

Tracking algorithm for the stable spin polarization field in storage rings using stroboscopic averaging

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Polarized protons have never been accelerated to more than about 25 GeV. To achieve polarized proton beams in the Relativistic Heavy Ion Collider (RHIC, 250 GeV), the Hadron Electron Ring Accelerator (HERA, 820 GeV), and Fermilab's TeV accelerator (TEVATRON, 900 GeV), ideas and techniques applicable to accelerator physics are needed. In this publication we will stress an important aspect of very high energy polarized proton beams, namely, the fact that the equilibrium polarization direction can vary substantially across the beam in the interaction region of a high energy experiment when no countermeasure is taken. Such a divergence of the polarization direction would not only diminish the average polarization available to the particle physics experiment, but it would also make the polarization involved in each collision analyzed in a detector strongly dependent on the phase space position of the interacting particle. In order to analyze and compensate for this effect, methods for computing the equilibrium polarization direction are needed. In this paper we introduce the method of stroboscopic averaging, which computes this direction in a very efficient way. Since only tracking data are needed, our method can be implemented easily in existing spin tracking programs. Several examples demonstrate the importance of the spin divergence and the applicability of stroboscopic averaging. [S1063-651X(96)12510-1]

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INTRODUCTION

In order to maximize the number of collisions of stored particles in a storage ring system, one tries to maximize the total number of particles in the bunches, and tries to minimize the emittances so that the particle distribution across phase space is narrow and the phase space density is high. At equilibrium the phase space distribution does not change in time, and is therefore periodic in the machine azimuth.

If, in addition, the beam is spin polarized, one requires that the polarization is high. As first emphasized by Barber and co-workers [1,2], for energies of the order of 1 TeV, a fundamental limitation to the polarization of particle beams becomes important. To put our work in context, we repeat the arguments here.

Spins traveling with particles in electromagnetic fields precess according to the Thomas-Bargmann-Michel-Telegdi equation (TBMT) to be discussed below. The guide fields in storage rings are produced by dipole and quadrupole magnets. The dipole fields constrain the particles to almost circular orbits and the quadrupole fields focus the beam, thus ensuring that the particles do not drift too far away from the central orbit.

In horizontal dipoles, spins precess only around the vertical field direction. The quadrupoles have vertical and horizontal fields and additionally cause the spins to precess away from the vertical direction. The strength of the spin precession and the precession axis in machine magnets depends on the trajectory and energy of the particle. Thus in one turn around the ring the effective precession axis can deviate from the vertical, and will depend on the initial position of the particle in six-dimensional phase space. From this it is

clear that if an equilibrium spin distribution exists, i.e., if the polarization vector at every phase space point is periodic in the machine azimuth, it will vary across the orbital phase space. This field of equilibrium spin directions in phase space does not change from turn to turn when particles propagate through the accelerator, although after each turn the particles find themselves at different positions in phase space. These directions, which we denote by the unit vector $\vec{n}(\vec{z}, \theta)$, where \vec{z} denotes the position in the six dimensional phase space of the beam and θ is the generalized azimuth, were first introduced by Derbenev and Kondratenko [3] in the theory of radiative electron polarization. Note that $\vec{n}(\vec{z}, \theta)$ is usually not an eigenvector of the spin transfer matrix at some phase space point, since the spin of a particle changes after one turn around the ring, but the eigenvector would not change. Thus, once we know this direction $\vec{n}(\vec{z}, \theta)$, the phase space dependent polarization $p(\vec{z}, \theta)$ in this direction, and the phase space density function $\rho(\vec{z}, \theta)$, we have a complete specification of the polarization state of a beam of spin- $\frac{1}{2}$ particles.

The maximizing of the polarization of the ensemble implies two conditions: the polarization $p(\vec{z}, \theta)$ at each point in phase space should be high, and the polarization vector $\vec{n}(\vec{z}, \theta)$ at each point should be almost parallel to the average polarization vector of the beam. According to the TBMT equation, the rate of spin precession is roughly proportional to $a\gamma$, where $a = (g-2)/2$ is the anomalous part of the spin g factor and γ is the Lorentz factor. At very high energy, as for example in the HERA proton ring [4,1], it could happen that on average $\vec{n}(\vec{z}, \theta)$ deviates by tens of degrees from the phase space average of \vec{n} . Thus even if each point in phase space were 100% polarized the average polarization could be much smaller than 100%. Clearly it is very important to have accurate and efficient methods for calculating $\vec{n}(\vec{z}, \theta)$, and

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for ensuring that the spread of $\vec{n}(\vec{z}, \theta)$ is as small as possible.

However, although it has been straightforward to define $\vec{n}(\vec{z}, \theta)$, this vector is not easy to calculate in general, and much effort has been expended on this topic, but mainly for electrons at energies up to 46 GeV. Except for the Fourier expansion formalism introduced in [5], all other methods developed so far are explicitly perturbative, and either do not go to high enough order [6–8] or have problems with convergence at high order and high energy [9,10].

In this paper we describe an alternative method for obtaining $\vec{n}(\vec{z}, \theta)$. It is based on multiturn tracking and the averaging of the spin viewed stroboscopically from turn to turn. Since this innovative approach only requires tracking data, it is fast and very easy to implement in existing tracking codes. We will show that the convergence speed promises rather quick execution when simulating realistic accelerators. However, probably the main advantage over other methods is the fact that stroboscopic averaging does not have an inherent problem with either orbit or spin-orbit resonances due to its nonperturbative nature. This allows the behavior of the periodic spin solution close to resonances to be analyzed.

I. THE SPIN-ORBIT SYSTEM

The motion of the spin of particles traveling in electromagnetic fields is governed by the equations of motion

$$\frac{d\vec{z}}{d\theta}(\theta) = \vec{v}(\vec{z}(\theta), \theta), \quad (1)$$

$$\frac{d\vec{s}}{d\theta}(\theta) = \vec{\Omega}(\vec{z}(\theta), \theta) \times \vec{s}(\theta). \quad (2)$$

Here θ is an independent variable parametrizing the d -dimensional particle phase space trajectory $\vec{z}(\theta)$ and the spin trajectory $\vec{s}(\theta)$. In circular accelerators, θ is the azimuth. The rest frame spin vector \vec{s} has three components, and we normally deal with orbital phase space vectors \vec{z} which have six components. In accelerator physics these components are usually the positions and momenta of a particle combined with its energy and the time of flight.

We neglect the Stern-Gerlach forces since they are very small in comparison with the Lorentz force. Equation (2) is the TBMT equation along an orbit parametrized by $\vec{z}(\theta)$ [11,12]. Because we deal with a circular accelerator at fixed energy, $\vec{v}(\vec{z}, \theta)$ and $\vec{\Omega}(\vec{z}, \theta)$, which depend on the guide fields, are periodic in θ with period 2π corresponding to the circumference of the ring. Due to the precession, Eq. (2), the length of the spin vector \vec{s} , does not change along the azimuth.

The dynamical system (1) and (2) allows us to formulate the following partial differential equation for the evolution of a field $\vec{f}(\vec{z}, \theta)$:

$$\frac{d\vec{f}}{d\theta} = \frac{\partial \vec{f}}{\partial \theta} + \sum_{j=1}^d v_j \frac{\partial \vec{f}}{\partial z_j} = \vec{\Omega} \times \vec{f}, \quad (3)$$

where the three components of the solution \vec{f} depend on \vec{z} and θ . In our applications \vec{f} will describe the propagation of a spin distribution associated with a particle beam, and this physical interpretation can be adopted because of the following. A solution $\vec{f}(\vec{z}, \theta)$ to Eq. (3) can be found by specifying an arbitrary $\vec{f}(\vec{z}, \theta_0)$ at initial azimuth θ_0 and propagating it to θ by integrating equations (1) and (2). In fact,

$$\vec{s}(\theta) = \vec{f}(\vec{z}(\theta), \theta) \quad (4)$$

solves Eq. (2) if \vec{f} solves Eq. (3). We say that \vec{f} is normalized if

$$|\vec{f}| = \sqrt{f_1^2 + f_2^2 + f_3^2} = 1. \quad (5)$$

We call every normalized solution of Eq. (3) a spin field.

The \vec{n} -axis introduced in the Introduction is a special spin field which is periodic in θ with period 2π [3]:

$$\vec{n}(\vec{z}, \theta + 2\pi) = \vec{n}(\vec{z}, \theta). \quad (6)$$

Since Eq. (2) represents a pure rotation, the propagation of the spin vector can be described by a 3×3 orthogonal matrix. We denote this rotation matrix which propagates initial spins $\vec{s}(\theta_0)$ along a given orbit trajectory $\vec{z}(\theta)$ by $\underline{R}(\vec{z}(\theta_0), \theta, \theta_0)$, so that

$$\vec{s}(\theta) = \underline{R}(\vec{z}(\theta_0), \theta, \theta_0) \cdot \vec{s}(\theta_0). \quad (7)$$

Because for a spin field \vec{f} the spin trajectory (4) solves Eq. (2), we obtain

$$\vec{f}(\vec{z}(\theta), \theta) = \underline{R}(\vec{z}(\theta_0), \theta, \theta_0) \cdot \vec{f}(\vec{z}(\theta_0), \theta_0). \quad (8)$$

If $\vec{f}(\vec{z}, \theta)$ is a spin field, then $\vec{f}(\vec{z}, \theta + 2\pi m)$ is a spin field (m is an integer). This follows from Eq. (3) because \vec{v} and $\vec{\Omega}$ are periodic in θ . Thus Eq. (8) generalizes to

$$\vec{f}(\vec{z}(\theta), \theta + 2\pi m) = \underline{R}(\vec{z}(\theta_0), \theta, \theta_0) \cdot \vec{f}(\vec{z}(\theta_0), \theta_0 + 2\pi m) \quad (m \text{ integer}) \quad (9)$$

Since an \vec{n} axis is a periodic spin field, we observe by (6) and (9) that

$$\vec{n}(\vec{z}(\theta + 2\pi), \theta) = \underline{R}(\vec{z}(\theta), \theta + 2\pi, \theta) \cdot \vec{n}(\vec{z}(\theta), \theta). \quad (10)$$

Alternatively this equation can be used for defining the \vec{n} axis [13]. The matrix $\underline{R}(\vec{z}(\theta), \theta + 2\pi, \theta)$ is called the one turn spin transfer matrix for the trajectory $\vec{z}(\theta)$.

In the special case where the orbital motion is determined by a Hamiltonian, we have

$$\vec{v}(\vec{z}, \theta) = \{\vec{z}, H_{\text{orb}}(\vec{z}, \theta)\}, \quad (11)$$

where H_{orb} denotes the orbital Hamiltonian. Furthermore in this case one can define for the whole spin-orbit system a Hamiltonian given by [14]

$$H(\vec{z}, \vec{s}, \theta) = H_{\text{orb}}(\vec{z}, \theta) + H_{\text{spin}}(\vec{z}, \vec{s}, \theta), \quad (12)$$

where

$$H_{\text{spin}}(\vec{z}, \vec{s}, \theta) = \vec{s} \cdot \vec{\Omega}(\vec{z}, \theta). \quad (13)$$

The Poisson brackets of this Hamiltonian lead to Eqs. (1) and (2) if the Stern-Gerlach forces are neglected [15].

II. CONSTRUCTION OF PERIODIC SPIN FIELDS BY STROBOSCOPIC AVERAGING

To find solutions of Eq. (3) which are periodic in θ by our method, one first constructs an arbitrary spin field \vec{f} . One then constructs the following stroboscopic average of \vec{f} :

$$\langle \vec{f} \rangle(\vec{z}, \theta) = \lim_{n \rightarrow \infty} \left[\frac{1}{n+1} \sum_{m=0}^n \vec{f}(\vec{z}, \theta + 2\pi m) \right]. \quad (14)$$

Since the convergence and differentiability of the sequence in Eq. (14) can in general not be guaranteed, the limit is only taken formally. The problem of the convergence will be analyzed in more detail in Sec. IV B. From Eq. (14) it follows that $\langle \vec{f} \rangle$ is formally periodic in θ . Moreover, because Eq. (3) is a linear equation and because \vec{v} and $\vec{\Omega}$ are periodic in θ , we observe for any spin field \vec{f} that $\langle \vec{f} \rangle$ is formally also a solution of Eq. (3). Hence we come to a first conclusion that if \vec{f} is a spin field, then $\langle \vec{f} \rangle$ is a solution of Eq. (3) which is periodic in θ . If for \vec{f} the stroboscopic average $\langle \vec{f} \rangle$ vanishes nowhere in the $d+1$ dimensional space, then we define

$$\langle \vec{f} \rangle_{\text{norm}} = \langle \vec{f} \rangle / |\langle \vec{f} \rangle|. \quad (15)$$

In general, $\langle \vec{f} \rangle$ is not normalized, but the modulus of $\langle \vec{f} \rangle$ is conserved, and $\langle \vec{f} \rangle_{\text{norm}}$ is a spin field which is periodic in

θ . Hence we come to the following second conclusion of this section: if \vec{f} is a spin field with the property that $\langle \vec{f} \rangle$ vanishes nowhere in the $d+1$ -dimensional space, then $\langle \vec{f} \rangle_{\text{norm}}$ has all the properties of an \vec{n} axis. This result shows that an \vec{n} axis can be obtained from a spin field \vec{f} for which $\langle \vec{f} \rangle$ vanishes nowhere. In Sec. III we will derive a tracking algorithm based on this.

One practical choice of \vec{f} is characterized for all \vec{z} by

$$\vec{f}(\vec{z}, \theta_0) = \vec{n}_0(\theta_0), \quad (16)$$

where $\vec{n}_0(\theta)$ denotes the so called closed orbit spin axis defined by

$$\vec{n}_0(\theta) = \underline{R}(\vec{z}_{\text{c.o.}}(\theta), \theta + 2\pi, \theta) \cdot \vec{n}_0(\theta), \quad (17)$$

where $\vec{z}_{\text{c.o.}}(\theta)$ is the closed orbit.

III. TRACKING ALGORITHM FOR THE \vec{n} AXIS USING STROBOSCOPIC AVERAGING

In this section we develop a tracking algorithm which provides an efficient way to evaluate an \vec{n} axis at $\vec{z} = \vec{z}_0, \theta = \theta_0$. Choosing a spin field \vec{f} and replacing θ_0 by $\theta_0 - 2\pi m$ in Eq. (9) for every integer m we obtain

$$\begin{aligned} \vec{f}(\vec{z}(\theta), \theta + 2\pi m) \\ = \underline{R}(\vec{z}(\theta_0 - 2\pi m), \theta, \theta_0 - 2\pi m) \cdot \vec{f}(\vec{z}(\theta_0 - 2\pi m), \theta_0). \end{aligned} \quad (18)$$

If we choose an orbit with $\vec{z}(\theta_0) = \vec{z}_0$, then inserting this into Eq. (14) results in

$$\langle \vec{f} \rangle(\vec{z}_0, \theta_0) = \lim_{n \rightarrow \infty} \left[\frac{1}{n+1} \sum_{m=0}^n \underline{R}(\vec{z}(\theta_0 - 2\pi m), \theta_0, \theta_0 - 2\pi m) \cdot \vec{f}(\vec{z}(\theta_0 - 2\pi m), \theta_0) \right]. \quad (19)$$

Normalization of $\langle \vec{f} \rangle$ yields an \vec{n} axis at (\vec{z}_0, θ_0) .

To apply the tracking algorithm, the infinite sum involved in the stroboscopic average (19) is replaced by a finite sum of $N+1$ terms, so that we approximate

$$\langle \vec{f} \rangle(\vec{z}_0, \theta_0) \approx \langle \vec{f} \rangle_N(\vec{z}_0, \theta_0) = \frac{1}{N+1} \sum_{m=0}^N \vec{f}(\vec{z}_0, \theta_0 + 2\pi m) = \frac{1}{N+1} \sum_{m=0}^N \underline{R}(\vec{z}(\theta_0 - 2\pi m), \theta_0, \theta_0 - 2\pi m) \cdot \vec{f}(\vec{z}(\theta_0 - 2\pi m), \theta_0), \quad (20)$$

which yields the following approximation of the \vec{n} axis:

$$\vec{n}(\vec{z}_0, \theta_0) \approx \frac{\langle \vec{f} \rangle_N(\vec{z}_0, \theta_0)}{|\langle \vec{f} \rangle_N(\vec{z}_0, \theta_0)|}. \quad (21)$$

The stroboscopic average $\langle \vec{f} \rangle_N$ in Eq. (20) has a very simple physical interpretation which illustrates its practical impor-

tance. If a particle beam is approximated by a phase space density, disregarding its discrete structure we can associate a spin field $\vec{f}(\vec{z}, \theta_0)$ with the particle beam at the azimuth θ_0 . If one installs a pointlike gedanken polarimeter at a phase space point $\vec{z}_0 = \vec{z}(\theta_0)$ and azimuth θ_0 , then this polarimeter initially measures $\vec{f}(\vec{z}_0, \theta_0)$. When the particle beam passes the azimuth θ_0 after one turn around the ring, the polarimeter measures

$$\begin{aligned} & \underline{R}(\vec{z}(\theta_0 - 2\pi), \theta_0, \theta_0 - 2\pi) \cdot \vec{f}(\vec{z}(\theta_0 - 2\pi), \theta_0) \\ &= \vec{f}(\vec{z}_0, \theta_0 + 2\pi). \end{aligned}$$

After the beam has traveled around the storage ring N times and the polarization has been measured whenever the beam passed the gedanken polarimeter, one only has to average over the different measurements in order to obtain $\langle \vec{f} \rangle_N$. If the particles of a beam are polarized parallel to $\vec{n}(\vec{z}, \theta_0)$ at every phase space point, then the spin field of the beam is invariant from turn to turn due to the periodicity property in Eq. (6). But, in addition, even for beams which are not polarized parallel to \vec{n} , we see that the polarization observed at a phase space point \vec{z} and azimuth θ_0 is still parallel to $\vec{n}(\vec{z}, \theta_0)$, if one averages over many measurements taken when the beam has passed the azimuth θ_0 .

For the special choice $\vec{f}(\vec{z}, \theta_0) = \vec{n}_0(\theta_0)$, we can simplify $\langle \vec{f} \rangle_N$ to

$$\begin{aligned} & \langle \vec{f} \rangle_N(\vec{z}_0, \theta_0) \\ &= \frac{1}{N+1} \sum_{m=0}^N \underline{R}(\vec{z}(\theta_0 - 2\pi m), \theta_0, \theta_0 - 2\pi m) \cdot \vec{n}_0(\theta_0). \end{aligned} \quad (22)$$

Equations (20) and (21) define an algorithm for obtaining an \vec{n} axis. We see that the only information needed from tracking is the set of the $N+1$ phase space points $\vec{z}(\theta_0), \vec{z}(\theta_0 - 2\pi), \dots, \vec{z}(\theta_0 - 2\pi N)$ and the N matrices $\underline{R}(\vec{z}(\theta_0 - 2\pi), \theta_0, \theta_0 - 2\pi), \underline{R}(\vec{z}(\theta_0 - 4\pi), \theta_0, \theta_0 - 4\pi), \dots, \underline{R}(\vec{z}(\theta_0 - 2\pi N), \theta_0, \theta_0 - 2\pi N)$. Each matrix is a product of one turn spin transfer matrices $\underline{R}(\vec{z}, \theta_0 + 2\pi, \theta_0)$. This means that one tracks along the orbit $\vec{z}(\theta)$ to obtain the spin transfer matrix $\underline{R}(\vec{z}(\theta), \theta_0, \theta)$ and stores it at the N instants, where $\theta = \theta_0 - 2\pi N, \dots, \theta = \theta_0 - 4\pi, \theta = \theta_0 - 2\pi$. The function $\vec{f}(\vec{z}, \theta_0)$ is chosen independently of the tracking results [for example, one can take the choice $\vec{f}(\vec{z}, \theta_0) = \vec{n}_0(\theta_0)$ of Eqs. (16) and (22)].

The following two kinds of pathologies can occur:

(i) The \vec{n} axis is not unique: if the proposed algorithm converges, then the result could depend on the choice of $\vec{f}(\vec{z}, \theta_0)$.

(ii) The stroboscopic average $\langle \vec{f} \rangle$ vanishes for $N \rightarrow +\infty$ or the sequence in Eq. (14) does not converge.

Both pathologies can be studied with the algorithm. The first situation occurs for systems on spin-orbit resonances [16]. In all examples studied so far, the stroboscopic average seems to converge, implying the existence of an \vec{n} axis. In the second situation, the pointlike polarimeter at \vec{z}_0 mentioned above monitors an averaged polarization which either vanishes or fluctuates indefinitely.

IV. EFFICIENT IMPLEMENTATION ONLY USING ONE TURN INFORMATION

In the previously outlined formalism for evaluating an \vec{n} axis by stroboscopic averaging, it became apparent that only

knowledge about one turn spin transfer matrices is required. One can therefore reformulate the algorithm of Sec. III in terms of one turn maps which are to be taken at a fixed but arbitrary azimuth value θ_0 , and thereby obtain a more practical algorithm. Thus we introduce the one turn orbit transfer map \vec{M} which maps initial coordinates \vec{z}_i into final coordinates $\vec{z}_f = \vec{M}(\vec{z}_i)$. Then in our notation we have $\vec{z}(\theta_0 + 2\pi) = \vec{M}(\vec{z}(\theta_0))$. To describe the transport of particles with spins \vec{s} , we write for simplification the one turn spin transfer matrix $\underline{R}(\vec{z}, \theta_0 + 2\pi, \theta_0)$ as $\underline{R}(\vec{z})$, so that we have $\vec{s}_f = \underline{R}(\vec{z}_i) \cdot \vec{s}_i$. All other quantities which depend on θ are taken at the specified azimuth θ_0 . For simplification, this azimuth is not indicated in the following. As already mentioned, Eq. (10) can be used to define the \vec{n} axes at θ_0 . This condition, will be called the periodicity condition and now reads

$$\underline{R}(\vec{z}) \cdot \vec{n}(\vec{z}) = \vec{n}(\vec{M}(\vec{z})). \quad (23)$$

A. Recipe

To illustrate the process of evaluating an \vec{n} -axis at \vec{z}_0 and θ_0 in the case of linear orbit motion, we establish a recipe.

(1) Compute the linearized one turn phase space transfer map $\vec{M} = \underline{M} \cdot \vec{z}_i$.

(2) Define the set of $N+1$ phase space points

$$C = \{\vec{c}_j = (\underline{M}^{-1})^j \cdot \vec{z}_0 | j \in \{0, \dots, N\}\}. \quad (24)$$

(3) Compute the rotation matrix $\underline{R}(\vec{z}_{c.o.})$ which describes the spin motion for particles on the closed orbit $\vec{z}_{c.o.}(\theta)$, and extract the corresponding rotation vector \vec{n}_0 . This is the periodic spin solution for particles on the closed orbit.

(4) Starting with a spin parallel to \vec{n}_0 at every phase space point in C , track until the phase space point \vec{z}_0 is reached. For a given j this requires tracking j turns starting at \vec{c}_j .

(5) Define the set of spin tracking results as

$$\begin{aligned} B = \{ & \vec{b}_0(\vec{z}_0) = \vec{n}_0, \vec{b}_j(\vec{z}_0) = \underline{R}(\vec{c}_1) \cdots \underline{R}(\vec{c}_j) \cdot \vec{n}_0 | j \\ & \in \{1, \dots, N\}\}. \end{aligned} \quad (25)$$

(6) Define the sum of the elements in B as $\vec{s}_N(\vec{z}_0) = (1)/(N+1) \sum_{j=0}^N \vec{b}_j(\vec{z}_0)$, and for $|\vec{s}_N| \neq 0$ define $\vec{n}_N = \vec{s}_N / |\vec{s}_N|$.

The vector $\vec{s}_N(\vec{z}_0)$ is equivalent to $\langle \vec{f} \rangle_N(\vec{z}_0, \theta_0)$ in Sec. III, if the initial distribution of spins is given by \vec{n}_0 as in Eq. (22).

B. Convergence speed

It will now be shown that if the angle between \vec{n}_0 and $\vec{b}_j(\vec{z}_0)$ is smaller than some positive number $\xi < \pi/2$ for all $j \in \{1, \dots, N+1\}$, then \vec{n}_N satisfies the periodicity condition (23) for the \vec{n} axis up to an error which is smaller than or

equal to $2 \sec(\xi/2)\tan(\xi)/(N+1)$. Since evaluating B by the recipe of Sec. IV A requires tracking $T=(N+1)N/2$ turns, the accuracy is bounded by $\sqrt{2/T}\sec(\xi/2)\tan(\xi)$. This slow convergence with the square root of T is a very serious limitation, and in Sec. IV C we will demonstrate how the convergence can be considerably improved.

The proof of this convergence property goes along the following lines. The average \vec{s}_N has been defined by

$$\vec{s}_N(\vec{z}_0) = \frac{1}{N+1} \sum_{j=0}^N \prod_{k=1}^j \underline{R}(\vec{c}_k) \cdot \vec{n}_0. \quad (26)$$

Here we adopt the convention $\prod_{k=1}^0 \underline{R}(\vec{c}_k) = \prod_{k=1}^0 \underline{R}(\vec{c}_{k-1}) = \underline{1}$, and $\prod_{k=1}^j \underline{R}(\vec{c}_k)$ is taken to mean the following order of multiplication: $\underline{R}(\vec{c}_1) \cdots \underline{R}(\vec{c}_j)$. To check how well \vec{s}_N satisfies the periodicity conditions (23) of the \vec{n} axis, we calculate

$$\begin{aligned} \vec{s}_N(\underline{M} \cdot \vec{z}_0) &= \frac{1}{N+1} \sum_{j=0}^N \prod_{k=1}^j \underline{R}(\vec{c}_{k-1}) \cdot \vec{n}_0 \\ &= \frac{1}{N+1} \left(\vec{n}_0 + \sum_{j=0}^{N-1} \prod_{k=0}^j \underline{R}(\vec{c}_k) \cdot \vec{n}_0 \right), \end{aligned} \quad (27)$$

$$\underline{R}(\vec{z}_0) \cdot \vec{s}_N(\vec{z}_0) = \frac{1}{N+1} \sum_{j=0}^N \prod_{k=0}^j \underline{R}(\vec{c}_k) \cdot \vec{n}_0, \quad (28)$$

$$\begin{aligned} \underline{R}(\vec{z}_0) \cdot \vec{s}_N(\vec{z}_0) - \vec{s}_N(\underline{M} \cdot \vec{z}_0) &= \frac{1}{N+1} (\underline{R}(\vec{z}_0) \cdot \vec{b}_N(\vec{z}_0) - \vec{n}_0) \\ &= \frac{1}{N+1} (\vec{b}_{N+1}(\underline{M} \cdot \vec{z}_0) - \vec{n}_0). \end{aligned} \quad (29)$$

The length $|\vec{b}_{N+1}(\underline{M} \cdot \vec{z}_0) - \vec{n}_0|$ is smaller than $2 \sin(\xi/2)$, as shown in Fig. 1. The length of \vec{s}_N is at least $\cos(\xi)$, and here it becomes essential that there is a limit of $\pi/2$ on the angle ξ . This information is sufficient to establish the following inequality:

$$\begin{aligned} |\underline{R}(\vec{z}_0) \cdot \vec{n}_N(\vec{z}_0) - \vec{n}_N(\underline{M} \cdot \vec{z}_0)| &= \left| \underline{R}(\vec{z}_0) \cdot \frac{\vec{s}_N(\vec{z}_0)}{|\vec{s}_N(\vec{z}_0)|} - \frac{\vec{s}_N(\underline{M} \cdot \vec{z}_0)}{|\vec{s}_N(\underline{M} \cdot \vec{z}_0)|} \right| \\ &= \frac{1}{|\vec{s}_N(\vec{z}_0)|} \left| \underline{R}(\vec{z}_0) \cdot \vec{s}_N(\vec{z}_0) - \vec{s}_N(\underline{M} \cdot \vec{z}_0) + \frac{|\vec{s}_N(\underline{M} \cdot \vec{z}_0)| - |\vec{s}_N(\vec{z}_0)|}{|\vec{s}_N(\underline{M} \cdot \vec{z}_0)|} \vec{s}_N(\underline{M} \cdot \vec{z}_0) \right| \\ &\leq \frac{1}{|\vec{s}_N(\vec{z}_0)|} (|\underline{R}(\vec{z}_0) \cdot \vec{s}_N(\vec{z}_0) - \vec{s}_N(\underline{M} \cdot \vec{z}_0)| + ||\underline{R}(\vec{z}_0) \vec{s}_N(\vec{z}_0)| - |\vec{s}_N(\underline{M} \cdot \vec{z}_0)||) \\ &\leq \frac{2}{|\vec{s}_N(\vec{z}_0)|} |\underline{R}(\vec{z}_0) \cdot \vec{s}_N(\vec{z}_0) - \vec{s}_N(\underline{M} \cdot \vec{z}_0)| \\ &\leq \frac{4 \sin(\xi/2)}{(N+1) \cos(\xi)}. \end{aligned} \quad (30)$$

The error by which the vector $\vec{n}_N(\vec{z}_0)$ violates the periodicity condition (23) of the \vec{n} axis is therefore smaller than $2 \sec(\xi/2)\tan(\xi)/(N+1)$, and converges to 0 for large N .

If one can prove the existence of a suitable number $\xi < \pi/2$ for some spin transport system, one has proven the existence of functions \vec{n}_N for this system which satisfy the defining equation for the \vec{n} axis to arbitrary precision. Since, however, these functions \vec{n}_N do not necessarily converge, this does not prove the existence of an \vec{n} axis for such a system.

If the orbit motion can be described in terms of action-angle variables, as is always the case for linear motion, and the orbital angle advances for one turn (2π times the orbit tunes) are not in resonance, then two important conclusions about this tracking algorithm with $\xi < \pi/2$ can be drawn.

(1) If an \vec{n} axis $\vec{n}(\vec{z})$ exists, then the sequence \vec{n}_N converges to $\vec{n}(\vec{z})$ linearly in $1/N$.

(2) If an \vec{n} axis exists and the spin rotation angle in one turn is not a linear combination of orbit phase advances modulo 2π , then the \vec{n} axis is unique up to a sign.

The proof is adapted from [5,16,17]. The first step will be to show how to define a spin rotation angle which depends only on orbital action variables. We assume that an \vec{n} axis exists and introduce two unit vectors $\vec{u}_1(\vec{z})$ and $\vec{u}_2(\vec{z})$ to create a right handed coordinate system $(\vec{n}, \vec{u}_1, \vec{u}_2)$. The vectors \vec{u}_1 and \vec{u}_2 are therefore defined up to a rotation around the \vec{n} axis by an arbitrary phase space dependent angle $\phi(\vec{z})$. We express the spin vectors \vec{s} in terms of this coordinate system by $\vec{s} = s_1 \vec{n} + s_2 \vec{u}_1 + s_3 \vec{u}_2$. The coefficient s_1 does

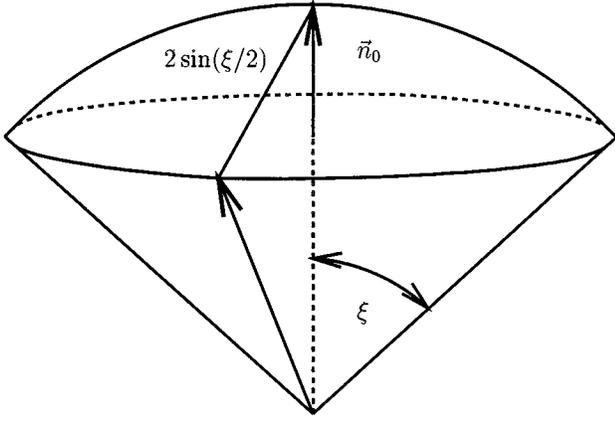


FIG. 1. Estimation of the convergence speed. To guarantee convergence of the stroboscopic average, the angle ξ between the direction of \vec{n}_0 and the transported spins has to stay below $\pi/2$.

not change during the particle motion around the ring since the particle transfer matrix $\underline{R}(\vec{z})$ is orthogonal, and ensures that $(\vec{s} \cdot \vec{n})$ is invariant. The spin motion described by the $\underline{R}(\vec{z})$ matrix is therefore simply a rotation around the \vec{n} -axis by a phase space dependent angle $\nu(\vec{z})$:

$$\begin{pmatrix} s_{f1} \\ s_{f2} \\ s_{f3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\nu(\vec{z})) & \sin(\nu(\vec{z})) \\ 0 & -\sin(\nu(\vec{z})) & \cos(\nu(\vec{z})) \end{pmatrix} \begin{pmatrix} s_{i1} \\ s_{i2} \\ s_{i3} \end{pmatrix}. \quad (31)$$

If we now introduce the complex quantity $\hat{s} = e^{-i\phi(\vec{z})}(s_2 + is_3)$, where $\phi(\vec{z})$ is the arbitrary angle, then the spin transport is described by

$$s_{f2} + is_{f3} = e^{-i\nu(\vec{z})}(s_{i2} + is_{i3}), \quad (32)$$

$$e^{i\phi(\vec{M}(\vec{z}))}\hat{s}_f = e^{i(-\nu(\vec{z}) + \phi(\vec{z}))}\hat{s}_i. \quad (33)$$

Now we introduce orbital action-angle variables \vec{J} and $\vec{\Phi}$ as well as the angle advances \vec{Q} for one turn around the accelerator. Note that the symbol \vec{Q} is 2π times the orbital tunes. In these variables the one turn transport is characterized by $\vec{J}_f = \vec{J}_i$ and $\vec{\Phi}_f = \vec{\Phi}_i + \vec{Q}$. Since the actions remain invariant during the particle motion, we use the symbols $\nu_j(\vec{\Phi})$ and $\phi_j(\vec{\Phi})$ to indicate the spin rotation angle and the free phase of the coordinate system for fixed actions \vec{J} ,

$$\hat{s}_f = e^{i(-\nu_j(\vec{\Phi}) + \phi_j(\vec{\Phi}) - \phi_j(\vec{\Phi} + \vec{Q}))}\hat{s}_i. \quad (34)$$

We now show how $\phi_j(\vec{\Phi})$ can be chosen so that the spin motion characterized by the exponent becomes simplified. As with any function of phase space, the rotation $e^{i\phi_j(\vec{\Phi})}$ is 2π periodic in all components Φ_j . Therefore, the rotation angle $\phi_j(\vec{\Phi})$ has a periodic contribution $\phi_{\circ j}(\vec{\Phi})$ and a linear contribution in the phases

$$\phi_j(\vec{\Phi}) = \phi_{\circ j}(\vec{\Phi}) + \vec{j} \cdot \vec{\Phi}, \quad (35)$$

with some vector \vec{j} that has integer components. The phase space dependent spin rotation $e^{i\nu_j(\vec{\Phi})}$ is also a periodic function of phase space. But since on the closed orbit ($\vec{J}=0$) the spin motion does not depend on $\vec{\Phi}$, therefore $\nu_j(\vec{\Phi})$ only has a periodic component, and no component linear in $\vec{\Phi}$.

If the orbital angle advances \vec{Q} are not in resonance, then $\phi_j(\vec{\Phi})$ can be chosen to eliminate the phase dependence of the exponent in Eq. (34) completely. This can be seen by Fourier transformation of the periodic functions $\nu_j(\vec{\Phi})$ and $\phi_{\circ j}(\vec{\Phi})$, leading to the requirement

$$\check{\nu}_j(\vec{k}) = \check{\phi}_{\circ j}(\vec{k})(1 - e^{i\vec{k} \cdot \vec{Q}}) \quad (36)$$

for the Fourier coefficients $\check{\nu}_j(\vec{k})$ and $\check{\phi}_{\circ j}(\vec{k})$. Therefore, $\check{\phi}_{\circ j}(\vec{k})$ can be chosen to eliminate all Fourier coefficients except for $\vec{k}=0$. With this choice the exponent reduces to $\check{\nu}_j(0) + \vec{j} \cdot \vec{Q}$. The vector \vec{j} can in general not be used to simplify this expression and we therefore usually choose $\vec{j}=0$. With $\nu(\vec{J}) = \check{\nu}_j(0)$, the spin rotation of Eq. (34) then simplifies to

$$\hat{s}_f = e^{-i\nu(\vec{J})}\hat{s}_i. \quad (37)$$

We have now achieved the goal of constructing a spin rotation depending only on orbital actions. It is interesting to note that if for amplitudes \vec{J} the integer coefficients of \vec{j} can be chosen so that

$$\nu(\vec{J}) + \vec{j} \cdot \vec{Q} = 0 \pmod{2\pi}, \quad (38)$$

one can eliminate the spin rotation completely. Here we only analyze the case where this resonance condition is not satisfied.

The stroboscopic average is now performed by the recipe of Sec. IV A, but this time in the coordinate system $(\vec{n}, \vec{u}_1, \vec{u}_2)$ just constructed by the described choice of $\phi_j(\vec{\Phi})$. We first establish the tracking points \vec{c}_j and note that $\nu = \nu(\vec{c}_j)$ is the same for all tracking points. In the coordinate system $(\vec{n}, \vec{u}_1, \vec{u}_2)$, the vector components of the periodic spin \vec{n}_0 on the closed orbit are not constant, but depend on the phase space position $(\vec{J}, \vec{\Phi})$. This vector is transported from the phase space points \vec{c}_j to $\vec{z}_0 = (\vec{\Phi}, \vec{J})$ by the rotation matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(j\nu) & \sin(j\nu) \\ 0 & -\sin(j\nu) & \cos(j\nu) \end{pmatrix}, \quad (39)$$

leading to the stroboscopic average in the coordinate system

$$\vec{s}_N(\vec{\Phi}, \vec{J}) = \frac{1}{N+1} \sum_{j=0}^N \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(j\nu) & \sin(j\nu) \\ 0 & -\sin(j\nu) & \cos(j\nu) \end{pmatrix} \times \begin{pmatrix} n_{0,1\vec{j}}(\vec{\Phi} - j\vec{Q}) \\ n_{0,2\vec{j}}(\vec{\Phi} - j\vec{Q}) \\ n_{0,3\vec{j}}(\vec{\Phi} - j\vec{Q}) \end{pmatrix}. \quad (40)$$

The first component of \vec{s}_N is $\sum_{j=0}^N n_{0,1\vec{j}}(\vec{\Phi} - j\vec{Q})/(N+1)$; the second and third components in complex notation are

$$\hat{s}_N = s_{N2} + i s_{N3} = \frac{1}{N+1} \sum_{j=0}^N e^{-i(j\nu)} \hat{n}_{0,\vec{j}}(\vec{\Phi} - j\vec{Q}), \quad (41)$$

where $\hat{n}_{0,\vec{j}}(\vec{\Phi}) = n_{0,2\vec{j}}(\vec{\Phi}) + i n_{0,3\vec{j}}(\vec{\Phi})$. In terms of the Fourier components $\check{n}_{0,\vec{j}}(\vec{k})$ of $\hat{n}_{0,\vec{j}}(\vec{\Phi})$, one obtains the inequality

$$\begin{aligned} |\hat{s}_N| &= \left| \frac{1}{N+1} \sum_{j=0}^N e^{-ij(\nu + \vec{k} \cdot \vec{Q})} \sum_{\vec{k}} \check{n}_{0,\vec{j}}(\vec{k}) e^{i\vec{k} \cdot \vec{\Phi}} \right| \\ &= \frac{1}{N+1} \left| \sum_{\vec{k}} \frac{1 - e^{-i(N+1)(\nu + \vec{k} \cdot \vec{Q})}}{1 - e^{-i(\nu + \vec{k} \cdot \vec{Q})}} \check{n}_{0,\vec{j}}(\vec{k}) e^{i\vec{k} \cdot \vec{\Phi}} \right| \\ &\leq \frac{1}{N+1} \sum_{\vec{k}} \frac{2}{|1 - e^{-i(\nu + \vec{k} \cdot \vec{Q})}|} |\check{n}_{0,\vec{j}}(\vec{k})|. \end{aligned} \quad (42)$$

Similarly, for the first component of \vec{s}_N one obtains

$$s_{N,1} = \check{n}_{0,1\vec{j}}(0) + \frac{1}{N+1} \sum_{k \neq 0} \frac{1 - e^{-i(N+1)\vec{k} \cdot \vec{Q}}}{1 - e^{-i\vec{k} \cdot \vec{Q}}} \check{n}_{0,1\vec{j}}(\vec{k}) e^{i\vec{k} \cdot \vec{\Phi}}. \quad (43)$$

Since $|\vec{s}_N| > \cos(\xi)$ and $\xi < \pi/2$, it is guaranteed that \vec{s}_N does not converge to 0. For large N , $|\vec{s}_N|$ is given by $\check{n}_{0,1\vec{j}}$, which therefore does not vanish. Since the resonance condition (38) is not satisfied for any vector of integers \vec{j} , none of the denominators is zero, and $s_{N,1}$ and \hat{s}_N converge linearly with $1/N$. At this point we impose the condition on \vec{n} that the sum on the right hand side of Eq. (42) converges. From the convergence of \vec{s}_N , the convergence of \vec{n}_N to \vec{n} follows from

$$\begin{aligned} |\vec{n} - \vec{n}_N|^2 &= \left| \vec{n} - \frac{\vec{s}_N}{|\vec{s}_N|} \right|^2 = \left(1 - \frac{s_{N,1}}{|\vec{s}_N|} \right)^2 + \left(\frac{|\hat{s}_N|}{|\vec{s}_N|} \right)^2 \\ &= 2 \frac{|\vec{s}_N| - s_{N,1}}{|\vec{s}_N|} < 2 \frac{|\vec{s}_N| - s_{N,1}}{|s_{N,1}|} < 2 \left(\frac{|\hat{s}_N|}{|s_{N,1}|} \right)^2. \end{aligned} \quad (44)$$

There is a number N^* such that the absolute value of the N dependent part of $s_{N,1}$ in Eq. (43) is smaller than $\check{n}_{0,1\vec{j}}(0)/2$. If the number of turns N is larger than N^* , we conclude that

$$|\vec{n}_N - \vec{n}| < 2\sqrt{2} \frac{|\hat{s}_N|}{|n_{0,1\vec{j}}(0)|}, \quad (45)$$

which, together with Eq. (42), establishes that \vec{n}_N converges linearly with $1/N$.

The second conclusion to be proven is the uniqueness of the \vec{n} axis. In the coordinate system $(\vec{n}, \vec{u}_1, \vec{u}_2)$, the periodicity condition (23) reads

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\nu) & \sin(\nu) \\ 0 & -\sin(\nu) & \cos(\nu) \end{pmatrix} \cdot \vec{n}(\vec{z}) = \vec{n}(\vec{M}(\vec{z})), \quad (46)$$

with the obvious solution $\vec{n}(\vec{z}) = (1, 0, 0)^T$ for all \vec{z} . If another \vec{n} axis $\vec{n}_2(\vec{z})$ exists, then $\vec{n}_2 - \vec{n}(\vec{n} \cdot \vec{n}_2)$ is nonzero at least at one phase space point and on all iterates of this point which can be reached during particle motion. We normalize this difference vector at these phase space points and write it as $\cos(\alpha(\vec{z}))\vec{u}_1 + \sin(\alpha(\vec{z}))\vec{u}_2$. In orbital action-angle variables, the function $\alpha_{\vec{j}}(\vec{\Phi})$ again has a periodic contribution and a linear contribution $\vec{j} \cdot \vec{\Phi}$.

In complex notation, the periodicity condition (46) reads

$$e^{i(-\nu(\vec{J}) + \alpha_{\vec{j}}(\vec{\Phi}))} = e^{i\alpha_{\vec{j}}(\vec{\Phi} + \vec{Q})}. \quad (47)$$

Since $\nu(\vec{J})$ does not depend on $\vec{\Phi}$, the periodic part of α has to vanish. The resulting $\alpha_{\vec{j}} = \vec{j} \cdot \vec{\Phi}$ can only solve the periodicity condition if there is a vector \vec{j} with integer coefficients fulfilling the resonance equation (38), and then a second \vec{n} axis exists. For linear orbit motion with phase advances which are not on resonances of type (38), Eq. (47) does not have a solution, and the \vec{n} axis is unique. There is so far no proof for nonlinear orbits, but nevertheless the method of stroboscopic averaging can also be used for exploring nonlinear orbit motion.

C. Improved recipe with faster convergence

In Sec. IV A an algorithm was introduced which converged with the square root of the number T of turns tracked. However, it is possible to obtain convergence linear in T , when one takes advantage of the orthogonality of the spin transfer matrix. To illustrate this, we again establish a recipe.

(1) Define, as before,

$$C = \{\vec{c}_j = (\underline{M}^{-1})^j \cdot \vec{z}_0 | j \in \{0, \dots, N\}\}. \quad (48)$$

(2) Define three orthogonal unit vectors $\{\vec{e}^{(1)}, \vec{e}^{(2)}, \vec{e}^{(3)}\}$.

(3) Obtain the sets S_j of the three vectors $\vec{s}_j^{(1)}, \vec{s}_j^{(2)}$, and $\vec{s}_j^{(3)}$ by tracking the $\vec{e}^{(k)}$ for $N-j$ turns,

$$\vec{s}_j^{(k)} = \underline{R}(\vec{c}_{j+1}) \cdot \dots \cdot \underline{R}(\vec{c}_N) \cdot \vec{e}^{(k)}. \quad (49)$$

(4) From the set of vectors S_j and the set S_0 , one can obtain the spin transfer matrix from the phase space point \vec{c}_j to \vec{z}_0 denoted by $\underline{R}(\vec{c}_j)$. This becomes clear when one realizes that $\vec{s}_0^{(k)} = \underline{R}(\vec{c}_j) \cdot \vec{s}_j^{(k)}$ for all k . Obtaining these $N+1$ transport matrices requires the propagation of three spins around the circular accelerator for N turns.

(5) Now we can compute the set $B = \{\underline{R}(\vec{c}_j) \cdot \vec{n}_0 | j \in \{0, \dots, N\}\}$. This is obviously identical to the set denoted by B in Sec. IV B.

(6) The normalized average of B , denoted by \vec{n}_N , can now again be computed as mentioned above, and solves the defining equation for an \vec{n} axis up to the small error discussed previously.

In this approach one only has to track three initial spin directions over N turns, leading to $T = 3N$. The error is therefore bounded by $6 \sec(\xi/2) \tan(\xi)/T$. This implies convergence linear in the number T of turns tracked. The following example illustrates the speed of this method: when the angle ξ happens to be 45° , and we require an accuracy at the 10^{-3} level, this linear convergence approach only requires 6500 tracking turns; when the angle ξ is small, fewer iterations are needed.

D. Backward tracking

In the two recipes of Secs. IV A and IV B, we needed to find the set C of $N+1$ backwards tracked phase space points, and then launch spins at these points and track forward so as to compute set B . In the case of linear motion, it is trivial to obtain these backward tracked phase space points. One simply transforms \vec{z}_0 into the action-angle variables of the linear motion, and determines the phase advance per turn of the linear motion. Counting back these phase advances and transforming the action-angle variables back into phase space leads to the points \vec{c}_j . In the nonlinear case this would actually require us to track for N more turns around the ring.

In the case of the linearly convergent method of Sec. IV C, this extra effort becomes unnecessary. One can start with the phase space point \vec{z}_0 and launch three particles with spins along $\vec{e}^{(1)}$, $\vec{e}^{(2)}$, and $\vec{e}^{(3)}$. Tracking backward in the azimuth defines the $N+1$ sets P_j of the spins $\vec{p}_j^{(1)}$, $\vec{p}_j^{(2)}$, and $\vec{p}_j^{(3)}$ with

$$\vec{p}_j^{(k)} = \underline{R}^{-1}(\vec{c}_j) \cdot \dots \cdot \underline{R}^{-1}(\vec{c}_1) \cdot \vec{e}^{(k)}. \quad (50)$$

From the sets P_j and P_0 one can again compute the spin transfer matrix $\underline{R}(\vec{c}_j)$, and with these matrices one obtains the set B with the elements $\vec{b}_j = \underline{R}(\vec{c}_j) \cdot \vec{n}_0$, which again leads to \vec{n}_N by averaging.

E. Forward tracking

There is an even more fundamental problem in the case of nonlinear motion than the computation time. When the lattice or the effect of separate nonlinear elements are computed by nonlinear transfer maps, the inverses of these maps might not be known. In this case the required phase space points \vec{c}_j cannot be computed at all. Nevertheless our method can be used to obtain the vector \vec{n}_N as follows. The arguments of Sec. IV D can simply be repeated for tracking forward. One establishes the phase space points $\vec{c}_j = \vec{M}(\vec{c}_{j-1})$ with $\vec{c}_0 = \vec{z}_0$ for $j \in \{1, \dots, N\}$, and simultaneously the sets S_j by tracking the three unit vectors $\vec{s}_{j-1}^{(k)}$ for one turn with

$\vec{s}_0^{(k)} = \vec{e}^{(k)}$. As in the fourth step of the recipe in Sec. IV C one can then obtain the spin transfer matrix $\underline{R}(\vec{c}_j)$ from the phase space point \vec{z}_0 to \vec{c}_j . The inverse of this transfer matrix is simply obtained by transposition leading to the vectors $\vec{b}_j = \underline{R}(\vec{c}_j)^T \cdot \vec{n}_0$. The normalized average of the vectors \vec{b}_j is then the stroboscopic average $\vec{n}_{\text{inv}, N}$ of the inverse motion. For this average the error of the periodicity condition for the inverse motion converges linearly to zero. Fortunately, an \vec{n} axis of the inverse motion $\vec{n}_{\text{inv}}(\vec{z})$ is also an \vec{n} axis of the forward motion, since the periodicity condition of the inverse motion reads as

$$\underline{R}^{-1}(\vec{M}^{-1}(\vec{z})) \cdot \vec{n}_{\text{inv}}(\vec{z}) = \vec{n}_{\text{inv}}(\vec{M}^{-1}(\vec{z})), \quad (51)$$

and from this it follows that $\vec{n}_{\text{inv}}(\vec{z})$ also obeys the periodicity condition (23) of the forward motion.

F. Tracking rotation angles and rotation vectors

One can represent a rotation matrix by its rotation vector $\vec{\beta}$ and its rotation angle $\phi \in [0, \pi]$. The simplest representation is in terms of the vector $\vec{\gamma} = \sin(\phi/2) \vec{\beta}$ and $\kappa = \cos(\phi/2)$. The rotation matrix is then given by

$$R_{ij} = 2(\kappa^2 \delta_{ij} + \gamma_i \gamma_j - \kappa \epsilon_{ijk} \gamma_k) - \delta_{ij}. \quad (52)$$

This representation can be easily transported through an accelerator by means of the equations

$$\vec{\kappa} = \kappa_1 \kappa_2 - \vec{\gamma}_1 \cdot \vec{\gamma}_2, \quad \kappa = |\vec{\kappa}|, \quad (53)$$

$$\vec{\gamma} = (\vec{\gamma}_1 \kappa_2 + \vec{\gamma}_2 \kappa_1 + \vec{\gamma}_2 \times \vec{\gamma}_1) \frac{\vec{\kappa}}{\kappa}. \quad (54)$$

It is therefore even more appropriate and faster by about a factor of 3 to track rotation matrices represented by $\vec{\gamma}$ and κ directly, than to track three spin vectors.

V. APPROXIMATION OF LINEAR SPIN-ORBIT MOTION

In this section we consider the special case of linear spin-orbit motion as a way to illustrate and confirm the chief features of our algorithm [18]. For this purpose we define two periodic vectors \vec{m} and \vec{l} orthogonal to \vec{n}_0 in order to obtain an orthonormal right-handed dreibein at the previously specified azimuth θ_0 . We then write a spin vector \vec{s} in the form

$$\vec{s} = \sqrt{1 - \alpha^2 - \beta^2} \vec{n}_0 + \alpha \vec{m} + \beta \vec{l}. \quad (55)$$

In the case where both α and β are much smaller than unity, we may treat the spin in a first order approximation in α and β so that the spin vector (55) can be written as

$$\vec{s} = \vec{n}_0 + \alpha \vec{m} + \beta \vec{l}, \quad (56)$$

and is normalized in linear approximation. We combine the two-component vector

$$\vec{s} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (57)$$

and the orbit vector \vec{z} into a single vector \vec{z} of $d+2$ components. By linearizing the equations of motion with respect to \vec{z} , an initial spin-orbit coordinate \vec{z}_i at azimuth θ_0 is mapped into a final coordinate \vec{z}_f after one turn around the ring [7] by the $(d+2) \times (d+2)$ spin-orbit transfer matrix $\underline{\tilde{M}}$,

$$\vec{z}_f = \underline{\tilde{M}} \cdot \vec{z}_i, \underline{\tilde{M}} = \begin{pmatrix} \underline{M} & \underline{0} \\ \underline{G} & \underline{D} \end{pmatrix}, \quad (58)$$

where \underline{M} again denotes the $d \times d$ dimensional orbit transfer matrix. The matrices \underline{G} and \underline{D} are $2 \times d$ and 2×2 matrices, respectively, with

$$\vec{s}_f = \underline{G} \cdot \vec{z}_i + \underline{D} \cdot \vec{s}_i. \quad (59)$$

In order to simplify the following formulas, we introduce the $2 \times d$ -dimensional component \underline{G}_m of the transfer matrix $\underline{\tilde{M}}^m$ for m turns around the ring. We note that the two-component vector corresponding to \vec{n}_0 is $\vec{n}_0 = 0$ so that propagating this initial spin vector at a phase space point \vec{z}_i once around the ring leads to the final spin $\vec{s}_f = \underline{G}_m \cdot \vec{z}_i$. Therefore, we define the two-component vector \vec{s}_N analogous to Eq. (26) by

$$\vec{s}_N(\vec{z}_0) = \sum_{m=0}^N \underline{G}_m \cdot \underline{M}^{-m} \cdot \vec{z}_0, \quad \underline{\tilde{M}}^m = \begin{pmatrix} \underline{M}^m & \underline{0} \\ \underline{G}_m & \underline{D}^m \end{pmatrix}. \quad (60)$$

$$\sum_{m=0}^N \underline{G}_m \cdot \underline{M}^{-m} \cdot \vec{y}_j = \lambda_j [(N+1) \underline{1} - (\underline{1} - \lambda_j^{N+1} \underline{D}^{N+1}) \cdot (\underline{1} - \lambda_j \underline{D})^{-1}] \cdot (\underline{1} - \lambda_j \underline{D})^{-1} \cdot \underline{G} \cdot \vec{y}_j, \quad (66)$$

so that

$$\lim_{N \rightarrow \infty} \left[\frac{1}{N+1} \sum_{m=0}^N \underline{G}_m \cdot \underline{M}^{-m} \cdot \vec{y}_j \right] = \lambda_j (\underline{1} - \lambda_j \underline{D})^{-1} \cdot \underline{G} \cdot \vec{y}_j. \quad (67)$$

Combining this with Eqs. (60) and (62), we obtain

$$\vec{n}(\vec{z}_0) = \sum_{j=1}^d a_j \lambda_j (\underline{1} - \lambda_j \underline{D})^{-1} \cdot \underline{G} \cdot \vec{y}_j, \quad (68)$$

where \vec{n} is the two-component vector corresponding to \vec{n} . For $d=6$ this yields the well known expression of the \vec{n} axis for the case of linear spin-orbit motion [9,15]. This confirms the method of stroboscopic averaging for the linear case. Also, our predictions about convergence speed are confirmed as follows. In storage rings the particle motion can only be

From Eq. (58), it follows that

$$\underline{G}_m \cdot \underline{M}^{-m} = \sum_{k=0}^{m-1} \underline{D}^k \cdot \underline{G} \cdot \underline{M}^{-(k+1)}. \quad (61)$$

We expand the phase space coordinate \vec{z}_0 in eigenvectors \vec{y}_j of the one turn transfer matrix \underline{M} ,

$$\vec{z}_0 = \sum_{j=1}^d a_j \vec{y}_j, \quad (62)$$

with expansion constants a_j and eigenvalues λ_j^{-1} ,

$$\underline{M} \cdot \vec{y}_j = \lambda_j^{-1} \vec{y}_j. \quad (63)$$

Thus we obtain

$$\underline{G}_m \cdot \underline{M}^{-m} \cdot \vec{y}_j = \sum_{k=0}^{m-1} \lambda_j^{k+1} \underline{D}^k \cdot \underline{G} \cdot \vec{y}_j. \quad (64)$$

We do not consider the case of spin-orbit resonances, so that the matrices $\underline{1} - \lambda_j \underline{D}$ are nonsingular. Thus Eq. (64) simplifies to

$$\underline{G}_m \cdot \underline{M}^{-m} \cdot \vec{y}_j = \lambda_j (\underline{1} - \lambda_j^m \underline{D}^m) \cdot (\underline{1} - \lambda_j \underline{D})^{-1} \cdot \underline{G} \cdot \vec{y}_j. \quad (65)$$

From this follows

stable if the eigenvalues have $|\lambda_j| \leq 1$. To analyze the convergence speed, one has to realize that \underline{D} is a rotation matrix, and therefore that

$$|(\underline{1} - \lambda_j^{N+1} \underline{D}^{N+1}) \vec{s}| \leq (1 + |\lambda_j|^{N+1}) |\vec{s}| \leq 2 |\vec{s}| \quad \text{for all } \vec{s}, \quad (69)$$

$$|\vec{n}_N(\vec{z}_0) - \vec{n}(\vec{z}_0)| \leq \frac{2}{N+1} \sum_{j=1}^d |a_j| |(\underline{1} - \lambda_j \underline{D})^{-2} \cdot \underline{G} \cdot \vec{y}_j|. \quad (70)$$

This inequality also shows that for linear spin-orbit motion one finds convergence linear in $1/N$, and it can also be seen that the convergence speed decreases with the orbital amplitudes a_j and becomes very slow close to first order spin-orbit resonances.

VI. NUMERICAL EXAMPLES

In order to illustrate which quantities can be computed and how effective stroboscopic averaging can be in numeri-

cal computations, we apply it to a model accelerator with a known \vec{n} axis. Since this model is somewhat artificial, we also apply our method to the vertical motion of the HERA proton ring. For simplicity, the vertical bends at the interaction regions have been ignored.

A. Comparison with an analytically solvable model

In this section we consider a special model [19] with $d=2$ and $\vec{z}=(\Phi, J)$. The equations of motion are given by

$$\begin{aligned} \frac{d\Phi}{d\theta}(\theta) &= \frac{Q}{2\pi}, \quad \frac{dJ}{d\theta}(\theta) = 0, \\ \frac{d\vec{s}}{d\theta}(\theta) &= \vec{\Omega}(\Phi(\theta), J) \times \vec{s}(\theta), \end{aligned} \quad (71)$$

with

$$\vec{\Omega}(\Phi, J) = \frac{1}{2\pi} (\nu_0 \vec{e}_1 + \mu \sqrt{J} (\vec{e}_3 \sin\Phi + \vec{e}_2 \cos\Phi)), \quad (72)$$

where μ is a real parameter. Hence initial coordinates \vec{z}_i are taken into final coordinates \vec{z}_f by $\vec{z}_f = \vec{M}(\vec{z}_i)$ with $\Phi_f = \Phi_i + Q$ and $J_f = J_i$. In the following we assume $\nu_0 \neq Q$ and that $\nu_0/2\pi$ is not integer. This model corresponds to the rotating field approximation often used to discuss spin resonance in solid state physics [20]. We now introduce the orthogonal matrix $\underline{T}(\vec{e}, \varphi)$ describing a rotation around a unit vector \vec{e} by an angle φ . Transforming the spin components of \vec{s} into a rotating frame by introducing $\vec{s}_{\text{rot}} = \underline{T}(\vec{e}_1, \Phi) \cdot \vec{s}$, one obtains the simplified equation of the spin motion,

$$\frac{d\vec{s}_{\text{rot}}}{d\theta}(\theta) = \vec{\Omega}_{\text{rot}}(J) \times \vec{s}_{\text{rot}}(\theta), \quad \vec{\Omega}_{\text{rot}}(J) = \frac{1}{2\pi} \begin{pmatrix} \nu_0 - Q \\ \mu \sqrt{J} \\ 0 \end{pmatrix}. \quad (73)$$

If in this frame a spin field is oriented parallel to $\vec{\Omega}_{\text{rot}}$, this field does not change from turn to turn. Therefore $\vec{n}_{\text{rot}} = \vec{\Omega}_{\text{rot}}/|\vec{\Omega}_{\text{rot}}|$ is an \vec{n} axis. In the original frame this \vec{n} axis is

$$\vec{n}(\Phi, J) = \frac{\nu_0 - Q}{|\nu_0 - Q|} \frac{1}{\Lambda} ((\nu_0 - Q) \vec{e}_1 + \mu \sqrt{J} (\vec{e}_3 \sin\Phi + \vec{e}_2 \cos\Phi)), \quad (74)$$

$$\Lambda = \sqrt{(\nu_0 - Q)^2 + \mu^2 J}, \quad (75)$$

where the sign factor $(\nu_0 - Q)/|\nu_0 - Q|$ has been chosen so that on the closed orbit ($J=0$) the \vec{n} axis $\vec{n}(\Phi, 0)$ coincides with $\vec{n}_0 = \vec{e}_1$. As with any function of phase space this \vec{n} axis is a periodic function in Φ .

We now perform the stroboscopic average by the recipe of Sec. IV A to compute an \vec{n} axis at $\vec{z}_0 = (\Phi, J)$ and θ_0 ,

$$\vec{n}_N = \frac{1}{N+1} \sum_{j=0}^N \prod_{k=1}^j \underline{R}(\Phi - kQ, J) \cdot \vec{n}_0 \quad (76)$$

$$= \frac{1}{N+1} \sum_{j=0}^N \prod_{k=1}^j \underline{T}(\vec{e}_1, -\Phi) \cdot \underline{T}(\vec{n}_{\text{rot}}, \Lambda) \cdot \underline{T}(\vec{e}_1, \Phi) \cdot \vec{n}_0 \quad (77)$$

$$= \frac{1}{N+1} \underline{T}(\vec{e}_1, -\Phi) \cdot \sum_{j=0}^N \underline{T}(\vec{n}_{\text{rot}}, j\Lambda) \cdot \vec{e}_1. \quad (78)$$

If $\Lambda/2\pi$ is not an integer, after some tedious manipulations one obtains

$$|\vec{n}_N(\Phi, J) - \vec{n}(\Phi, J)| = \sqrt{2} \sqrt{1 - \tau_N}, \quad (79)$$

$$\tau_N = \left(1 + \frac{\mu^2 J}{(N+1)^2 (\nu_0 - Q)^2} \frac{1 - \cos((N+1)\Lambda)}{1 - \cos(\Lambda)} \right)^{-1/2}. \quad (80)$$

One sees that $|\vec{n}_N - \vec{n}|$ is an oscillating function of J whose local maxima increase with J , reflecting the fact that large orbital amplitudes reduce the convergence speed. This behavior is plotted in Fig. 2. In this and the other figures concerned with the solvable model we used the parameters $\nu_0 = 0.6\pi$, $Q = 0.46\pi$, $\mu = 0.2\pi$, and $\Phi = 0.32$. For large N , Eqs. (79) and (80) predict that the convergence is indeed linear in $1/N$, as illustrated by the slope of -1 in Fig. 3. Also one sees that $|\vec{n}_N - \vec{n}|$ vanishes when $\Lambda/2\pi$ is not an integer, and if $(N+1)\Lambda/2\pi$ is an integer. When $\Lambda/2\pi$ is an integer we have

$$\tau_N = \left(1 + \frac{\mu^2 J}{(\nu_0 - Q)^2} \right)^{-1/2}. \quad (81)$$

Therefore, in this case \vec{n}_N does not converge to \vec{n} . This is no surprise, since the condition where $\Lambda/2\pi$ is an integer amounts to the resonance condition (38) which leads to this nonuniqueness of the \vec{n} axis. It is interesting that the local maxima of $|\vec{n}_N - \vec{n}|$ do not occur at these resonance points.

Figure 4 shows the variation of the opening angle of the polarization field as a function of orbital amplitude. Since $\vec{n}_0 \cdot \vec{n} = |\nu_0 - Q|/\Lambda$, as well as $\vec{n}_0 \cdot \vec{n}_N = (1)/(N+1) \sum_{j=0}^N T_{11}(\vec{n}_{\text{rot}}, j\Lambda)$, do not depend on the angle variable Φ , the depicted angles are equivalent to the phase averaged opening angle. The good agreement with the analytically computed opening angle of the \vec{n} axis shows that accurate values for the field can be obtained with a very limited number of turns.

This analytically solvable model can also be used to illustrate the construction of a phase independent spin rotation angle $\nu(J)$. Having obtained an \vec{n} axis, one can transform the spin components of \vec{s} into a rotated coordinate system $(\vec{n}, \vec{u}_1, \vec{u}_2)$. With the simple choice

$$\vec{u}_1(\Phi, J) = \frac{1}{|\vec{n} \times \vec{e}_1|} (\vec{n} \times \vec{e}_1) = \frac{\nu_0 - Q}{|\nu_0 - Q|} (\vec{e}_2 \sin\Phi - \vec{e}_3 \cos\Phi), \quad (82)$$

$$\vec{u}_2(\Phi, J) = \vec{n} \times \vec{u}_1, \quad (83)$$

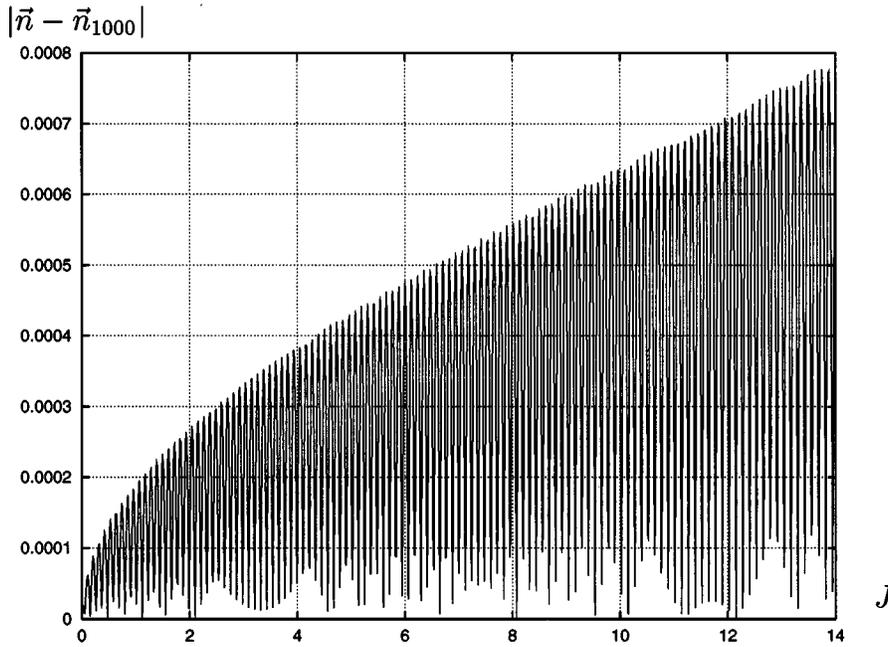


FIG. 2. Referring to the analytically solvable model: The deviation of the stroboscopic average \vec{n}_N from the analytically calculated \vec{n} as a function of the amplitude J in phase space.

both of which are clearly periodic in Φ , one can show that the spin rotation angle is independent of Φ ; that is,

$$\nu(J) = \frac{\nu_0 - Q}{|\nu_0 - Q|} \Lambda. \quad (84)$$

However, $\nu(J)$ is free up to multiples of the orbit tune as outlined in Sec. IV B. We can use this freedom to obtain a $\nu(J)$ which reduces to ν_0 on the closed orbit ($J=0$). The only choice which satisfies this condition is obtained by an additional rotation of \vec{u}_1 and \vec{u}_2 around \vec{n} by $-\Phi$,

$$\begin{aligned} \vec{u}_1(\Phi, J) = & \vec{e}_1 \frac{\mu\sqrt{J}}{\Lambda} \sin\Phi + \vec{e}_2 \cos\Phi \sin\Phi \left(\frac{\nu_0 - Q}{|\nu_0 - Q|} - \frac{\nu_0 - Q}{\Lambda} \right) \\ & - \vec{e}_3 \left(\frac{\nu_0 - Q}{|\nu_0 - Q|} \cos^2\Phi + \frac{\nu_0 - Q}{\Lambda} \sin^2\Phi \right), \end{aligned} \quad (85)$$

$$\vec{u}_2(\Phi, J) = \vec{n} \times \vec{u}_1. \quad (86)$$

Both are again periodic in Φ , and again we obtain a spin rotation angle independent of Φ , viz.

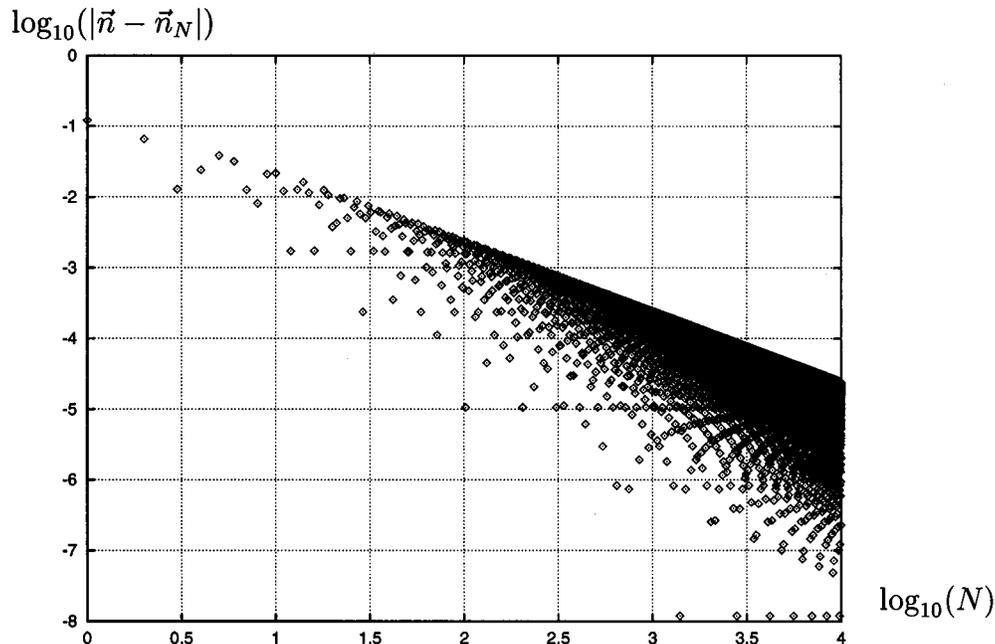


FIG. 3. Referring to the analytically solvable model: Convergence of the stroboscopic average \vec{n}_N to the analytically calculated \vec{n} with the number N of turns tracked for $J=14$.

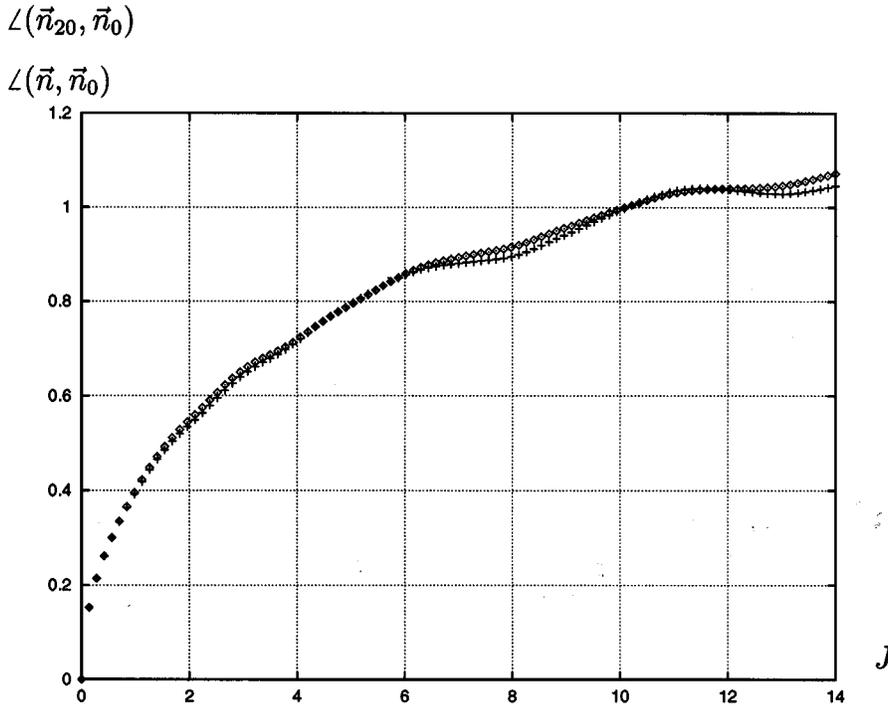


FIG. 4. Opening angle of the analytically calculated \vec{n} (diamonds) and the stroboscopic average \vec{n}_N for $N=20$ (+) as a function of phase space amplitude J .

$$\nu(J) = \frac{\nu_0 - Q}{|\nu_0 - Q|} \Lambda + Q. \quad (87)$$

Correspondingly, on the closed orbit, \vec{u}_1 and \vec{u}_2 reduce to

$$\vec{u}_1(\Phi, 0) = -\vec{e}_3 \frac{\nu_0 - Q}{|\nu_0 - Q|}, \quad \vec{u}_2(\Phi, 0) = \vec{e}_2 \frac{\nu_0 - Q}{|\nu_0 - Q|}. \quad (88)$$

B. Application to the HERA proton ring

The HERA proton ring stores protons with an energy of 820 GeV. At this energy, the proton spin rotates about 1566.85 times around the vertical direction in a flat ring during one turn. The HERA ring is not completely flat. Nevertheless this number illustrates the complexity of spin motion. If a change of the phase space position of a particle initiates a relative change of the one turn spin rotation by only one part in 10^4 , the angle of precession has changed by 56 degrees. This simple observation already hints at the strong dependence of the equilibrium spin direction on the phase space coordinates pointed out in the Introduction. To illustrate this fact, we restricted ourselves to vertical motion with a tune of 0.29, ignored the vertical bends to obtain a flat ring, and computed the third order expansion of the spin transfer matrix with respect to the vertical phase space coordinates using the differential algebra code COSY INFINITY [21]. Finally we stored the third order phase space expansion of the corresponding rotation vector $\vec{\gamma}$. This power expansion was then used by the program SPRINT to obtain the \vec{n} axis. Due to the strong phase space dependence of the spin motion, this expansion does not represent the real spin motion in HERA very well, but it is nevertheless useful to demonstrate the

applicability of stroboscopic averaging. We transported spins by means of this rotation vector, and approximated the orbit motion by the linear transport matrix.

In a realistic accelerator, the existence of the \vec{n} axis cannot be guaranteed, but an approximately invariant spin field can be found if the series \vec{n}_N converges. To indicate the convergence of this series, we plot $|\vec{n}_N - \vec{n}_{20000}|$ for the phase space point with $y_i = 0.4$ mm and $y'_i = 0$ in the East interaction region. This corresponds to an emittance of 69π mm mrad, and is therefore a particle at approximately 4σ of the beam distribution. The slope of -1 in the double logarithmic scale of Fig. 5 illustrates clearly that the convergence is linear in $1/N$ as derived in Sec. IV B.

In that section, convergence could only be guaranteed if the angle between the vectors \vec{b}_N and \vec{n}_0 stayed smaller than $\pi/2$ during tracking. As an example we checked this requirement for $y_i = 0.4$ mm and found that this condition is violated as illustrated in Fig. 6, and convergence cannot be guaranteed *ad hoc*. Nevertheless, an approximately periodic spin field is obtained.

One can easily check this by tracking a spin which is initially parallel to \vec{n}_N for many turns. In order for \vec{n}_N to approximate an \vec{n} axis, the tracked spins have to lie approximately on a closed curve on the unit sphere. Four such closed curves were created by computing \vec{n}_{12000} at the phase space points $y_i = 0.1$ mm, $y_i = 0.2$ mm, $y_i = 0.3$ mm, and $y_i = 0.4$ mm with $y'_i = 0$, and tracking for a further 600 turns. In Fig. 7 we display the projections of these curves on the horizontal plane.

Optimization of the average polarization of a particle beam requires that the equilibrium polarization direction for every particle is almost parallel to the average polarization direction. We therefore averaged the angle between \vec{n}_N and

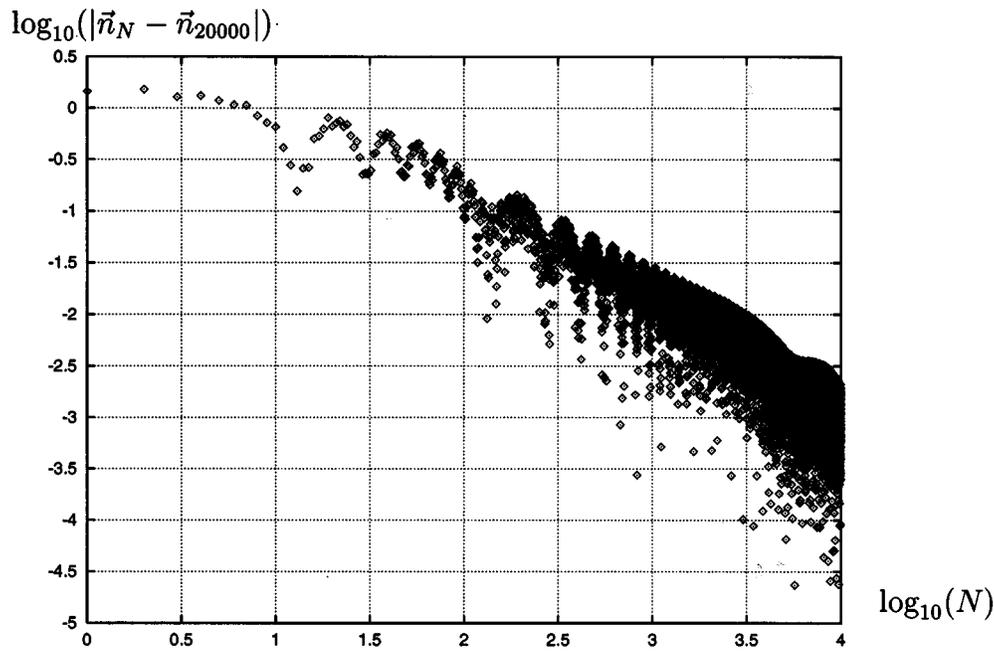


FIG. 5. Indication of convergence for a realistic accelerator: The slope of -1 in the double logarithmic scale illustrates that the convergence is linear in $1/N$.

\vec{n}_0 over the orbital phases and displayed this divergence for different phase space amplitudes in Fig. 8.

Yet another way of illustrating the importance of the \vec{n} axis is illustrated in Fig. 9. Particles at 100 different phases at a normalized emittance of about 4π mm mrad, corresponding to $y_i=0.1$ mm and $y'_i=0$, have been tracked through HERA for 500 turns while the beam was initially polarized 100% parallel to \vec{n}_0 . Similar kinds of tracking re-

sults were presented in [22]. Since this polarization distribution is not the equilibrium distribution, the averaged polarization exhibits a strong beat. This figure also shows that when spins at phase space coordinate \vec{z} are initially parallel to $\vec{n}(\vec{z})$, the averaged polarization stays constant. Therefore, by starting simulations with spins parallel to the \vec{n} axis, one can perform a much cleaner analysis of beam polarization in accelerators.

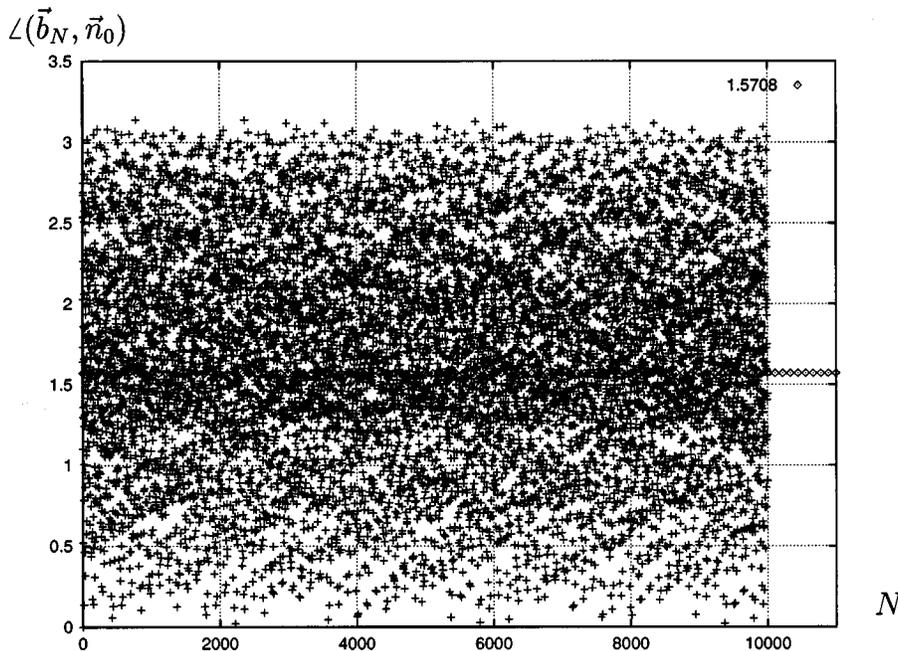


FIG. 6. Angle between the tracked spins and \vec{n}_0 for $y_i=0.4$ mm. To guarantee convergence, this angle has to stay between 0 and $\pi/2$. For many spins this angle is substantially larger; nevertheless we do obtain a good convergence speed.

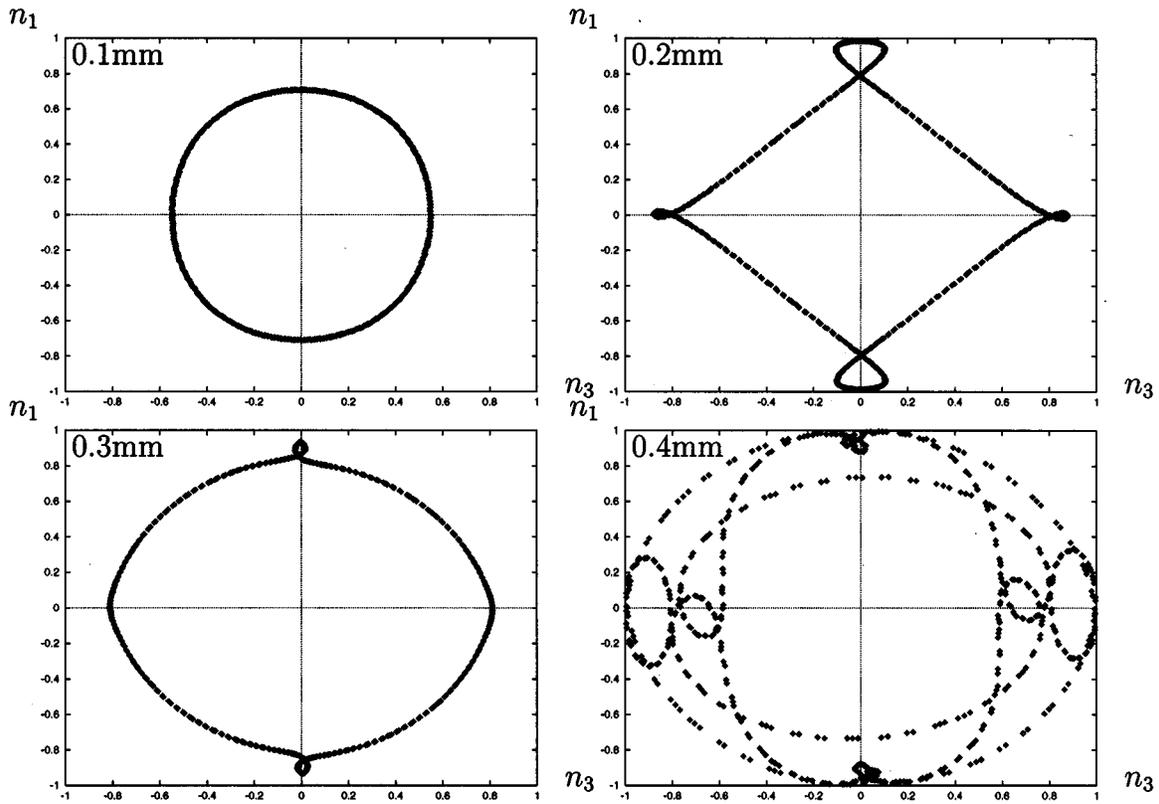


FIG. 7. Referring to the flat HERA model: The stroboscopic average \vec{n}_N for $N=12\,000$ tracked for a further 600 turns for $y_i=0.1, 0.2, 0.3,$ and 0.4 mm.

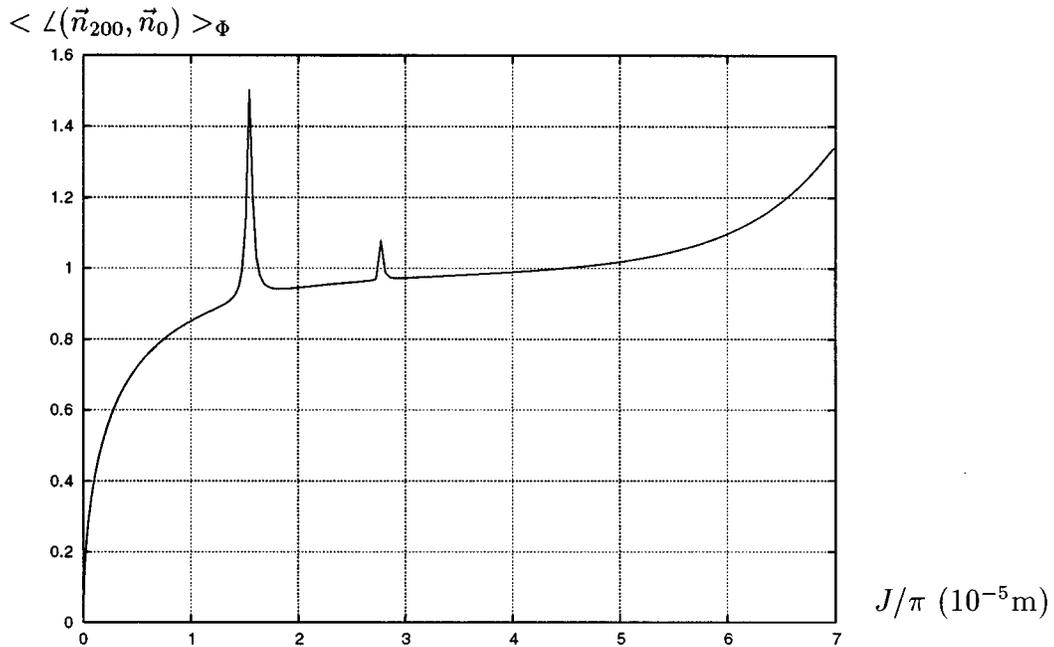


FIG. 8. Referring to the flat HERA model: Phase averaged opening angle of the stroboscopic average \vec{n}_N for $N=200$ as a function of the vertical phase space amplitude J .

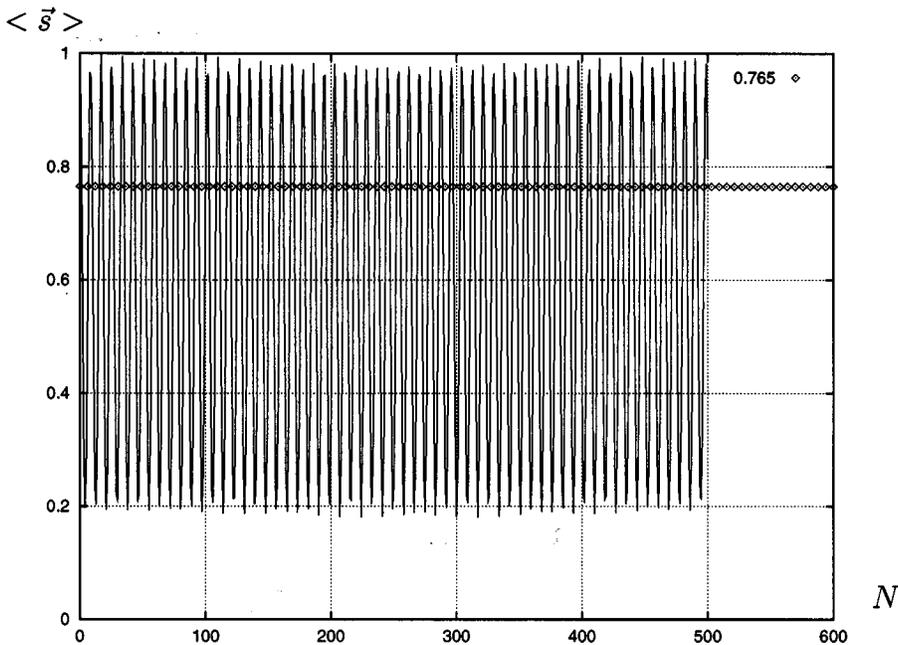


FIG. 9. Propagation of a beam that is initially completely polarized parallel to \vec{n}_0 leads to a fluctuating average polarization. For another beam that is initially polarized parallel to the periodic spin solution \vec{n} , the average polarization stays constant, in this case equal to 0.765.

In this paper we do not want to discuss in detail the possible countermeasures which can be taken to reduce the opening angle of the polarization distribution in a given accelerator. Nevertheless, with one example we want to demonstrate the possibility of such counter measures. The pictures in Fig. 10 were calculated for the actual nonflat HERA lattice, and compare the \vec{n} axis before and after the introduction of four Siberian snakes [4] into the accelerator.

CONCLUSION

We introduced an algorithm for computing the Derbenev-Kondratenko spin axis (\vec{n} axis) from straightforward spin phase space tracking data with the following features:

- (i) It can be implemented in any existing spin tracking program.
- (ii) For an accuracy on the 10^{-3} level typically less than 3000 turns have to be tracked.

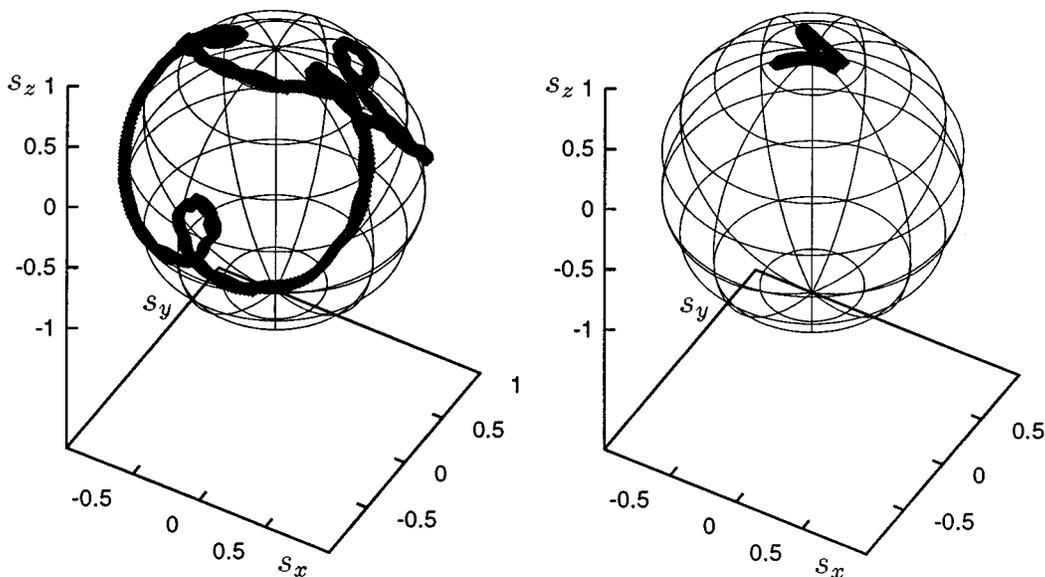


FIG. 10. Referring to the actual HERA lattice as of 1996 (left) and after incorporation of a suitable choice of four Siberian snakes (right): Variation of $\vec{n}(\vec{z})$ over a 2σ vertical phase space ellipse.

(iii) No artificial damping is needed.

(iv) Since our method is nonperturbative, no resonance factors appear in the algorithm and it is applicable even near spin-orbit resonances.

With the method of stroboscopic averaging, important features of accelerators can now be analyzed. One very significant field of study will be the computation of spin tune spreads, which previously could hardly be analyzed and are now easily accessible. Diffusion and damping terms could be added to the analysis, and $\gamma \partial \vec{n} / \partial \gamma$ [5] can be computed for the simulation of electron polarization in storage rings. In addition, numerical checks of uniqueness, convergence near

resonances, and the effect of nonlinear orbit motion should be made.

It is the opinion of the authors that now, after the introduction of stroboscopic averaging, spin tracking in storage rings should always be initialized with spins parallel to the equilibrium distribution, since much clearer analysis becomes possible.

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