frac{y}{2}B_z'\\ B_z\end{pmatrix} \longrightarrow \begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \frac{qB_z}{m\gamma}\begin{pmatrix}\dot{y}\\ -\dot{x}\end{pmatrix} + \frac{qB_z'\dot{z}}{2m\gamma}\begin{pmatrix}y\\ -x\end{pmatrix} \longrightarrow \ddot{w} = -i\frac{qB_z}{m\gamma}\dot{w} - i\frac{q\dot{B}_z}{2m\gamma}w$

In a coordinate system which rotates with $\boxed{g = \frac{qB_z}{2m\gamma}}$: $\boxed{w_0 = w\, e^{i\int_0^t g\,dt}}$

$\ddot{w}_0 = (\ddot{w} + i2g\dot{w} + i\dot{g}w - g^2 w)e^{i\int_0^t g\,dt} = -g^2 w_0$

$\ddot{x}_0 = -g^2 x_0 \quad , \quad \ddot{y}_0 = -g^2 y_0$

Solenoids are used to focus low γ beams

The solenoids rotation $\dot{\varphi} = -\frac{qB_z}{2m\gamma}$ of the beam should be compensated by a reversed solenoid called compensator.

$$\ddot{\varrho} = -\left(\frac{qB_z}{2m\gamma}\right)^2 \varrho$$

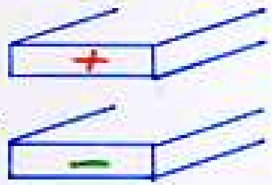
Weak focusing, compared to the deflections created by a transverse magnetic field.

Transverse fields : $\vec{B} = B_x \cdot \vec{e}_x + B_y \cdot \vec{e}_y$

$$m\gamma \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q \begin{pmatrix} \dot{x} \\ \dot{y} \\ V_z \end{pmatrix} \times \begin{pmatrix} B_x \\ B_y \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \frac{qV_z}{m\gamma} \begin{pmatrix} -B_y \\ B_x \end{pmatrix} \Rightarrow$$ **strong focusing**

$\nu = 1$ $\quad \Psi = \Psi_1 \, \mathrm{Im}\{x - iy\} = -\Psi_1 \cdot y \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\Psi = \Psi_1 \cdot \vec{e}_y$

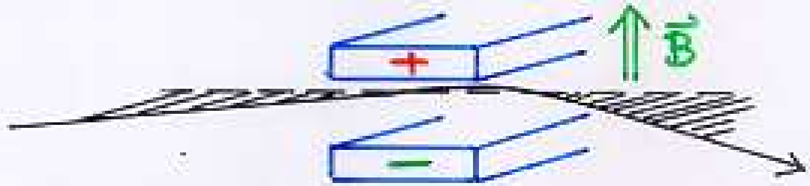
C_1 Symmetry



$(+,-)$ in Ψ

$\Downarrow$

(S,N) in $\vec{B}$



Dipole magnets are used for steering the beams direction.

$$\frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B}$$

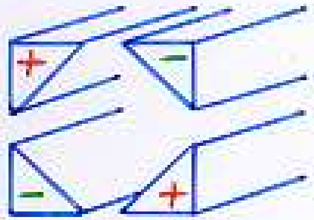
$$\frac{|d\vec{p}|}{|\vec{p}|} = d\varphi = \frac{qB_\perp}{p} dl \Rightarrow$$

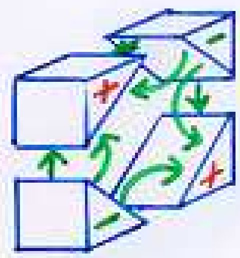
Curvature:

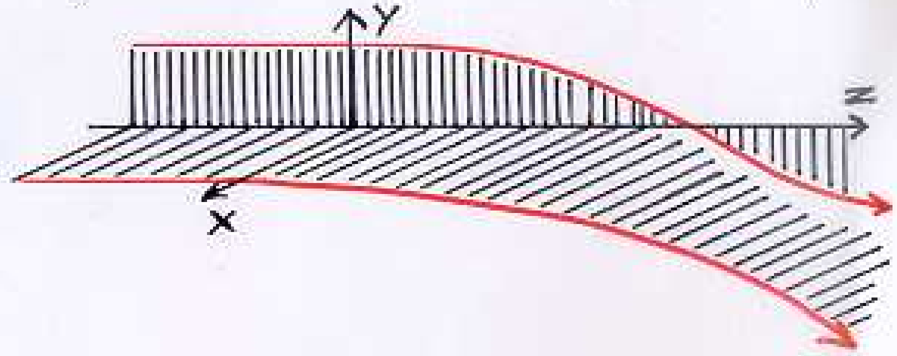
$$\frac{1}{\varrho} = \frac{d\varphi}{dl} = \frac{qB_\perp}{p}$$

$\underline{\nu = 2}$ $\quad \psi = \Psi_2 \, \mathrm{Im}\{(x - iy)^2\} = -\Psi_2 \cdot 2xy \quad \Rightarrow \quad \vec{B} = \Psi_2 \cdot 2 (y\vec{e}_x + x\vec{e}_y)$

C_2
Symmetry





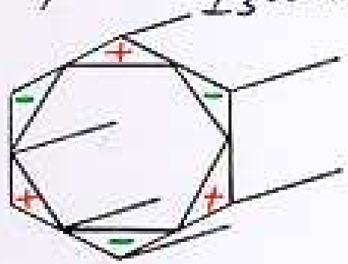


$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \frac{q v_z}{m\gamma} \begin{pmatrix} -B_y \\ B_x \end{pmatrix} = \frac{q v_z}{m\gamma} 2 \Psi_2 \begin{pmatrix} -x \\ y \end{pmatrix}$$

In a quadrupole particles are focused in one plane and defocused in the other plane.
Other modes of strong focusing are not possible.

$\underline{\nu = 3}$ $\quad \psi = \Psi_3 \, \mathrm{Im}\{(x - iy)^3\} = \Psi_3 (y^3 - 3x^2 y) \quad \Rightarrow \quad \vec{B} = 3\Psi_3 (2xy\vec{e}_x + (x^2 - y^2)\vec{e}_y)$

C_3
Symmetry



i) Sextupole fields hardly influence the particles close to the center, where one can linearize in x and y.

ii) In linear approximation a by Δx shifted sextupole has an additional quadrupole field.

$$\vec{B} = 3\Psi_3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

$$x \to \Delta x + x$$

$$\vec{B} = 6 \Psi_3 \Delta x \begin{pmatrix} y \\ x \end{pmatrix} + \mathcal{O}(x^2, y^2, \Delta x^2)$$

Particle motion in circular accelerators

Linearization in x and y

Quadrupoles focus the beam by strong alternating gradient focusing

$x_2 = x_1 - \frac{x_1}{f} d$; $x'_{end} = -\frac{x_1 - x_2}{f} = -\frac{x_1}{f^2/d}$

- Total focal length: $\frac{f^2}{d} > 0 \Rightarrow$ focusing $\forall f \in \mathbb{R}$ in x and y
- Horizontal (x) and vertical (y) motion are independent

The equation of motion

$$\frac{d}{dt}(m\gamma\dot{\vec{r}}) = q(\dot{\vec{r}} \times \vec{B} + \vec{E}) \xrightarrow{\frac{d}{dt} = (\dot{z})\frac{d}{dz}} \frac{d^2}{dz^2}\vec{r} = \vec{f}(\vec{r}, \vec{r}\,', z)$$

Since x and y are independent : $x'' = f_x(x, x', z)$; $y'' = f_y(y, y', z)$

After linearization : $x'' = a(z) \cdot x + b(z) \cdot x'$

1) Let $C(z)$ be the solution for initial conditions $(x(0) = 1, x'(0) = 0)$
2) Due to linearity $C(z) \cdot x_i$ is the solution for $(x(0) = x_i, x'(0) = 0)$
3) Let $S(z) \cdot x'_i$ be the solution for " $(x(0) = 0, x'(0) = x'_i)$
4) Due to linearity $x(z) = C(z) \cdot x_i + S(z) x'_i$ for $(x(0) = x_i, x'(0) = x'_i)$

The linearized equation of motion

$$\left.\begin{aligned} x(z) &= C(z)\cdot x(0) + S(z)\cdot x'(0) \\ x'(z) &= C'(z)\cdot x(0) + S'(z)\cdot x'(0) \end{aligned}\right\} \begin{pmatrix} x \\ x' \end{pmatrix}_{(z)} = \underline{M}(z\leftarrow 0)\begin{pmatrix} x \\ x' \end{pmatrix}_{(0)}$$

$\underline{M}(z\leftarrow 0)$ is the transport matrix from 0 to z.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{(L)} = \underline{M}(L\leftarrow z)\begin{pmatrix} x \\ x' \end{pmatrix}_{(z)} = \underbrace{\underline{M}(L\leftarrow z)\,\underline{M}(z\leftarrow 0)}\begin{pmatrix} x \\ x' \end{pmatrix}_{(0)} = \underline{M}(L\leftarrow 0)\begin{pmatrix} x \\ x' \end{pmatrix}_{(0)}$$

Linearized particle motion is described by matrix multiplication for all possible initial conditions.

Liouville's theorem in the x-x' plane

In magnetic fields: $p = const.$ linear in x': $x' = \frac{p_x}{p} \propto p_x$

(x, p_x) are canonically conjugate $\Rightarrow$ (x, x') are canonically conjugate

$$\int_{F(z)} dx\,dx' = \int_{F_0} |\det(\underline{M}(z\leftarrow 0))|\,dx_0\,dx_0' \stackrel{!}{=} \int_{F_0} dx_0\,dx_0'$$

Liouville's theorem $\Rightarrow$ $\underline{\underline{\det(M(z\leftarrow 0)) = 1}}$

Parametrization of the one turn map

$\underline{M}(0) = \underline{M}(L \leftarrow 0)$ for the circumference L

$$\begin{pmatrix} x_n \\ x'_n \end{pmatrix} = \underline{M}_{(0)} \begin{pmatrix} x_{n-1} \\ x'_{n-1} \end{pmatrix} = \underline{M}_{(0)}^{n} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\underline{M}_{(0)} = \begin{bmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{bmatrix} = \cos\mu \cdot \mathbb{1} + \sin\mu \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}, \ \det(\underline{M}_{(0)}) = \gamma\beta - \alpha^2 = 1$$

$$\cos\mu = \tfrac{1}{2}\mathrm{Tr}(\underline{M}_{(0)}) \ , \ c = \cos\mu, \ s = \sin\mu \ , \ \alpha = \frac{M_{11} - M_{22}}{2s}, \ \beta = \frac{M_{12}}{s}, \ \gamma = \frac{M_{21}}{s}.$$

$\mu \in \mathbb{R}$ for $|\mathrm{Tr}(\underline{M}_{(0)})| \le 2$.

Invariants of motion

$I(x, x') = \gamma x^2 + 2\alpha x x' + \beta x'^2 = (x, x') \begin{bmatrix} \gamma & \alpha \\ \alpha & \beta \end{bmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$ is an invariant

Proof:

$$\underline{I(x_{n+1}, x'_{n+1})} = \left[\left\{c \cdot \mathbb{1} + s \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}\right\} \begin{pmatrix} x_n \\ x'_n \end{pmatrix}\right]^T \begin{bmatrix} \gamma & \alpha \\ \alpha & \beta \end{bmatrix} \left\{c \cdot \mathbb{1} + s \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}\right\} \begin{pmatrix} x_n \\ x'_n \end{pmatrix}$$

$$= (x_n, x'_n) \left\{c \cdot \mathbb{1} + s \begin{bmatrix} \alpha & -\gamma \\ \beta & -\alpha \end{bmatrix}\right\} \left\{c \cdot \begin{bmatrix} \gamma & \alpha \\ \alpha & \beta \end{bmatrix} + s \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\} \begin{pmatrix} x_n \\ x'_n \end{pmatrix}$$

$$= (x_n, x'_n) \left\{c^2 \begin{bmatrix} \gamma & \alpha \\ \alpha & \beta \end{bmatrix} + s^2 \begin{bmatrix} \gamma & \alpha \\ \alpha & \beta \end{bmatrix} + s \cdot c \cdot \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)\right\} \begin{pmatrix} x_n \\ x'_n \end{pmatrix}$$

$$= (x_n, x'_n) \begin{bmatrix} \gamma & \alpha \\ \alpha & \beta \end{bmatrix} \begin{pmatrix} x_n \\ x'_n \end{pmatrix} \ \underline{= I(x_n, x'_n)}$$

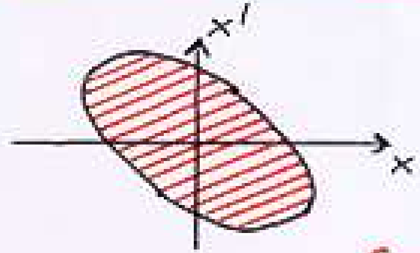
Particles on the curve $I(x, x') = \text{const.}$ stay on this curve every turn

The emittance

$\{\binom{x}{x'} \mid \binom{x}{x'} \in \mathbb{R}^2 \wedge I(x,x') = \varepsilon\}$ describes an ellipse with area $\pi \cdot \varepsilon$.

Finding the invariant curve by tracking

The invariant area

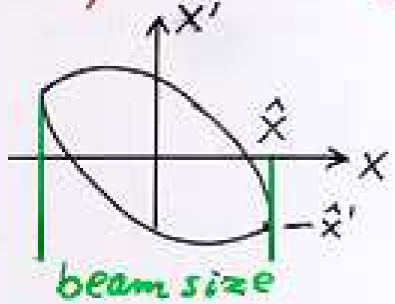


A beam which fills an invariant curve $I(x,x')=\varepsilon$ will always fill this curve. It has the emittance $\varepsilon \cdot \pi$.

$\{\underline{M}^n \binom{x_0}{x_0'} \mid n \in \mathbb{N}\}$ is on one invariant.

Emittance = Area which a beam fills in phase space.

The β- and γ-function

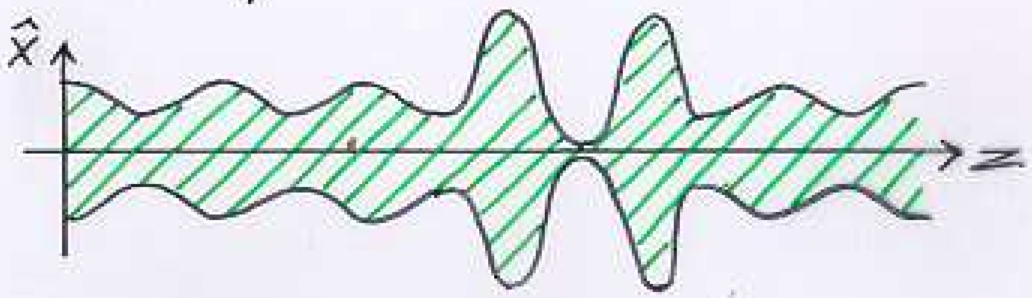


$$\partial_{x'} I(\hat{x},\hat{x}') = 2(\beta\hat{x}' + \alpha\hat{x}) = 0$$

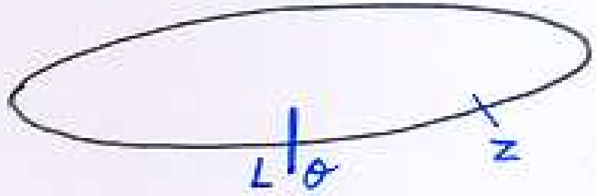
$$\begin{aligned} I(\hat{x},\hat{x}') &= (\beta\hat{x}' + \alpha\hat{x})\hat{x}' + \alpha\hat{x}\hat{x}' + \gamma\hat{x}^2 \\ &= \alpha\hat{x}(-\tfrac{\alpha}{\beta}\hat{x}) + \gamma\hat{x}^2 = (\gamma\beta - \alpha^2)\frac{\hat{x}^2}{\beta} \\ &= \hat{x}^2/\beta = \varepsilon \end{aligned}$$

Beam size : $\sqrt{\beta\varepsilon}$

Beam divergence: $\sqrt{\gamma\varepsilon}$



Propagation of β, γ, and μ



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{(z+L)} = \underline{M}(z \leftarrow 0) \begin{pmatrix} x \\ x' \end{pmatrix}_{(L)}$$

$$= \underline{M}(z \leftarrow 0)\, \underline{M}(L \leftarrow 0)\, \underline{M}(0 \leftarrow z) \begin{pmatrix} x \\ x' \end{pmatrix}_{(z)}$$

$$= \underline{M}(z \leftarrow 0)\, \underline{M}(0)\, \underline{M}^{-1}(z \leftarrow 0) \begin{pmatrix} x \\ x' \end{pmatrix}_{(z)}$$

$$\underline{M}(0) \Rightarrow \beta(0),\ \gamma(0),\ \cos\mu(0) = \tfrac{1}{2}\mathrm{Tr}(\underline{M}(0))$$

$$\underline{M}(z) = \underline{M}(z \leftarrow 0)\, \underline{M}(0)\, \underline{M}^{-1}(z \leftarrow 0) \Rightarrow \beta(z),\ \gamma(z)$$

$$\cos\mu(z) = \tfrac{1}{2}\mathrm{Tr}\left(\underline{M}(z \leftarrow 0)\, \underline{M}(0)\, \underline{M}^{-1}(z \leftarrow 0)\right) = \tfrac{1}{2}\mathrm{Tr}(\underline{M}(0)) = \cos\mu(0)$$

The tune $Q = \frac{\mu}{2\pi}$ is the same in all places of the ring

$$\begin{pmatrix} x_n \\ x_n' \end{pmatrix} = \underline{M}(0)^n \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \left\{\cos(n\mu)\cdot \mathbb{1} + \sin(n\mu) \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}\right\} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

Induction proof:

$$\underline{M}(0)^1 = \cos\mu \cdot \mathbb{1} + \sin\mu \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}^2 = \begin{bmatrix} \alpha^2 - \beta\gamma & 0 \\ 0 & \alpha^2 - \beta\gamma \end{bmatrix} = -\mathbb{1}$$

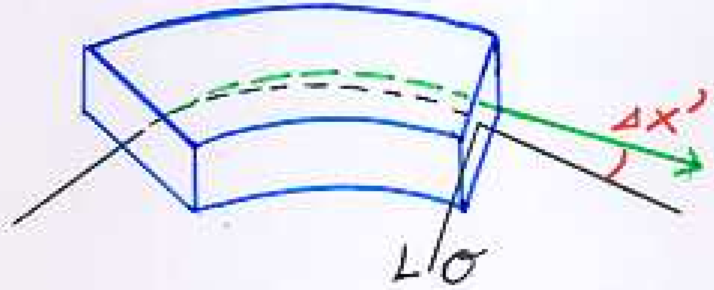
$$\underline{M}(0)^{n-1}\, \underline{M}(0) = \left\{\cos(n-1)\mu \cdot \mathbb{1} + \sin(n-1)\mu \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}\right\}\left\{\cos\mu \cdot \mathbb{1} + \sin\mu \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}\right\}$$

$$= (\cos(n-1)\mu \cos\mu - \sin(n-1)\mu \sin\mu)\, \mathbb{1} + (\sin(n-1)\mu \cos\mu + \cos(n-1)\mu \sin\mu) \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}$$

$$= \cos(n\mu)\cdot \mathbb{1} + \sin(n\mu) \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix} = \underline{M}(0)^n \quad \odot \Rightarrow \text{for } \mu \in \mathbb{R}:$$

$$x_0 = 0 \Rightarrow x_n = x_0' \beta \sin(n\mu) = x_0' \beta \sin(Q\omega_0 t)$$ Beams oscillation with $Q\omega_0$

Closed orbit deviations



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{(L)} = \underline{M}_{(0)} \begin{pmatrix} x \\ x' \end{pmatrix}_{(0)} + \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix}$$

Now a particle starting on the design curve $(x_0 = 0, x_0' = 0)$ has no periodic orbit since $x_1' = \Delta x'$ after one turn.

The new closed orbit starts with $\begin{pmatrix} x_{c.o.} \\ x'_{c.o.} \end{pmatrix}_{(L)} = \underline{\begin{pmatrix} x_{c.o} \\ x'_{c.o} \end{pmatrix}_{(0)}} = \underline{M}_{(0)} \begin{pmatrix} x_{c.o.} \\ x'_{c.o.} \end{pmatrix}_{(0)} + \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix} = \underline{(\mathbb{1} - \underline{M}_{(0)})^{-1} \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix}}$

Inverse of $\underline{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \frac{1}{\det(A)}$ since $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \mathbb{1}(ad - bc)$

$$\begin{pmatrix} x_{c.o.} \\ x'_{c.o.} \end{pmatrix} = (\mathbb{1} - \underline{M}_{(0)})^{-1} \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix} = \left\{ (1-c)\mathbb{1} - s \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \right\}^{-1} \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix}$$

$$= \frac{1}{\det(\mathbb{1} - M)} \left\{ (1-c)\mathbb{1} - s \begin{pmatrix} -\alpha & -\beta \\ \gamma & \alpha \end{pmatrix} \right\} \begin{pmatrix} 0 \\ \Delta x' \end{pmatrix}$$

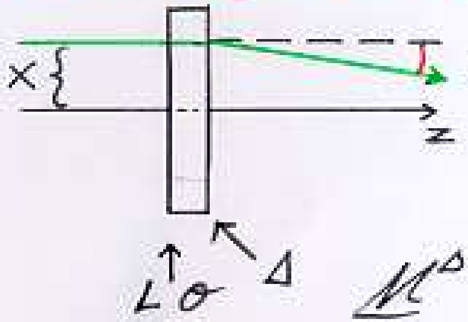
$$x_{c.o.} = + \frac{\sin\mu \cdot \beta \Delta x'}{\det(\mathbb{1} - \underline{M})} = + \frac{2 \sin\frac{\mu}{2} \cos\frac{\mu}{2} \cdot \beta \Delta x'}{\det(\mathbb{1} - \underline{M})} \; ; \; \det\left(\begin{bmatrix} 1 - M_{11} & M_{12} \\ M_{21} & 1 - M_{22} \end{bmatrix} \right) = 1 - \mathrm{Tr}(\underline{M}) + |\underline{M}|$$

$$= 2(1 - \cos\mu) = 4 \sin^2\frac{\mu}{2}$$

$$\underline{\underline{x_{c.o.}(z) = + \frac{\beta \Delta x'}{2 \,\mathrm{tg}(\pi Q)}}}$$

Close to integer tunes $Q \approx n \in \mathbb{N}$ the closed orbit is very sensitive.

Tune shift and quadrupole errors



$$x' = -\frac{x}{f} = -\Delta K \cdot x \qquad \begin{pmatrix} x \\ x' \end{pmatrix}_{(L)} = \underline{M}_{(0)} \begin{pmatrix} x \\ x' \end{pmatrix}_{(\Delta)} = \underbrace{\underline{M}_{(0)} \begin{bmatrix} 1 & 0 \\ -\Delta K & 1 \end{bmatrix}}_{\underline{M}^{\Delta K}} \begin{pmatrix} x \\ x' \end{pmatrix}_{(0)}$$

$$\underline{M}^{\Delta K} = \begin{bmatrix} C+\alpha S & \beta S \\ -\gamma S & C-\alpha S \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\Delta K & 1 \end{bmatrix} = \begin{bmatrix} C+\alpha S-\Delta K \beta S & \beta S \\ \cdots & C-\alpha S \end{bmatrix}$$

$$\frac{1}{2} Tr(\underline{M}^{\Delta K}) = \cos(\mu+\Delta\mu) = C - \frac{\Delta K \beta S}{2} = \cos\mu\cos\Delta\mu - \sin\mu\sin\Delta\mu \approx C - S\Delta\mu$$

$$\Delta Q = \frac{\Delta\mu}{2\pi} = \frac{\Delta K \cdot \beta}{4\pi}$$

β- beat

$$M_{12}^{\Delta K} = (\beta+\Delta\beta)\sin(\mu+\Delta\mu) = \underline{\beta S} = (\beta+\Delta\beta)(\sin\mu\cos\Delta\mu + \cos\mu\sin\Delta\mu)$$

$$\approx (\beta+\Delta\beta)(S + C\cdot\Delta\mu) \approx \underline{\beta S} + \underbrace{\Delta\beta\cdot S + \beta C \Delta\mu}_{0}$$

$$\Delta\beta = -\frac{\beta\cdot C}{S}\Delta\mu = -\frac{\Delta K \cdot \beta^2}{2 tg(2\pi Q)}$$

Change of β $\Longrightarrow$ change of beam shape

Close to $Q \approx n \cdot \frac{1}{2}$, $n \in \mathbb{N}$ the beam shape is very sensitive.

The periodic dispersion

Particles with higher momentum $p+\Delta p$ travel differently around the ring, since $\frac{1}{\varrho} = \frac{qB_\perp}{p+\Delta p} \approx \frac{qB_\perp}{p}(1-\frac{\Delta p}{p})$.

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \underline{M}_{(0)} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} + \vec{D}\frac{\Delta p}{p}$$ after linearization in $\frac{\Delta p}{p}$.

The closed orbit for particles with by Δp higher momentum

$$\begin{pmatrix} x_{c.o.} \\ x'_{c.o.} \end{pmatrix} = (\underline{1} - \underline{M}_{(0)})^{-1} \vec{D}\frac{\Delta p}{p} = \vec{\eta}\frac{\Delta p}{p}$$ periodic dispersion $= \eta_x$

Chromaticity correction

Quadrupoles focus less for higher momentum particles

$$K = \frac{1}{f} \Rightarrow K \cdot \frac{p}{p+\Delta p} \approx K(1-\frac{\Delta p}{p}) \quad ; \quad \Delta K = -K\frac{\Delta p}{p}$$

Tune shift: $\Delta Q = \sum_n \frac{\Delta K_n \beta_n}{4\pi} = -\sum_n \frac{K_n \beta_n}{4\pi} \cdot \frac{\Delta p}{p}$ Chromaticity $= \frac{\Delta Q}{\Delta p/p}$

Purpose of sextupoles:
- Particles with Δp go off center through sextupoles
- A shifted sextupole has a quadrupole field

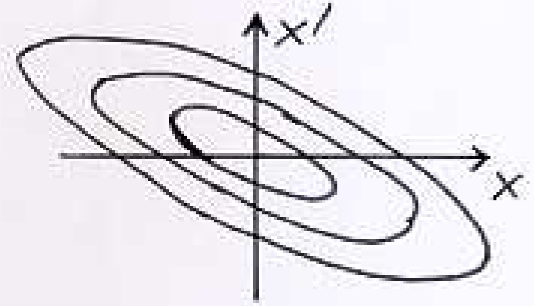
These Δp dependent quadrupole fields are used to correct for ΔQ

Nonlinear invariants of motion

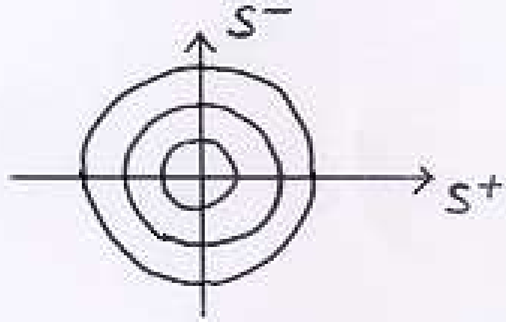
Linearization:

$$I(x,x') = \gamma x^2 + 2\alpha x x' + \beta x'^2 = \text{const.}$$

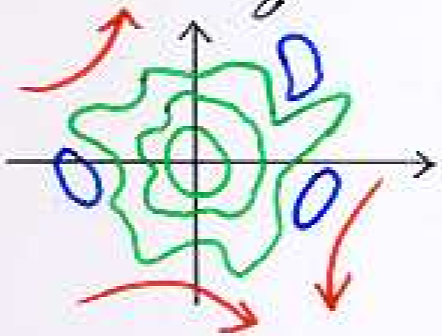
$$= \frac{1}{\beta}\left(s^{+2} + s^{-2}\right) \quad \text{for} \quad s^+ = x \text{ and } s^- = \alpha x + \beta x'.$$



$\xRightarrow{\text{Transformation to action angle variables}}$

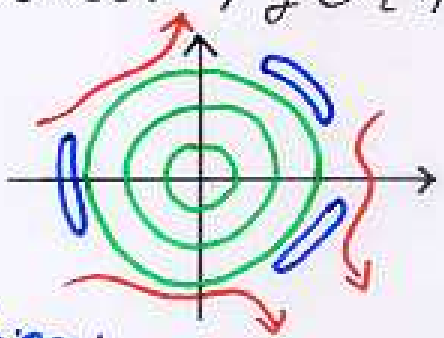


Nonlinear dynamics: $I_j(x,x',y,y',\tau,\Delta E) = \text{const.}, \; j \in \{1,2,3\}$



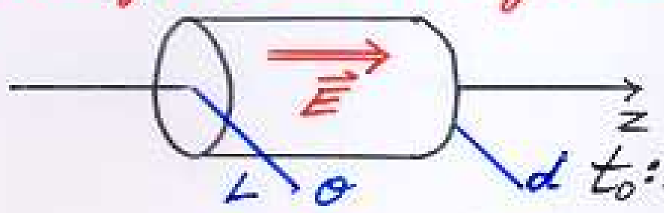
stable regions
islands
unstable regions

$\xRightarrow[\text{Normal form transformation:}]{}$



Accelerator physics contains applications of nonlinear dynamics in multidimensional phase space.

Longitudinal dynamics



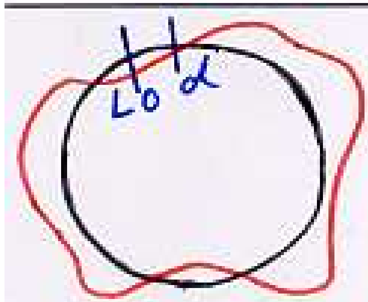
$$E(d) = E(0) + V_0 \sin(\omega t)$$

t_0: Time of a reference particle on the design curve

$$\tau = t - t_0 \quad : \quad \text{Small time difference}$$

$$\Delta E(d) = \Delta E(0) + V_0 \sin\omega(t_0+\tau) - V_0 \sin(\omega t_0)$$
$$= \Delta E(0) + V_0 \left(\sin(\omega t_0)\cos(\omega\tau) + \sin(\omega\tau)\cos(\omega t_0) - \sin(\omega t_0)\right)$$
$$\approx \Delta E(0) + V_0\,\omega\,\tau(0)\cos(\omega t_0)$$

$$\begin{pmatrix}\tau(d)\\ \Delta E(d)\end{pmatrix} = \begin{pmatrix}1 & 0\\ V_0\,\omega\cos(\omega t_0) & 1\end{pmatrix}\begin{pmatrix}\tau(0)\\ \Delta E(0)\end{pmatrix}$$



$$\eta_x \frac{\Delta p}{p} = \eta_x \frac{\Delta E}{\beta^2 E}\ , \ (\text{here } \beta = \tfrac{v}{c})$$

$\Delta E = 0$: Traveltime with $x = 0$: $\quad t_0 = T$

$\Delta E \neq 0$: Traveltime with $x = \frac{\eta_x}{\beta^2 E}\Delta E$: $\quad t = T(1 + \alpha_E \frac{\Delta E}{E})$

$$\tau(L) = \tau(d) + \frac{T}{E}\alpha_E\,\Delta E(d)$$

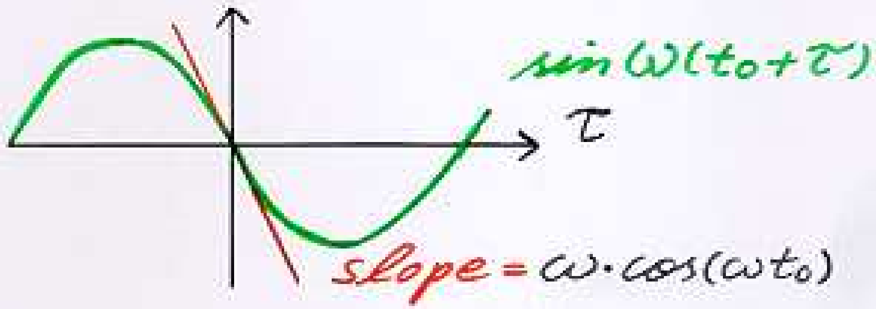
$$\begin{pmatrix}\tau\\ \Delta E\end{pmatrix}_{(L)} = \begin{bmatrix}1 & \frac{T}{E}\alpha_E\\ 0 & 1\end{bmatrix}\begin{pmatrix}\tau\\ \Delta E\end{pmatrix}_{(d)} = \begin{bmatrix}1 & \frac{T}{E}\alpha_E\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ V_0\,\omega\cos(\omega t_0) & 1\end{bmatrix}\begin{pmatrix}\tau\\ \Delta E\end{pmatrix}_{(0)}$$

$$= \underline{\underline{\begin{bmatrix}1 + V_0\,\omega\frac{T}{E}\alpha_E\cos(\omega t_0) & \frac{T}{E}\alpha_E\\ V_0\,\omega\cos(\omega t_0) & 1\end{bmatrix}\begin{pmatrix}\tau\\ \Delta E\end{pmatrix}_{(0)}}} = \underline{M_E}\begin{pmatrix}\tau\\ \Delta E\end{pmatrix}_{(0)}$$

$\underline{M}_{(0)} = \cos\mu + \sin\mu \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}, \quad \underline{M}_{(0)}^{n} = \cos(n\mu) + \sin(n\mu) \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}$

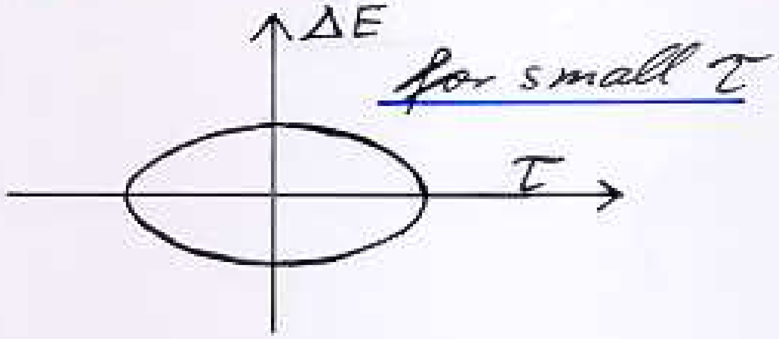
$\Rightarrow$ Particle motion can only be stable for $\mu \in \mathbb{R}$ $\Rightarrow$ $|Tr(\underline{M}_{(0)})| \leq 2$

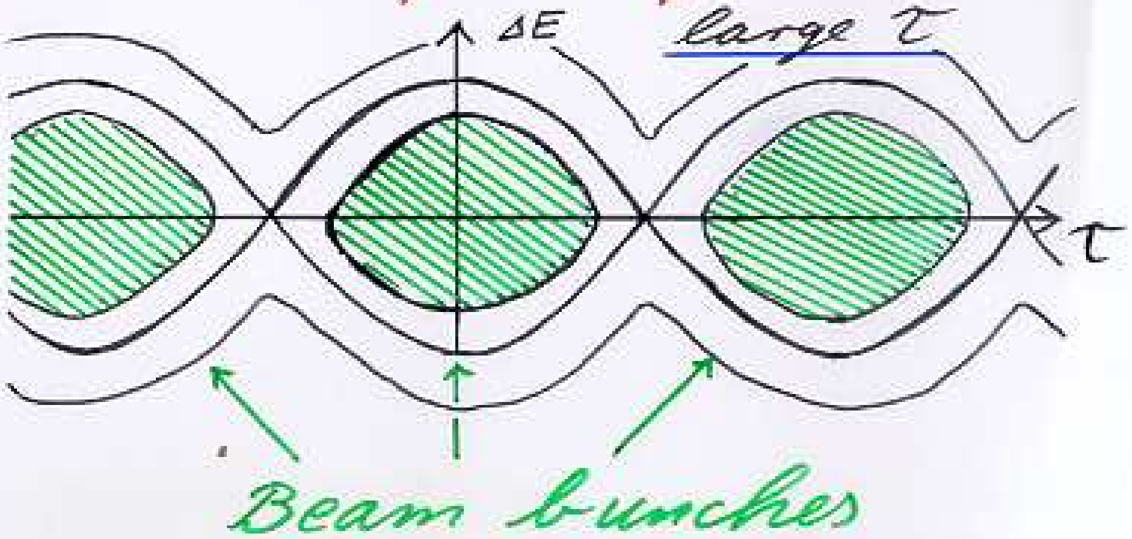
$Tr(\underline{M}_E) = 2 + V_0\, \omega \frac{T}{E} \alpha_E \cos(\omega t_0)$ $\Rightarrow$ Phase stability: $\alpha_E \cos(\omega t_0) \leq 0$



Phase stability means that a particle with $\Delta E > 0$ receives less energy than the reference particle.

Synchrotron oscillations in $\tau - \Delta E$ phase space





Many particle effects

a) Space charge

Rest frame field of a particle bunch $\vec{E}_R$

$$\vec{E} = \vec{E}_{R\parallel} + \gamma \vec{E}_{R\perp}\ , \quad \vec{B} = \gamma \vec{\beta} \times \vec{E}_R \frac{1}{c} = \frac{1}{c} \vec{\beta} \times \vec{E}$$

Force on a particle with velocity $\vec{\beta}_p$:

$$\vec{F} = q(\vec{E} + c \cdot \vec{\beta}_p \times \vec{B}) = q(\vec{E} + \vec{\beta}_p \times (\vec{\beta} \times \vec{E}_\perp)) = q(\vec{E} + \vec{\beta}(\vec{\beta}_p \vec{E}_\perp) - (\vec{\beta}\vec{\beta}_p)\vec{E}_\perp)$$

$$\vec{F}_\perp = q \vec{E}_\perp (1 - \vec{\beta} \cdot \vec{\beta}_p)$$

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b) Intra beam scattering: $\vec{\beta}_p \approx \vec{\beta} \Rightarrow \vec{F}_\perp$ is small but ΔE by $\vec{F}_\parallel$.

c) Beam beam effect: $\vec{\beta}_p \approx -\vec{\beta} \Rightarrow \vec{F}_\perp$ is big

d) Restgas scattering: Interaction of beam and gas in beampipe.

e) Beam wall interactions: Interaction of beam particles with the mirror charges in the chamber walls.

Accelerator physics contains applications of many particle dynamics and statistical mechanics.

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