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RIGOROUS STABILITY ESTIMATES

G. H. Hoffstätter and M. Berz

Abstract

The normal form method is used to give completely rigorous lower bounds on the number of turns which particles survive in a storage ring when the motion is described by a Taylor map. This can be achieved by computing the necessary invariants of motion with nonlinear normal form theory and performing the required global optimizations with interval arithmetic. Conventional interval arithmetic optimization turns out to be by far to inefficient for the complex optimization problems evolving from the normal form method. The concept of interval chains will be introduced which is faster by many orders of magnitude for the special functions that have to be optimized.

Introduction

The normal form method of long term estimates was used by several people in the past and is lined out in the accompanying report [1]. Nonlinear normal form theory yields pseudo invariants of motion f which are polynomials of order $2n$ if the Taylor map $\vec{M}$ has order n. The deviation of being invariant under application of $\vec{M}$ is described by the deviation function δ.

$$f(\vec{M}) = f + \delta \ . \tag{1}$$

The deviation function δ has only contributions of orders higher than n. For weakly nonlinear problems and high expansion orders n, the contribution of δ is very small and due to (1), f is an approximate invariant of the map $\vec{M}$.

In the region of multidimensional phase space described by

$$\{\vec{z}|r_{min} < f(\vec{z}) < r_{max}\} \ , \tag{2}$$

the global maximum $\overline{\delta}$ of δ has to be found. It can then be guaranteed that the number of turns N_{max} which particles survive in the storage ring is certainly bigger than $N_N = (r_{max} - r_{min})/\overline{\delta}$. For the reasoning behind this statement, please refer to the literature given in [1]. Figure (1a) shows the structure of the function δ which has to be maximized for the PSR II lattice. The optimization has to be performed in a four dimensional space; we can only depict a two dimensional section.

Interval arithmetic is a means to obtain an interval $G(I)$ which contains all possible values of a function g on an interval I [2].

$$G(I) \supseteq g(I) = \{g(x)|x \in I\} \tag{3}$$

The upper bound $\overline{G}$ of the interval G is then a guaranteed upper bound of the function g on the interval I. Using interval arithmetic, the global maximum $\overline{\delta}$ can be rigorously bound to garantie the following estimate:

$$N_{max} \le N_N \le N_I = \frac{r_{max} - r_{min}}{\overline{\Delta}} \ . \tag{4}$$

If the interval I is big, interval arithmetic tents to over–estimate $g(I)$ substantially. Therefore it is often necessary to divide the interval I into many subintervals I_i and find the maximum of all the maxima on the I_i. Figure (1a) shows the function δ on a section of four dimensional phase space. The relevant region in phase space had to be divided into 10^{18} interval blocks in order to reduce blow–up to a useful amount. Figure (1b) shows a section of phase space on which the maximum of δ is bound by interval arithmetic on many interval blocks. The substantial interval blow–up that occurs when $\delta = f(\vec{M}) - f$ is being evaluated is due to cancellation of the contributions up to order n. Although it is known that these contributions vanish, they still cause blow–up during their computation. The deviation function δ is a polynomial of order $2n^2$, which typically is around 200. The blow–up caused by the computations of all orders higher than n is less critical than the low order blow–up, due to the weakly nonlinear structure of the problem.

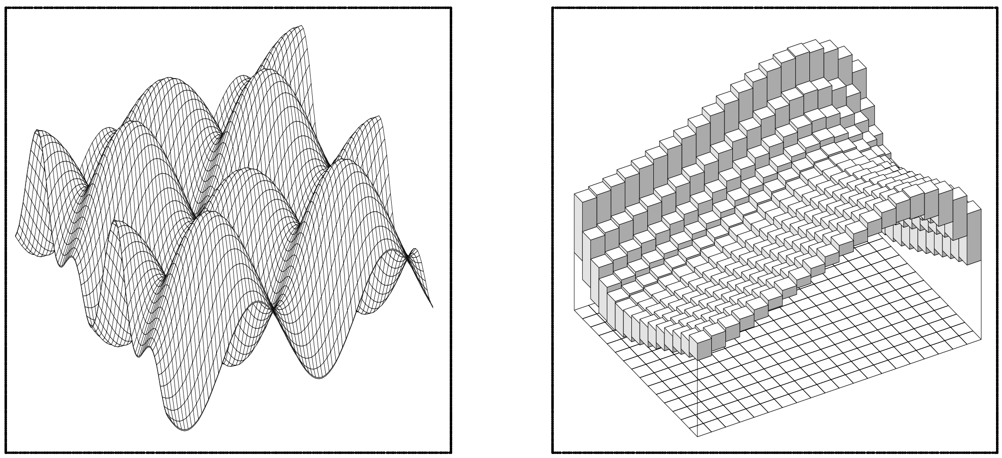

Figure 1: a: The deviation function δ. The maximum $\overline{\delta}$ of this function has to be found. In this example the function varies between $-0.5 \cdot 10^{-6}$ and $0.5 \cdot 10^{-6}$. b: δ bounded by intervals on a small part of phase space.

Maximizing the deviation function with interval chains

The concept of interval chains takes advantage of the knowledge that δ has no contributions up to order n. An interval chain I consists of a finite sequence of intervals I_i, $i \in \{0, \ldots, n+1\}$:

$$J = (J_0, J_1, J_2, \ldots, J_{n+1}) \tag{5}$$

where J_i is called the ith order of the interval chain. With a special arithmetic [3,4] one can obtain an interval chain with intervals J_i, $i \leq n$ that contain all contributions to a given polynomial of order i. J_{n+1} contains all contributions to the polynomial of orders higher than n. In the case of the polynomial δ, J_{n+1} contains all function values, whereas the lower orders of the interval chain contain all the blow–up due to low order cancellation. Thus, the bound of the function value and all blow–up due to lower order computations are strictly separated.

But even with these simplifications, the resulting objective functions have a tendency to exhibit interval blow–up because of complexity, while the bounds of the function have to be determined rather tightly in order to guarantee large numbers of stable turns.

The results in the next section were obtained by choosing 630 intervals for the examples with one degree of freedom and 1000188 for the example with two degrees of freedom. Without the concept of interval chains a realization af the described method was virtually impossible. For examples with two degrees of freedom approximately 10^{12} times more intervals would have been needed for similar results.

Results

Order of Invariant	Interval Bounding (guaranteed)	Interval Chains (guaranteed)	Conventional Rastering (optimistic)
3	11252	743,667	849,195
4	11252	743,667	849,195
5	11306	876,059,284	982,129,435
6	11306	876,059,284	982,129,435
7	11306	432,158,877,713	636,501,641,854
8	11306	432,158,877,713	636,501,641,854

Table 1: Predictions of the number of stable turns as a function of the order of the polynomials describing the normal form transformation for the physical pendulum $d^2/dt^2\phi + \sin(\phi) = 0$ for a time step of t=1 and an amplitude of 1/10 rad. Because of energy conservation, the map is known to be permanently stable for any amplitude.

Using the technique discussed in the previous section, several nonlinear systems were studied using the interval chain rastering methods to provide upper bounds for the invariant defects. In order to get a feeling for the quality of these upper bounds, the numbers were compared with approximations for the maximal invariant defects obtained by a rather tight rastering in real arithmetic. Because of the large number of local maxima, this method proved to be the most robust noninterval way to estimate the absolute maxima of the functions involved. Lower bounds on the number of stable turns obtained by conventional intervals are given in the tables 1 to 3 in order to illustrate the usefulness of interval chains. When conventional intervals were used, the deviation function was simplified as much as possible by accounting for cancellations up to second order analytically. The number of conventional intervals and the number of interval chains used in the bounding are equivalent. In all of the examples below, the choice of r_{max} is given, and r_{min} was chosen half as large.

As the first example to check the method, we used a one–dimensional physical pendulum. This is a good test case since energy conservation requires the nonlinear motion to be stable. Table 2 shows the results of the stability analysis for this case. As is to be expected, the number of stable turns predicted increases with the order and hence accuracy of the approximate invariants. While the approximate scanning method can take full advantage of this increased accuracy, the interval bounding method shows a saturation at 11306 turns. This asymptotic behavior is connected to the size of the intervals because of the unavoidable blow–up of intervals. The blow–up in third order dominates the calculation, causing the higher order improvements to not materialize. The method of interval chains takes care of all the low order cancellations and consequently the estimate is much better.

As another example, we chose the Henon map, which is a standard test case for the analysis of nonlinear motion because it exhibits almost all of the phenomena encountered in Hamiltonian nonlinear dynamics. These include stable and unstable regions, chaotic motion, and periodic elliptic fixed points. The Henon map can even serve as a very simplistic model of an accelerator under the presence of sextupoles for chromaticity correction. The results of these calculations are shown in table 3. Similar to the previous case, the number of predicted turns increases with order. In the case of interval bounding, the number of periodic turns shows asymptotic behavior limited by blow–up. Again the superiority of strict bounding with interval chains is obvious.

Order of Invariant	Interval Bounding (guaranteed)	Interval Chains (guaranteed)	Conventional Rastering (optimistic)
2	895	891	1,086
3	1736	9,926	11,450
4	1668	54,016	65,667
5	1674	678,725	809,612
6	1670	3,389,641	4,351,679
7	1671	42,640,927	52,474,387
8	1671	192,650,961	263,904,035

Table 2: Predictions of the number of stable turns for the Henon map at tune 0.13, strength parameter k = 1.1, and starting position of .01 as a function of the order of the polynomials in the normal form transformation.

Order of Invariant	Interval Bounding (guaranteed)	Interval Chains (guaranteed)	Conventional Rastering (optimistic)
3	179	16,137	38,385
4	179	18,197	38,857
5	173	309,356	560,309
6	173	347,312	613,135
7	171	925,531	2,184,998
8	171	1,004,387	2,248,621

Table 3: Predictions of the number of stable turns as a function of order of the approximate invariant for the Los Alamos PSR II storage ring for the motion in a phase space of 100 mm mrad.

In the final example, we study a realistic accelerator, the Los Alamos PSR II. The same data are shown as for the two previous, more academic examples. To limit the calculation time, the intervals used for the optimization were 5 times as wide as the intervals used for the previous two tables.

References

1. G. H. Hoffstätter and M. Berz, Refinement of the normal form method for long term stability estimates, Annual Report, NSCL/Michigan State University, 1993
2. E. R. Hansen, Global optimization using interval analysis – the multidimensional case, Numerische Mathematik, 34(1980), 247–270
3. M. Berz and G. H. Hoffstätter, Exact bounds on the long term stability of weakly nonlinear systems applied to the design of large storage rings, Interval Computations, 1994, in print
4. M. Berz and G. H. Hoffstätter, Computation an application of Taylor polynomials with remainder bounds, submitted to Interval Computations