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1 Fringe Field Approximations - G. Hoffstätter and M. Berz

The fringe fields of particle optical elements have a strong effect on optical properties. So far their transfer maps can only be calculated accurately using numerical integrators, which is very time consuming. We developed a new symplectic approximation method and implemented it in COSY INFINITY [1].

Often the effect of fringe fields is approximated by fringe field integrals to save computation time. This was done in second order for TRANSPORT [2] and in third order for GIOS [3]. There is an attempt being made to expand to fifth order [4] for some particle optical elements. The approximation by fringe field integrals, however, has some serious disadvantages:

- It is non-symplectic and therefore not especially suited for circular machines, where symplectic tracking can be advantageous.
- It represents the fringe effect well only if the region of the fringe field is not much bigger than the dimension of the beam diameter.
- It is restricted to low orders.

In order to speed up the fringe field calculation in the arbitrary order code COSY INFINITY we searched for a method that does not have those drawbacks, works fast, and to all orders.

In the following we will use TRANSPORT notation for the map [5], which means the variables of motion are the cartesian coordinates and slopes x, x', y, y', the path length difference l, and the relative momentum deviation from a reference momentum δ_p. Those six components build the vector $\vec{z}$. The transfer map describes the variables of motion $\vec{z}_f$ behind an optical element as a function of the variables $\vec{z}_i$ in front of it.

$$\vec{z}_f = \vec{M}^P(\vec{z}_i) \tag{1}$$

The index P indicates that the map $\vec{M}$ depends on certain parameters, like the momentum p, mass m, and charge z of the reference particle and the aperture A of the optical element.

From now on we will be concerned with magnetostatic elements, although parts of the procedure are applicable to electrostatic elements, too. The equation for the Lorentz force yields that the bending radius of the path of a particle with momentum p at field B is

$$R = \frac{p}{qB\sin\phi} \tag{2}$$

where ϕ denotes the angle between momentum and field direction. All maps are identical that describe particles with equivalent bending radii along their path. If the map for a specific beam is known as a function of the field B at the pole tip, the transfer map for all other beams can be computed:

$$\vec{M}^{p^*,m^*,z^*,B^*} = \vec{M}^{p,m,z}(B)\big|_{B=B^*\frac{z^*p}{p^*z}} \tag{3}$$

The map on the right hand side is known as a function of the field B, whereas the left hand side describes a map that is calculated at a certain field B^*. This is just another way of saying that the map depends only on the ratio of field B to magnetic rigidity p/z, as long as saturation is not important. At this point it is important to use TRANSPORT notation with δ_p as a variable, rather than the relative energy deviation δ_E, because the relative momentum deviation is equivalent to a relative deviation from the bending radius at momentum p as well as at any momentum p^*. Therefore the dependence of $\vec{z}_f$ on δ_p is the same at p and at p^*. With δ_E as a variable this would not be the case. Furthermore it is helpful that the TRANSPORT notation uses path length as a variable, rather than time of flight, because the time of flight is different for different reference momenta, whereas the path length stays the same.

Now let us consider two similar magnetostatic elements that differ by a scaling factor α, only. If the bending radii also differ by a factor of α, the maps are similar. Equation (2) shows that this is the case whenever the α-times bigger element has a $1/\alpha$ times stronger field. After scaling the coordinates x, y, l, we obtain

$$\begin{pmatrix} M_x^{B/\alpha,A\alpha} \\ M_{x'}^{B/\alpha,A\alpha} \\ M_l^{B/\alpha,A\alpha} \\ M_{\delta_p}^{B/\alpha,A\alpha} \end{pmatrix}_{(x_i,x_i',l,\delta_p)} = \begin{pmatrix} \alpha M_x^{B,A} \\ M_{x'}^{B,A} \\ \alpha M_l^{B,A} \\ M_{\delta_p}^{B,A} \end{pmatrix}_{(x_i/\alpha,x_i',l/\alpha,\delta_p)} \tag{4}$$

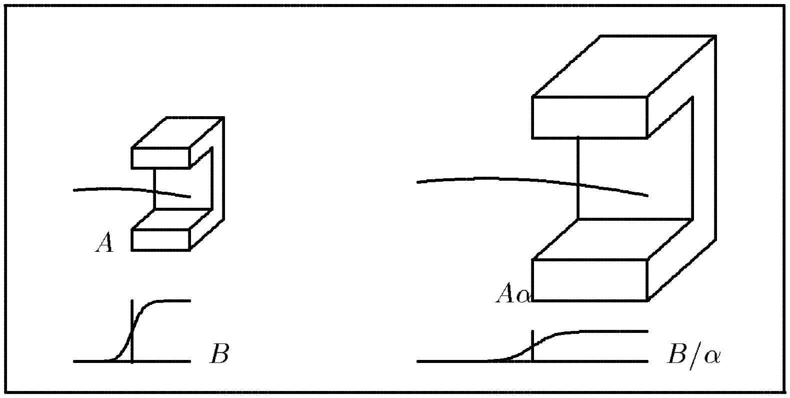

Figure 1: Particle coordinates in two elements scale with a factor α, if the elements scale with a factor α, and the fields scale with a factor $1/\alpha$.

The second dimension (y, y') is not mentioned, for it has the same properties as (x, x').

As a conclusion, we state that the knowledge of the transfer map as a function of the field strength at the pole tip, for particles with a specific magnetic rigidity, and for an element with a specific aperture, is sufficient to know all transfer maps of similar elements, for all energies, masses, and charges. In fact the map does not even have to be known as a function of the field, because the dependence on the field can be obtained by equation (3) from the dependence of the map on the momentum, which is already contained in the map, as momentum is one of the six variables.

In general it is useful and customary to work with the canonical coordinates $(x, \frac{p_x}{p_0}, y, \frac{p_y}{p_0}, \frac{E_0}{p_0}(t_0 - t), \delta_E)$ [6], therefore a transformation routine had to be implemented that transforms canonical notation into TRANSPORT notation and back.

$$M^{p,z} = T_{(E,m)} M^{E,m,z} T^{-1}_{(E,m)} \tag{5}$$

This transformation T depends on the reference energy and mass.

COSY INFINITY can readily compute the map of a fringe field of certain aperture $\vec{M}^{E,m,z,A}$, with δ_E being the sixth variable. Now we would be finished. The map $\vec{M}^{E,m,z,A}$ of a certain fringe field for a certain beam had to be stored once as a function of B in order to compute maps of similar fields and all kind of beams. COSY INFINITY, however, can only approximate a function by its Taylor series, therefore the scaled map will only be approximated. The accuracy depends on the chosen order of the Taylor expansion and on the relative difference of the bending radii in the scaled and the saved element. We can live with that for we only strive for an approximation. The approximated map will only be approximately symplectic. Symplecticity, however, is an intrinsic symmetry of canonical motion that arises from the special structure of Hamilton's equations. It should not be violated, especially when long term behaviour is of interest.

This drawback can be eliminated by storing the reference map $\vec{M}^{E,m,z,A}$ in a symplectic representation [7]. Either in form of a generating function

$$\vec{z}_{i,f} = SJ\partial_{\vec{z}_{i,f}} F(B) \tag{6}$$

where S is a matrix with only 1 or -1 on the diagonal, or in form of a Lie factorization

$$\vec{z}_f = L(B)e^{:P(B):}\vec{z}_i \tag{7}$$

where $:P:$ denotes a Poisson bracket waiting to happen. In higher orders the first representation is slow because a map inversion is required. The second one has the disadvantage that the matrix L(B) can only be approximated and is not exactly symplectic. It is most efficient to represent the non-linear part by $e^{:P:}$ and the linear part by the generating function that is most accurate for the given matrix L.

Now all pieces are assembled to display figure (2) which describes the whole symplectic scaling procedure. To do the required manipulation of parameter dependent maps certain procedures like the Lie transformation, the generating functions, and the TRANSPORT notation had to be added to COSY INFINITY.

Figure (3) shows the dependence of the expansion coefficient (x, xxa) as a function of the field B at the pole tip of a quadrupole. Because functions like this can be closely approximated by polynomials, this procedure is quite accurate. The following speed and accuracy comparisons will be made to the standard COSY Runge Kutta of eighth order which for the sake of speed is usually set to an accuracy of 10^{n-9} for order n. This integrator is thought to be most efficient for the differential equations of particle dynamics [8].

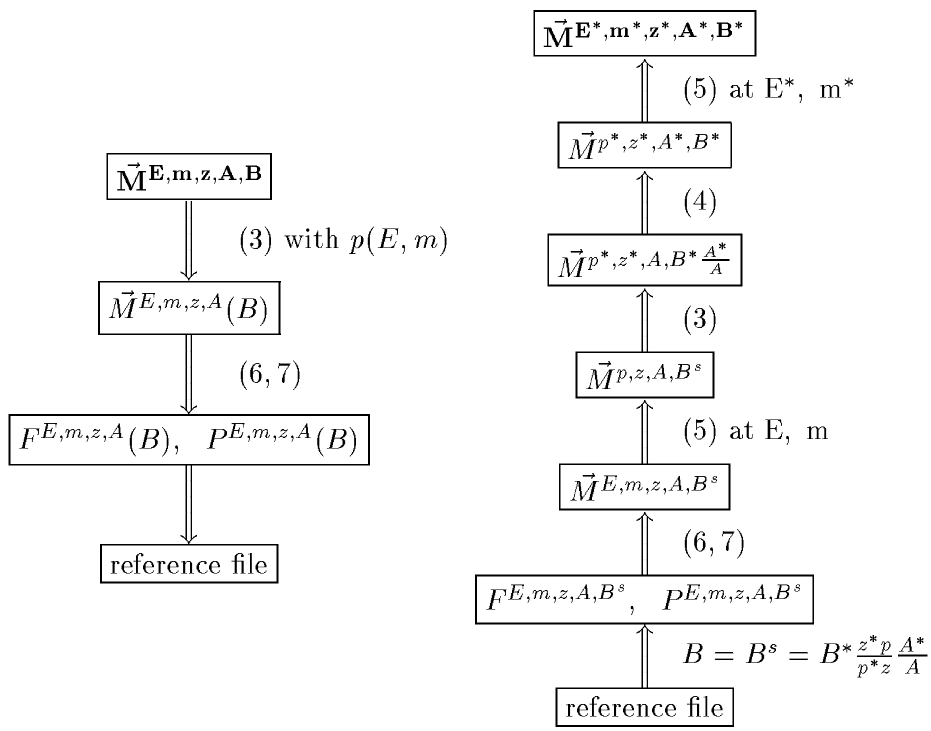

Figure 2: Shows how a map for arbitrary beam parameters E^*, m^*, z^*, fields B^*, and aperture A^* can be computed from the map of a similar element using symplectic scaling.

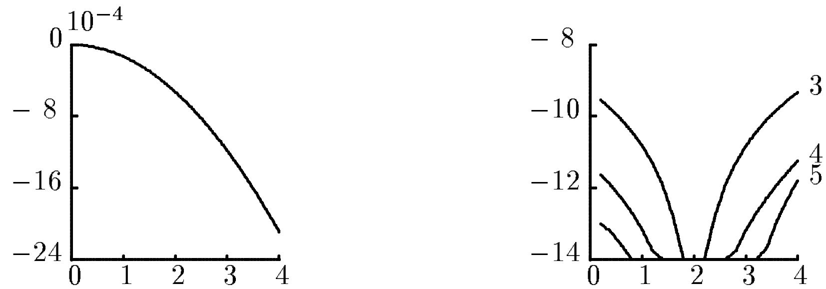

Figure 3: left: (x, xxa) for a quadrupole as a function of the field at the pole tip. right: Logarithmic error of the approximation of (x, xxa) with different expansion orders for the reference representation. The reference representation was computed at $B = 2T$.

Even at the border of the range in figure (3), the presented method is more accurate than the COSY standard integrator. Inside the region the accuracy increases drastically. The results in figure (3) were obtained by evaluating the symplectic reference representation to third, fourth, and fifth order. The accuracy can be further improved by increasing this order which of course increases the computation time that has to be invested for creating the reference map in advance. This investment can be very much rewarding, especially when beamlines or spectrometers are being fitted so that maps of similar fringe fields are needed over and over again with only slightly different parameters.

In typical cases the presented method is faster by a factor of 45 for first order and 80 for third order matrix elements, for higher orders the speed advantage increases rapidly.

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