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Survival times of particles in storage rings

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Storage rings are designed to hold particles for a long time. In the SSC this time is in the order of one day, which relates to $3 \cdot 10^8$ turns around the machine. To ensure that the machine design is capable of holding the required emittance for that number of turns, it is important to develop a method to find out how long a particle with a given initial condition will remain inside the ring. This could be done by tracking the paths of particles through 10^8 turns, which in an accurate way is far too time consuming with todays computing power, and the stability of motion can only be checked for a limited number of particles. There are, however, some programs available that follow this approach using kick approximations for the optical elements to speed up the computation [1]. Other approaches look at the one turn transfer map that relates initial phase space coordinates $\vec{z}_i$ to final coordinates after one turn $\vec{z}_f = \vec{M}(\vec{z}_i)$. This one turn map contains all information about particle motion after many turns since many turns are described by successive action of the one turn map.

The transfer map can be approximated in different ways. Recently an appropriate choice of spline functions and fourier series has been applied [2]. More Commonly, the Taylor expansion of the function is used. This Taylor map can be obtained automatically to an arbitrary order when DA codes are used [3]. Time considerations often restrict calculations to about order 12. Usually the Taylor map approach is justified since in accelerators motion near the closed orbit is weakly nonlinear. Once the one turn map is obtained, particles can be tracked through the map to find out how long they stay inside the accelerator. Applying high order maps the required number of turns can still be very time consuming and, like in the case of element by element tracking, the stability can only be checked for a very limited number of particles.

We want to propose a method which analyses the one turn map directly without tracking through it several times. Furthermore this method will not only test single particles but will give information about all particles in a given region of phase space. We assume that there is a closed orbit in the ring. Particles with phase space coordinates near the closed orbit will not be lost, particles which are too far away from the closed orbit will be lost during their motion around the ring. We therefore divide the phase space $\mathcal{P}$ into the allowed region $\mathcal{O}$ and the forbidden region $\mathcal{P} \backslash \mathcal{O}$.

The question we want to answer is: How many turns does a particle which

starts in a given region of phase space $\mathcal{N}$ circle the ring without leaving the accelerator. We therefore look for the number

$$N_{\max} = \max\{n | \vec{M}^n(\mathcal{N}) \subseteq \mathcal{O}\} \tag{1}$$

where $\vec{M}^n(\mathcal{N}) = \{\vec{M}^n(\vec{z}) | \vec{z} \in \mathcal{N}\}$, and $\vec{M}^n(\vec{z})$ means applying the map n times. The different regions are shown in figure 1 a). With the following method we will find a strict lower bound N for $N_{\max}$.

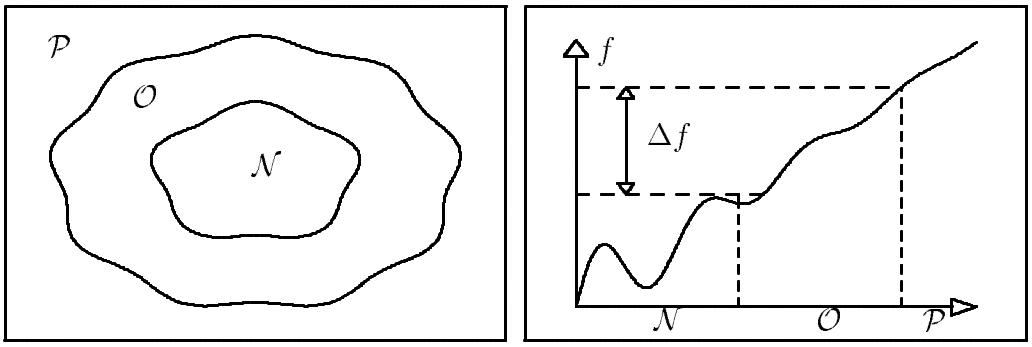

Figure 1: a) The initial region $\mathcal{N}$ and the allowed region $\mathcal{O}$ of phase space $\mathcal{P}$ with $\mathcal{N} \subset \mathcal{O} \subset \mathcal{P}$. b) The gap Δf that has to be bridged.

If we find a real valued test function $f(\vec{z})$ which does not have common values in $\mathcal{N}$ and in $\mathcal{P}\backslash\mathcal{O}$, then successive action for the map must bridge a gap Δf as shown in figure 1 b). Particles start to bridge this gap by entering the phase space region $\mathcal{S}_i = \vec{M}(\mathcal{N})\backslash\mathcal{N}$. The gap is bridged when a particle has reached the region $\mathcal{S}_f = \vec{M}(\mathcal{O})\backslash\mathcal{O}$. If $\mathcal{S}_i$ or $\mathcal{S}_f$ are empty, particles in $\mathcal{N}$ will never leave $\mathcal{O}$. If they are not empty, the gap goes from f_i to f_f with $f_i = \max\{f(\vec{z}) | \vec{z} \in \mathcal{S}_i\}$ and $f_f = \min\{f(\vec{z}) | \vec{z} \in \mathcal{S}_f\}$. The function $d(\vec{z}) = f(\vec{M}(\vec{z})) - f(\vec{z})$ describes how much the gap is closed by the action of the map. If we assume $f_f \geq f_i$, the maximum step from f_i towards f_f is

$$\delta = \max\{d(\vec{z}) | \vec{z} \in (\mathcal{O}\backslash\mathcal{N})\} \quad . \tag{2}$$

A particle that starts in $\mathcal{N}$ therefore survives at least N turns with

$$N = [\frac{f_f - f_i}{\delta}] \quad . \tag{3}$$

We are thus left with four problems:

1. finding a suitable test function $f(\vec{z})$ such that N becomes favorable.

2. finding f_i, the maximum of $f(\vec{z})$ on $\mathcal{S}_i$.

3. finding f_f, the minimum of $f(\vec{z})$ on $\mathcal{S}_f$.

4. finding δ, the maximum of the change of $f(\vec{z})$ under the action of the one turn map in the appropriate region.

Describing the initial region and the allowed region is essential to finding $\mathcal{S}_i$ and $\mathcal{S}_f$ and therefore to finding a function that changes substantially between those two regions. The allowed region of an accelerator is typically given by the acceptance in the x–p_x phase space, the y–p_y phase space, and the time–energy phase space while it is assumed that the liner map does not couple these phase spaces. The linear motion follows invariant ellipses in every plane. A canonical transformation can be applied, which transforms those ellipses to circles. The new coordinates are the linear normal form coordinates. Since the product of two circles is topologically a torus, the particle moves on the so called invariant torus, and the radii of those circles are the linear invariants of motion.

Giving those linear invariants of motion to describe the allowed region is equivalent to specifying the acceptances ϵ_i of the machine. The pictures a) and b) in figure 2 describe the ellipses which specify the boundary of the allowed region. The allowed region can also be expressed in figure 2 c) by drawing the allowed radii. The linear invariants are described by $I_i^{(1)}(\vec{z})$ and the conditions for the allowed region is $I_i^{(1)}(\vec{z}) \leq r_i = \sqrt{\epsilon_i}$ for all subspaces i. For $p_x = 0$ and $p_y = 0$ the radii are proportional to x and y. Since beamlines are circular, it is appropriate to change the allowed region in figure c) by requiring:

$$\sum_i \left(\frac{I_i^{(1)}(\vec{z})}{r_i}\right)^2 < 1 \quad . \tag{4}$$

To keep the notation simple we describe the initial region $\mathcal{N}$ in a similar way with radii αr_i where $\alpha < 1$. The boundaries are most easily described when a norm is introduced which measures the distance from the closed orbit according to the invariant torus on which the particle moves in linear approximation

$$||\vec{z}|| = \sum_i \left(\frac{I_i^{(1)}(\vec{z})}{r_i}\right)^2 \quad . \tag{5}$$

The different regions are given by:

$$\mathcal{N} = \{\vec{z} | \alpha^2 \geq ||\vec{z}||\} \quad , \mathcal{O} = \{\vec{z} | 1 \geq ||\vec{z}||\} \quad . \tag{6}$$

To make the desired estimate as good as possible, we should find a function $f(\vec{z})$, which tends to increase when $||\vec{z}||$ increases and should at the same time be close to an invariant to make δ as small as possible.

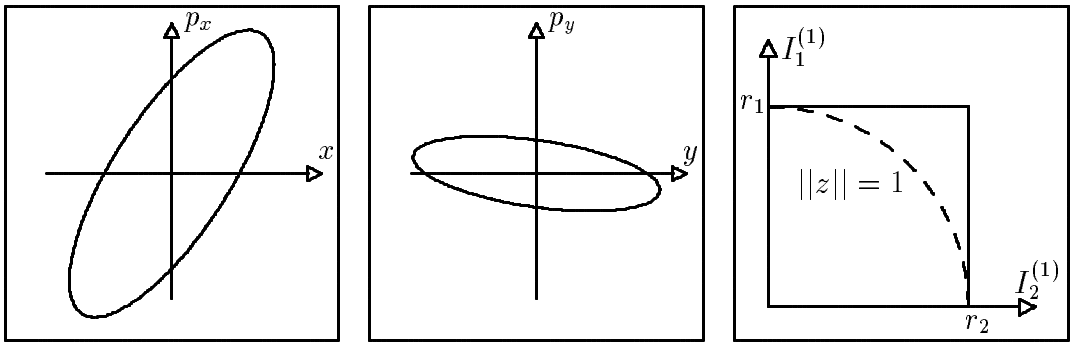

Figure 2: The motion on invariant ellipses in phase space: a) x–p_x, b) y–p_y. c) The allowed and the forbidden region and the definition of $||\vec{z}||$.

Such a function is easily obtained when the transformation to circles is not only performed for the linear map but for the complete nonlinear Taylor map up to the evaluation order n. The radii of those circles are invariant up to the evaluation order and are described by $I_i^{(n)}(\vec{z})$. Those normal form radii are easily obtained in the DA framework [4]. The desired function is therefore given by

$$f(\vec{z}) = \sum_i \left(\frac{I_i^{(n)}(\vec{z})}{r_i}\right)^2 \quad . \tag{7}$$

The remaining three problems are concerned with the finding of maxima. The regions $\mathcal{S}_i = \vec{M}(\mathcal{N})\backslash\mathcal{N}$ and $\mathcal{S}_f = \vec{M}(\mathcal{O})\backslash\mathcal{O}$ can not be represented as clearly as the regions $\mathcal{N} = \{\vec{z}|\alpha^2 \geq ||\vec{z}||\}$ and $\mathcal{O} = \{\vec{z}|1 \geq ||\vec{z}||\}$. This does not lead to a problem when auxiliary functions $g(\vec{z})$ and $h(\vec{z})$ are used,

$$g(\vec{z}) = \begin{cases} f(\vec{M}(\vec{z})) & \text{if } \vec{M}(\vec{z}) \notin \mathcal{N} \\ -K & \text{if } \vec{M}(\vec{z}) \in \mathcal{N} \end{cases} \tag{8}$$

$$h(\vec{z}) = \begin{cases} -f(\vec{M}(\vec{z})) & \text{if } \vec{M}(\vec{z}) \notin \mathcal{O} \\ K & \text{if } \vec{M}(\vec{z}) \in \mathcal{O} \end{cases} \tag{9}$$

with a sufficiently big parameter K. The required quantities are then given by the following equations:

$$\begin{aligned} f_i &= \max\{f(\vec{z})|\vec{z} \in \mathcal{S}_i\} = \max\{g(\vec{z})|\vec{z} \in \mathcal{N}\} \\ f_f &= \min\{f(\vec{z})|\vec{z} \in \mathcal{S}_f\} = \max\{h(\vec{z})|\vec{z} \in \mathcal{O}\} \\ \delta &= \max\{d(\vec{z})|\vec{z} \in (\mathcal{O}\backslash\mathcal{N})\} \quad . \end{aligned} \tag{10}$$

Those maxima can be found in a mathematically rigorous way using interval arithmetic [5]. Doing this would, together with equation 3, give a mathematically

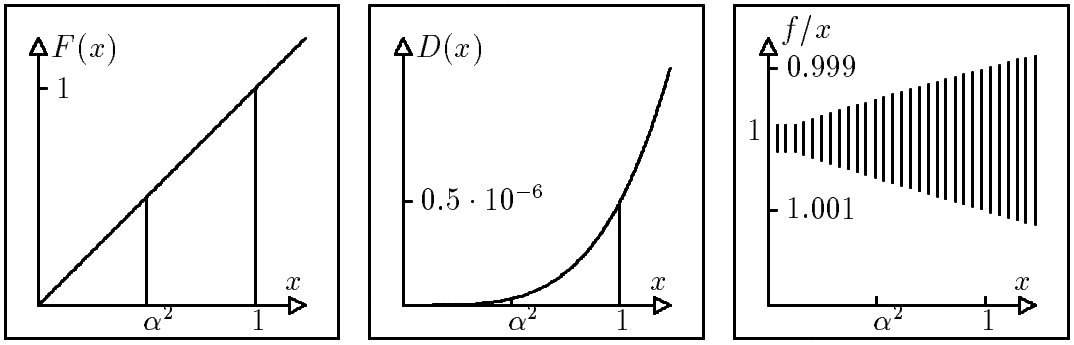

Figure 3: The functions a) $F(x)$ and b) $D(x)$ in the region $\mathcal{O}$. c) Variation of $f(\vec{z})$ relative to $||\vec{z}||$.

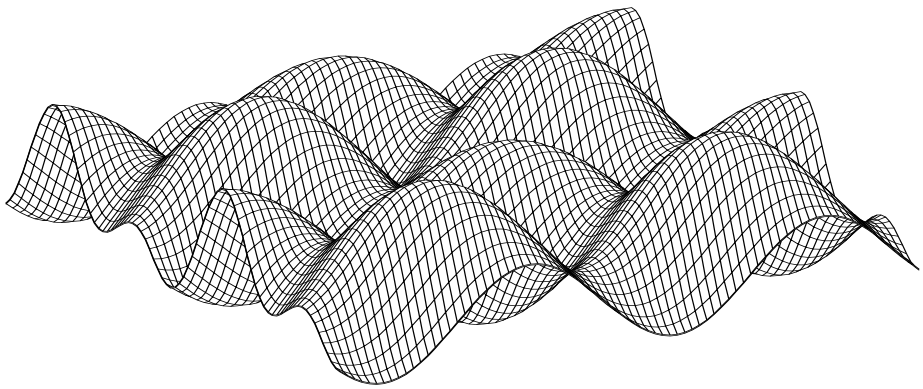

Figure 4: $d(\vec{z})$ on the boundary between allowed and forbidden region. The maximum δ of this function has to be found. In this example the function varies between $-0.5 \cdot 10^{-6}$ and $0.5 \cdot 10^{-6}$.

rigorous estimate for the survival time of particles propagated by the given one turn Taylor map.

The functions $f(\vec{z})$ and $d(\vec{z})$ have some properties which allow to make sensible simplifications. Those properties will be demonstrated for the proposed PSR II. The evaluation order is 6 and the acceptances are ϵ_i = 1mm mrad. As shown in figure 3 a), the function $F(x) = \max\{f(\vec{z})|x = ||\vec{z}||\}$ is typically growing monotonously with x so that the maximum of $f(\vec{z})$ on $\mathcal{S}_i$ occurs at $||\vec{z}|| = \alpha^2$ and the minimum of $f(\vec{z})$ on $\mathcal{S}_f$ occurs at $||\vec{z}|| = 1$. The function $D(x) = \max\{d(\vec{z})|x = ||\vec{z}||\}$ is also typically growing monotonously as shown in figure 3 b), so that the maximum δ occurs at the border of the allowed region ($||\vec{z}|| = 1$). Furthermore figure 3 b) shows that the variation of $f(\vec{z})$ relative to $||\vec{z}||$ is much smaller than $\Delta f = f_f - f_i$ which therefore is close to $1 - \alpha^2$. We obtain the estimate

$$N = \frac{1 - \alpha^2}{\max\{d(\vec{z})|1 = ||\vec{z}||\}} \tag{11}$$

which involves finding only one maximum on a subspace with constant $||\vec{z}||$. Figure 4 shows the function $d(\vec{z})$ on the border of the allowed region ($||\vec{z}|| = 1$). This function does not have sharp maxima so that sampling with 20 steps in each direction does give a good approximation of the maximum value. Table 1 gives N for different systems and for different evaluation orders for $\alpha^2 = 1/2$. For the pendulums the propagation after one second was taken as one turn map so that N describes the time in which the instruments certainly stays in the allowed region. Due to energy conservation, the pendulum and the coupled pendulums are stable for all times. The quality of our estimate is shown by the big numbers N which we obtain for those cases.

Order	Pendulum (r =1mm mrad)	Coupled Pendulums ($r_1 = r_2$ =1mm mrad)	PSR II ($r_1 = r_2$ =1mm mrad)
3	$1.37 \cdot 10^{10}$	$1.04 \cdot 10^{9}$	$3.22 \cdot 10^{4}$
4	$1.37 \cdot 10^{10}$	$1.02 \cdot 10^{9}$	$3.60 \cdot 10^{4}$
5	$1.20 \cdot 10^{15}$	$5.42 \cdot 10^{10}$	$6.37 \cdot 10^{5}$
6	$1.20 \cdot 10^{15}$	$4.20 \cdot 10^{12}$	$6.94 \cdot 10^{5}$

Table 1: Minimum number of turns N to move from $I_i^{(1)} = r_i/2$ to $I_i^{(1)} = r_i$.

The evaluation of the functions $f(\vec{z})$ and $d(\vec{z})$, using interval arithmetic to find the maximum values and therefore establishing a mathematically strict lower bound for the turn number N_{max}, is currently under investigation.

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