I. ELECTRON AND PROTON SPIN POLARISATION IN STORAGE RINGS — AN INTRODUCTION [a]

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This article provides a unified introduction to the theory of electron and proton spin polarisation in storage rings and it provides a common starting point for the written versions of the four talks that I gave at Monterey.

1 Foreword

Each of the four talks that I gave at Monterey had to do with spin polarisation in storage rings and accelerators and in each talk I covered the relevant and necessary aspects of the theory. Indeed, three of the talks essentially dealt only with theory and there was considerable repetition of the basics. If the written versions were to reflect the talks as I delivered them, there would again be repetition but there would also be an apparent lack of connection between those topics which were specific to each talk. Thus a reader who survived reading all four articles might still not have a solid view of the connections between the concepts covered. So it seems appropriate to provide a common introduction to the theory. That is the burden of this article. This also provides a suitable opportunity to present a synthesis of the various ways of describing the competition between polarisation build–up and depolarisation for electrons that I have come across or contributed to over the last decade. Moreover it is an opportunity to lay to rest some confusions that have crept into the subject. Owing to space limitations I will not attempt to maintain a high degree of mathematical rigour but aim instead to impart a feeling for the issues and for our current level of understanding [1]. I shall refer to this article as Article I.

The written versions of the talks themselves will be referred to as Articles II, III, IV, and V as follows:

II Longitudinal electron spin polarisation at 27.5 *GeV* in HERA.
(D.P. Barber *for the HERA Polarisation Group*)

III The permissible equilibrium polarisation distribution in a stored proton beam.
(D.P. Barber, K. Heinemann, M. Vogt and G.H. Hoffstätter)

IV Unruh effect, spin polarisation and the Derbenev-Kondratenko formalism. (D.P. Barber)

[a] Updated version of a contribution to the proceedings of the 15th ICFA Advanced Beam Dynamics Workshop: "Quantum Aspects of Beam Physics", Monterey, California, U.S.A., January 1998. Also in DESY Report 98–096, September 1998.

V The semiclassical FW transformation and the derivation of the Bloch equation for spin–1/2 polarised beams using Wigner functions.
(K. Heinemann and D.P. Barber)

2 Introduction

Spin behaviour in the electromagnetic guide fields of storage rings is dominated by two effects:

- Spin precession
- Spin flip due to synchrotron radiation emission [b].

In existing proton rings and those that will be built in the foreseeable future only spin precession is of significance since the synchrotron radiation power emitted by protons is negligible. However, as pointed out by Sokolov and Ternov in 1964, radiative spin flip can, for electrons, lead to a build up of polarisation [2]. This phenomenon is then commonly known as the Sokolov–Ternov (ST) effect [c]. At the time of writing, the only known practical way of obtaining a stored polarised proton beam is to inject a prepolarised beam provided by a suitable source [3] and then accelerate it. Nevertheless another method has been suggested and I will comment on that in Article III.

In the remainder of this article I will provide a unified overview of spin precession and spin flip and show how to arrive at an efficient description of their combined effect.

3 Spin precession

Spin precession for particles travelling in the electromagnetic fields in storage rings is most conveniently described in terms of the Thomas–Bargmann–Michel–Telegdi (T–BMT) equation [4, 5, 6]:

$$\frac{d}{dt}\,\vec{S} = \vec{\Omega} \wedge \vec{S} \tag{1}$$

where $\vec{S}$ is the 3–vector describing spin in the centre of mass frame and

$$\vec{\Omega} = \frac{e}{mc}\left[-\left(\frac{1}{\gamma}+a\right)\vec{B} + \frac{a\gamma}{1+\gamma}\frac{1}{c^2}(\dot{\vec{r}}\cdot\vec{B})\dot{\vec{r}} + \frac{1}{c}\left(a+\frac{1}{1+\gamma}\right)(\dot{\vec{r}}\wedge\vec{E})\right]. \tag{2}$$

The vector $\vec{B}$ is the magnetic field, $\vec{E}$ is the electric field and γ is the Lorentz factor. The vectors $\vec{r}$ and $\dot{\vec{r}}$ are the position and velocity and evolve according to the Lorentz

[b] But it will become clear later that the distinction between the two can become blurred in storage rings. Indeed resonant spin flip in nuclear magnetic resonance experiments can be viewed either as flip due to photon absorption or precession by π around an effective horizontal field.

[c] In these articles statements made about electrons will also apply to positrons except for appropriate trivial sign changes in mathematical expressions.

equation. The quantity $a = (g-2)/2$ is the gyromagnetic anomaly. For electrons $a \approx 0.0011596$ and for protons $a \approx 1.7928$. The other symbols used here and elsewhere have their usual meanings. The derivations of the T–BMT equation by its authors were purely classical in spirit. The derivation by BMT was based on the requirements of relativistic covariance. However, Thomas combined conventional notions of spin precession with the relativistic effect now called Thomas precession [4, 6] [d]. Note that Eqs. (1) and (2) reduce smoothly to the usual nonrelativistic limit. To obtain a clearer view of the implications of the T–BMT equation one can rewrite it in terms of the field components perpendicular and parallel to the orbit:

$$\begin{aligned} \frac{d\vec{S}}{dt} &= \frac{e\vec{S}}{mc\gamma} \wedge ((1+a)\vec{B}_{||} + (1+a\gamma)\vec{B}_{\perp}) \\ &= \frac{e\vec{S}}{mc} \wedge ((\frac{g}{2\gamma})\vec{B}_{||} + (\frac{1}{\gamma} - 1 + \frac{g}{2})\vec{B}_{\perp}) , \end{aligned} \tag{3}$$

where for this part of the discussion the effect of electric fields has been ignored. Eq. (3) shows that for motion perpendicular to the field, the spin precesses around the field at a rate $1 + a\gamma$ faster than the corresponding rate of orbit deflection:

$$\delta\theta_{spin} = (1+a\gamma)\delta\theta_{orbit} = a\gamma\delta\theta_{orbit} + \delta\theta_{orbit} \tag{4}$$

in an obvious symbolic notation. This precession rate is strongly influenced by the Thomas precession. This is contained in the term $1/\gamma - 1$. For electrons ($g \approx 2$) the total precession is strongly suppressed. For protons ($g \approx 5.58$) the relative suppression is much weaker.

However, 'spin' is a purely quantum mechanical concept. Moreover, we are not working in a regime where electron–positron creation and annihilation are important. Thus a two–component description of spin should suffice and one should therefore look for a Foldy–Wouthuysen transformation (Article V) of the Dirac Hamiltonian (containing a 'Pauli' term for the anomalous magnetic moment) appropriate for the semiclassical regime of a storage ring. By 'semiclassical' I mean that for the high energies involved it should only be necessary to keep terms up to first order in $\hbar$. A Hamiltonian of the required type was already written down in 1973 by Derbenev and Kondratenko (DK) [7] and takes the form [e]

$$h^{dk}_{op} = h^{dk}_{op,orb} + \frac{\hbar}{2}\vec{\sigma}_{op} \cdot \vec{\Omega}_{op} , \tag{5}$$

where:

$$h^{dk}_{op,orb} = J_{op} + e\phi_{op} , \tag{6}$$

[d] Thomas also provided covariant forms for his equation.

[e] The subscript 'op' is to remind the reader that we are dealing with operators. In this case they operate on two–component wavefunctions. The fields in h^{dk}_{op} are external fields. The derivation of this Hamiltonian is the subject of Article V.

and:

$$\vec{\Omega}_{op} = -\frac{e}{2mc}\Big(\frac{mc^2}{J_{op}} + \frac{g-2}{2}\Big)\vec{B}_{op} + \frac{ec(g-2)}{4m}\frac{1}{J_{op}(J_{op}+mc^2)}\vec{\pi}_{op}(\vec{B}_{op}\cdot\vec{\pi}_{op})$$
$$+\frac{e}{2m}\Big(\frac{g-2}{2J_{op}} + \frac{mc^2}{J_{op}(J_{op}+mc^2)}\Big)(\vec{\pi}_{op}\wedge\vec{E}_{op}) + \text{h.c.}\,, \qquad (7)$$

and where $\vec{\pi}_{op}$ and J_{op} are defined as:

$$\vec{\pi}_{op} = \vec{p}_{op} - \frac{e}{c}\vec{A}_{op}\,, \qquad J_{op} = \sqrt{c^2\vec{\pi}_{op}\cdot\vec{\pi}_{op} + m^2c^4}\,. \qquad (8)$$

Thus the DK Hamiltonian consists of a purely orbital part of zeroth order in $\hbar$ and a spin part of first order in $\hbar$. The orbital part resembles the familiar form of the classical relativistic Hamiltonian from the textbooks [8] and the spin part is reminiscent of a Stern–Gerlach (SG) dipole energy term. As will be noted in Article V, at second order in $\hbar$ this Hamiltonian gains just extra orbital terms. All in all, the DK Hamiltonian has a satisfying and physically transparent form. It is then no surprise that in first order in $\hbar$ the Heisenberg equation of motion (EOM) for the kinetic momentum $\vec{\pi}_{op}$ is the Lorentz equation with an additional term for the SG force. It is also clear that in first order in $\hbar$ the Heisenberg EOM for the spin $(\hbar/2)\vec{\sigma}_{op}$ is a precession equation with the same form as the T–BMT equation, Eqs. (1) and (2), since the operator $\vec{\Omega}_{op}$ has a structure equivalent to that of $\vec{\Omega}$ in Eq. (2). In a wave packet approximation and at first order in $\hbar$ the Heisenberg EOM lead to the T–BMT equation for the expectation value $\langle\vec{\sigma}_{op}\rangle$ (= the polarisation) and the EOM for the expectation value $\langle\vec{\pi}_{op}\rangle$ of the kinetic momentum of a wave packet is again the Lorentz equation modified by a SG term [9]. Thus we have now put the T–BMT equation on a firm quantum mechanical footing and have shown that it is the natural outcome of a semiclassical approximation. Moreover (see Article V), we know how to calculate beyond first order in $\hbar$ if necessary. Note that the magnetic SG terms differ from the familiar textbook forms for slowly moving particles but reduce to them at low energy: our terms contain Thomas precession contributions so that, for example, $g/2$ is replaced by $g/2 - 1 + 1/\gamma = a + 1/\gamma$. A detailed discussion on the SG terms in the DK Hamiltonian and on the SG forces allowed by covariance can be found in [10] where the EOM are given a classical interpretation. See also Article III.

The full Hamiltonian given by Derbenev and Kondratenko to include radiation effects is

$$h^{dk}_{tot} = h^{dk}_{op} + h^{dk}_{rad} + h^{dk}_{int} \qquad (9)$$

where h^{dk}_{rad} is the Hamiltonian of the free radiation field and where

$$h^{dk}_{int} = e(\phi_{rad} - \frac{\vec{v}}{c}\cdot\vec{A}_{rad}) + \frac{\hbar}{2}(\vec{\sigma}_{op}\cdot\vec{\Omega}_{rad}) \qquad (10)$$

describes the particle–radiation interaction. The operator $\vec{\Omega}_{rad}$ has the same structure as $\vec{\Omega}_{op}$ except that the external field operators (denoted by the subscript *'op'*) are replaced with radiation field operators (denoted by the subscript *'rad'*).

4 Spin distributions

In the last section it became clear that to first order in $\hbar$ the centres of wave packets move (classically) according to the usual Lorentz force modified by a SG term and that the accompanying $\langle(\hbar/2)\vec{\sigma}_{op}\rangle$ obeys the T–BMT equation. Thus for many purposes the particles and the expectation values of their spin operators can be treated as if they are classical objects and we are then in a position to move beyond single particles and to discuss classical spin and phase space distributions. Article V shows how to arrive at spin and particle distributions directly from the density operator.

To construct a classical treatment one uses the correspondences:

$$\langle\vec{r}_{op}\rangle \to \vec{r}\ , \qquad \langle\vec{\pi}_{op}\rangle \to \vec{\pi}\ , \qquad \langle\frac{\hbar}{2}\vec{\sigma}_{op}\rangle \to \vec{\xi} \tag{11}$$

where $\vec{\xi}$ is a classical spin of length $\hbar/2$. Then with the Hamiltonian:

$$\mathfrak{h}^{dk} = \mathfrak{h}^{dk}_{orb} + \vec{\xi}\cdot\vec{\Omega} \tag{12}$$

with

$$\mathfrak{h}^{dk}_{orb} = J + e\cdot\phi \tag{13}$$

and the Poisson bracket relations [f]:

$$\begin{aligned}
&\{r_j, p_k\} = \delta_{jk}\ , \qquad \{r_j, r_k\} = \{p_j, p_k\} = \{r_j, \xi_k\} = \{p_j, \xi_k\} = 0\ , \\
&\{\xi_j, \xi_k\} = \sum_{m=1}^{3} \varepsilon_{jkm}\ \xi_m\ , \qquad (j, k = 1, 2, 3)\ ,
\end{aligned} \tag{14}$$

and where semiclassically the $\dot{\vec{r}}$ in Eq. (2) equals $c^2\vec{\pi}_{op}/J$, the Lorentz (modified by a SG term) and T–BMT equations emerge from the canonical equations of motion:

$$\dot{\vec{r}} = \{\vec{r}, \mathfrak{h}^{dk}\}\ , \qquad \dot{\vec{\pi}} = \{\vec{\pi}, \mathfrak{h}^{dk}\} + \frac{\partial\vec{\pi}}{\partial t}\ , \qquad \dot{\vec{\xi}} = \{\vec{\xi}, \mathfrak{h}^{dk}\}\ . \tag{15}$$

Since storage rings and accelerators have accelerating cavities which subject the particles to time dependent fields and since the magnet geometry is fixed, particle dynamics is best described in terms of the canonical coordinates $\vec{u} = (x, p_x, z, p_z, \Delta t, \Delta E)$ where $x,\ p_x,\ z,\ p_z$ describe transverse motion with respect to the curved periodic orbit and $\Delta t, \Delta E$ are the time delay relative to a synchronous particle (at the centre of the bunch) and the energy deviation from the energy of a synchronous particle respectively. The independent variable is now the distance around the ring, s. There is a corresponding (classical) Hamiltonian, correct up to first order in $\hbar$,

$$\tilde{h} = \tilde{h}_{orb} + \vec{\xi}\cdot\vec{\tilde{\Omega}}\ , \tag{16}$$

[f] If we were working to second or higher order in $\hbar$ we would use the Moyal algebra [11, 12]. In the present case of first order in $\hbar$ this simplifies to the Poisson algebra.

which enables the EOM to be written in canonical form and this is derived from $\mathfrak{h}^{dk}$ by standard means [13]. If the ring is distorted (see below), $\vec{u}$ describes the position with respect to the resulting closed orbit.

We now make the idealisation that the beam phase space can be described in terms of a smooth continuous density, $w(\vec{u};s)$, which is a scalar function of $\vec{u}$ and the azimuth s [g]. It is normalised to unity. In the absence of dissipation and noise (e.g. due to synchrotron radiation) and ignoring the effect of the tiny SG forces on the orbital motion, w is constant along a phase space trajectory and obeys a relation of the Liouville type:

$$\frac{\partial w}{\partial s} = \{\tilde{h}_{orb}, w\}\,. \tag{17}$$

If the beam is stable, i.e. if w is the same from turn to turn, then it is periodic in s and we write it as w_{eq} so that $w_{eq}(\vec{u};s) = w_{eq}(\vec{u};s+C)$, where C is the ring circumference.

Having assigned a phase space density to each point in phase space we now assign a polarisation $\vec{P}(\vec{u};s)$ to each point [h]. $\vec{P}$ is the average over particles of the unit spins $2\vec{\xi}/\hbar$ at $(\vec{u};s)$. Since the T-BMT equation is linear in the spin and since in this picture the spins at $(\vec{u};s)$ all see the same $\vec{\tilde{\Omega}}(\vec{u};s)$, $\vec{P}(\vec{u};s)$ obeys the T-BMT equation

$$\frac{d\vec{P}}{ds} = \vec{\tilde{\Omega}}(\vec{u}(s);s) \wedge \vec{P}\ . \tag{18}$$

Because Eq. (18) describes precession, $|\vec{P}(\vec{u};s)|$ is constant along a phase space trajectory. To make closer contact with the synchrobetatron motion, we can rewrite Eq. (18) as [14, 15]:

$$\frac{\partial \vec{P}}{\partial s} = \{\tilde{h}_{orb}, \vec{P}\} + \vec{\tilde{\Omega}}(\vec{u};s) \wedge \vec{P} \tag{19}$$

which is analogous to Eq. (17) and assumes that $\vec{P}(\vec{u};s)$ is differentiable in all directions in phase space. Note that the polarisation of the whole beam as measured by a polarimeter at azimuth s is the average across phase space:

$$\vec{P}_{av}(s) = \int d^6u\ w(\vec{u};s)\vec{P}(\vec{u};s)\,. \tag{20}$$

If the spin distribution is stable, i.e. if $\vec{P}(\vec{u};s)$ is the same from turn to turn, then $\vec{P}(\vec{u};s)$ not only obeys the T-BMT equation, but it is also periodic in s and we write it as $\vec{P}_{eq}$ so that $\vec{P}_{eq}(\vec{u};s) = \vec{P}_{eq}(\vec{u};s+C)$. We denote the unit vector along $\vec{P}_{eq}(\vec{u};s)$ by $\hat{n}(\vec{u};s)$ [i]. This also obeys Eq. (18) and is periodic in s: $\hat{n}(\vec{u};s) = \hat{n}(\vec{u};s+C)$. On the (periodic) closed orbit $\hat{n}(\vec{u};s)$ becomes $\hat{n}(\vec{0};s)$ and we denote it by $\hat{n}_0(s)$ [j].

[g] Note that in Article V the phase space density is denoted by 'ρ'.

[h] This is equivalent to associating a spin density matrix with each point in phase space.

[i] With respect to $\hat{n}(\vec{u};s)$ the spin density matrix at $(\vec{u};s)$ is diagonal.

[j] Many authors make no clear distinction between $\hat{n}$ and $\hat{n}_0$ and many use the symbol $\hat{n}$ for $\hat{n}_0$. This can sometimes lead to confusion. In particular the original symbol for $\partial\hat{n}/\partial\delta$ (section 5.3) was $\gamma\partial\hat{n}/\partial\gamma$ [7] and some have erroneously understood $\gamma\partial\hat{n}/\partial\gamma$ to mean $E_0\partial\hat{n}_0/\partial E_0$ where E_0 is the design energy [16].

Obviously $\hat{n}_0(s)$ obeys the periodicity condition $\hat{n}_0(s) = \hat{n}_0(s + C)$, i.e. $\hat{n}_0(s)$ is the ring periodic solution of the T–BMT equation on the closed orbit. In general it is unique.

In the foregoing I introduced the *invariant (vector) spin field* $\hat{n}(\vec{u}; s)$ by appealing to physical intuition. The underlying assumption was that the field $\hat{n}(\vec{u}; s)$, which is supposed to obey Eq. (18) over the whole of the beam phase space, not only *exists* but is smooth (to correspond with our expectations of the spin distribution in a real beam) and is unique. However, the situation is not quite so simple as I will now explain by describing some qualitative aspects of spin motion.

If a circular accelerator only had vertical (dipole) fields, vertical spins would not be affected and $\hat{n}_0(s)$ would be vertical. Moreover, according to Eq. (3) a non-vertical spin would precess around $\hat{n}_0(s)$ $a\gamma$ times per turn with respect to the (periodic) design orbit. I call the quantity $a\gamma$ the 'naive spin tune'. It represents the natural spin precession frequency of this simple system. It increases by one unit for every $\approx 440\ MeV$ ($\approx 523\ MeV$) increase in the energy of electrons (protons). But some rings contain vertical bend magnets so that the design orbit is not flat. The ring might also contain solenoidal fields of particle detectors. In these cases a periodic T-BMT solution, $\hat{n}_0(s)$, on the design orbit still exists but is no longer everywhere vertical and it is given by the real eigenvector (with unit eigenvalue) of the one turn (orthogonal) 3 x 3 spin transfer matrix for this design orbit [k]. Indeed, for the HERA electron ring (Article II) $\hat{n}_0$ is made *longitudinal* at the east IP by means of spin rotators. The number of spin precessions around $\hat{n}_0(s)$ per turn is extracted from the complex eigenvalues of the matrix [21, 22]. We call this the 'real spin tune' or just the 'spin tune' and denote it by ν_{spin}. In general it deviates from $a\gamma$ [l].

If the spin tune were an integer, the one turn matrix would be a unit matrix and $\hat{n}_0(s)$ would not be unique. This lack of uniqueness also manifests itself in extreme sensitivity to field errors. The quadrupoles and other magnets in storage rings normally have unavoidable small misalignements so that the periodic (closed) orbit deviates from the design orbit. Likewise the real $\hat{n}_0(s)$ deviates from the design $\hat{n}_0(s)$ since a spin on the closed orbit now 'sees' horizontal dipole components in the quadrupoles. There is also a small shift in the real spin tune. The angle between the two $\hat{n}_0(s)$'s is roughly proportional to the amount of closed orbit distortion. But it becomes very large if the design spin tune is close to an integer [23, 24] since the spin motion is then coherent with the imperfection fields. The spins are then said to be near an integer resonance (sometimes called an 'imperfection resonance').

Particle bunches in storage rings have nonzero transverse dimensions and energy

[k] However, for $\vec{u} \neq \vec{0}$, the constraint $\hat{n}(\vec{u}; s) = \hat{n}(\vec{u}; s + C)$ is obviously *not* equivalent to an analogous eigenproblem for $\hat{n}(\vec{u}; s)$ since in general a spin at $(\vec{u}; s)$ set parallel to $\hat{n}(\vec{u}; s)$ does not map into itself over one turn. Thus the *naive* algorithm based on a one turn map (e.g. see page 27 in [17]) is incorrect; in general a '$\hat{n}$' constructed in that way would not obey the T–BMT equation everywhere along an orbit. It would also not exhibit resonant structure at the tunes specified by Eqs. (21a, b). As a result, a '$\hat{n}$' constructed in that way should not be used to obtain the vector $\partial\hat{n}/\partial\delta$ (e.g. see page 52 in [18]) needed, as in section 5.3, to describe radiative depolarisation of electrons. However, $\hat{n}$ can be obtained as an eigensolution of a *modified* eigenproblem [19, 20]. See also footnote *j*.

[l] Actually, the complex eigenvalues only deliver the fractional part of the spin tune. The integer part must be found by following the spin motion for one turn.

spread and the motion of a spin, compared to that of a spin on the closed orbit, depends on the position in phase space via the $\vec{u}$ in $\vec{\Omega}(\vec{u};s)$. For particles circulating for many turns the total disturbance to a spin can grow to become very big if there is coherence between the natural spin motion and the oscillatory motion in the beam characterised by the spin–orbit resonance condition:

$$\nu_{spin} = m + m_x\ Q_x + m_z\ Q_z + m_s\ Q_s \tag{21a}$$

where the m's are integers and the Q's are respectively the horizontal, vertical and longitudinal tunes of the orbital oscillations.

The integer resonances ($|m_x|+|m_z|+|m_s| = 0$ in Eq. (21a)) can normally be identified with the imperfection resonances already mentioned and driven by the periodic imperfection fields along the closed orbit. We have absorbed their influence into a large deviation of $\hat{n}_0(s)$ from the design direction. The spin-synchrobetatron resonances ($|m_x|+|m_z|+|m_s| \neq 0$ in Eq. (21a)) (sometimes called 'intrinsic resonances') are driven by the quasiperiodic fields seen by particles executing quasiperiodic synchrobetatron oscillations about the closed orbit. The sum $|m_x|+|m_z|+|m_s|$ is called the 'order' of the resonance. An imperfection resonance is then a zeroth order resonance. Although I have just been discussing the behaviour near resonance of arbitrary spins it should now be clear that $\hat{n}$, which is a special solution of Eq. (18) constrained to be periodic, should, just like $\hat{n}_0(s)$, also show extreme behaviour near resonances. This is confirmed by the analytical structure and numerical output from the algorithms used for its construction [23]. Near integer resonances in a distorted ring $\hat{n}_0$ deviates strongly from the nominal direction for the perfectly aligned ring and near intrinsic resonances the difference $\hat{n}(\vec{u};s) - \hat{n}_0$ becomes large and increases with the synchrobetatron amplitude $\vec{u}$ and with $a\gamma$ [m]. So far in this paragraph I have, for simplicity, assumed that the horizontal, vertical and longitudinal oscillations of the particles are linear and mutually uncoupled. However, in practice these motions are coupled to a greater or lesser extent for a variety of reasons. Then the spin–orbit resonance condition becomes

$$\nu_{spin} = m + m_I\ Q_I + m_{II}\ Q_{II} + m_{III}\ Q_{III} \tag{21b}$$

where Q_I, Q_{II} and Q_{III} are the eigentunes in the presence of coupling [n].

[m] Note that the terminology 'intrinsic' and 'imperfection' must be used with care since synchrobetatron motion can also give rise to zeroth order resonance phenomena [25, 26].

[n] It should be emphasized that coupling or nonlinear fields (e.g. see pages 26, 33 and 39 in [18]) are *not* prerequisites for the occurrence of high order spin–orbit resonances. In fact high order resonances *must* occur even for perfectly linear uncoupled orbital motion in spite of the fact that the T–BMT equation is linear, simply because rotations around different axes do not commute. The influence of noncommutation can be seen in the nontrivial structure of the integrals used to calculate $\hat{n}$ in the perturbative treatments in [27, 28, 29, 25, 26]. The first order integrals correspond to the SLIM approximation [23, 21, 22] and each describes the effect of a single orbital mode and exhibits first order resonant behaviour. The higher order integrals describe the *combined* effects of *independent* orbital modes such as, for example, the effect of the vertical fields in quadrupoles on a spin–like vector which has been tilted from the vertical by radial quadrupole fields. The higher order integrals exhibit higher order resonant behaviour. One also sees from the treatments just cited that

For 27.5 GeV electrons in HERA (see Article II) the r.m.s. angle between $\hat{n}(\vec{u};s)$ and $\hat{n}_0$ (obtained by averaging across phase space) is just a few milliradians away from intrinsic resonances and about 100 milliradians very near such resonances. For protons at about 800 GeV in HERA (see Article III) on the '1–σ' torus this angle is typically 60 degrees unless Siberian Snakes are employed. Figure 1 depicts invariant

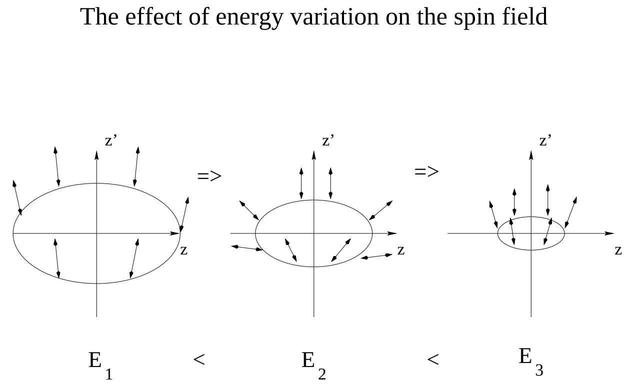

Figure 1: A typical $\hat{n}$–field at three energies, the second of which is close to resonance.

spin fields $\hat{n}$ 'attached' to vertical betatron phase space ellipses for three different fixed energies but for the same invariant vertical emittance. Other examples are given in Article III.

Although I introduced $\hat{n}$ via spin distributions, the history of $\hat{n}$ took a different course which provides more insight into its meaning and properties. It was first introduced by Derbenev and Kondratenko [30, 7] in the process of obtaining action–angle variables for combined spin–orbit motion by 'diagonalising' the Hamiltonian in Eq. (12) and this aspect was further illuminated by Yokoya in [31]. A similar approach can be used on the Hamiltonian in Eq. (16) [13, 28, 31, 32]. I now give a rough outline of the basic ideas.

It is assumed that the orbital motion is integrable and one makes an s dependent canonical transformation so that $\tilde{h}_{orb}$ is replaced by $\bar{h}_{orb} = \Sigma_i 2\pi \bar{Q}_i \bar{I}_i$ where the $\bar{Q}_i$ are the three orbital tunes and the three $\bar{I}_i$ are the components of the orbital action vector $\vec{\bar{I}}$. Then one describes the spin motion with respect to a set of orthonormal axes (a 'dreibein') $(\hat{n}_1(\vec{u};s),\, \hat{n}_2(\vec{u};s),\, \hat{n}_3(\vec{u};s))$ attached to each point in phase space and requires that $\vec{\xi}\cdot\hat{n}_3$ is a constant of motion. Clearly, $\hat{n}_3(\vec{u};s)$ must be a solution of the T–BMT equation at $(\vec{u};s)$ [o]. At the same time the dreibein is chosen so that

the so-called synchrotron sideband resonances comprise just one among several species of high order resonances which can be classified by examining the integrals in the *unified* treatments of resonances in [27, 26]. However,since Q_s is usually small, the synchrotron sideband resonances tend to be much stronger than other high order resonances so that they are often singled out for special attention and treated separately in various *ad hoc* ways. Naturally, the introduction of coupling and nonlinear fields does have consequencies. Coupling modifies the resonant ν_{spin} values (Eq. (21b)) and changes the strengths of resonant effects. Nonlinear fields can, of course, drive high order resonances but at high $a\gamma$ and for normal levels of nonlinearity, resonances resulting from noncommutation dominate.

[o]The angle between two T–BMT solutions following the same point on an orbit does not change in time. See also footnote k.

$\hat{n}_i(\vec{u};s) = \hat{n}_i(\vec{u};s+C)$ ($i = 1 \to 3$) and so that $\vec{\xi}$ precesses around $\hat{n}_3$ at a constant rate relative to $\hat{n}_1$ and $\hat{n}_2$. The rate, denoted by $\check{\nu}(\vec{\bar{I}})$, should depend only on the actions $\vec{\bar{I}}$. The vectors $\hat{n}_1$ and $\hat{n}_2$ are not solutions of the T–BMT equation. The vector $\hat{n}_3$ has just the properties of the vector $\hat{n}$ introduced earlier. This choice of the dreibein, which amounts to a $\vec{u}$ and s–dependent rotation of the axes for describing the spin motion, is achieved by a suitably designed $\vec{u}$ and s–dependent canonical transformation which delivers a final Hamiltonian (correct to first order in $\hbar$) with the 'diagonalised' form [31]

$$\check{h} = \Sigma_i \; 2\pi \check{Q}_i \check{I}_i + 2\pi \check{\nu} \check{I}_{spin} \tag{22}$$

where $\check{I}_{spin} = \vec{\xi} \cdot \hat{n}_3$ is now an integral of motion, the spin action, and the $\check{Q}_i$ and $\check{I}_i$ are the corresponding orbital tunes and actions. The $\check{I}_i$ differ from the $\bar{I}_i$ by SG terms [31]. Note that the concept of spin tune has now been generalised; instead of the closed orbit spin tune ν_{spin} we have a spin tune $\check{\nu}(\vec{\bar{I}})$ depending on the orbital actions (but not on $\check{I}_{spin}$) which differs slightly from ν_{spin} and which reduces to ν_{spin} for zero orbital actions [p]. Now, in retrospect, the definition of resonance must be refined; we should really use $\check{\nu}(\vec{\bar{I}})$ in Eqs. (21a, b) instead of ν_{spin}. It should now be clear why we sought a definition of spin precession rate, i.e. spin tune, which makes the latter independent of orbital phases and the azimuth s. Spin tune should tell us something about the degree of long term coherence between the spin motion and the orbital motion and allow us to express this coherence by means of resonance relations like Eqs. (21a, b) (with $\check{\nu}(\vec{\bar{I}})$). But if we work with a 'fake spin tune' such as that obtained from the one–turn eigenproblem (see footnote k and [33]) which depends on orbital phases so that the 'fake spin phase advance' per turn varies from turn to turn, we can make no statements about long term coherence. With this redefinition of spin tune the dreibein ($\hat{n}_1$, $\hat{n}_2$, $\hat{n}_3$) is unique except at spin–orbit resonances [31, 34] and by this uniqueness the vector $\hat{n}_3$ is just the vector $\hat{n}$ introduced earlier except for a possible difference of sign! The exotic (unstable) behaviour of $\hat{n}$ near resonance is a manifestation of lack of uniqueness at resonance.

Now I return to the questions of smoothness and existence of a $\hat{n}$ obeying Eq. (18). Since $\check{\nu}$ depends on orbital actions, $\hat{n}$ is *potentially* nonunique at almost all points in phase space because the resonance condition is satisfied almost everywhere if we include resonances of arbitrarily high order. Thus $\hat{n}$ might not be differentiable in all directions in phase space [35]. However, algorithms, both perturbative and nonperturbative, for constructing approximations to $\hat{n}$ are available [23] (see also Article III) and experience with calculating $\hat{n}$ by the author and colleagues seems to indicate that resonance effects rapidly become weak as the resonance order increases so that only a limited number of relatively low order resonances are likely to cause trouble. Therefore in the remainder of the article it will be assumed that the spin field $\hat{n}$ is a legitimate tool in practice. Nevertheless, the technical and interesting matters of existence and smoothness are under active investigation [36] and knowledge

[p] Note that for $\vec{\bar{I}} \neq \vec{0}$, $\check{\nu}(\vec{\bar{I}})$ *cannot* normally be obtained from a complex eigenvalue of the naive one–turn eigenproblem discussed in footnote k.

gained from this study will be incorporated in our treatment of spin distributions. An example of the dependence of $\check{\nu}$ on amplitude can be found in [33].

An extension of the numerical work reported in [33] but carried out just before this article was completed indicates that $\check{\nu}$ actually 'jumps over' resonant values as the orbit amplitude is changed [37, 38]. That work is based on the 'stroboscopic averaging' algorithm in the computer code SPRINT [34]. But even more recent results from a new version of the SODOM algorithm [19, 20] corroborates these findings. This implies, contrary to traditional expectations based on perturbation theory, that the spin–orbit resonance conditions of Eqs. (21a, b) is never exactly satisfied in non–perturbative calculations. However, near to resonance, $\hat{n}$ still exhibits exotic behaviour.

Although $\hat{n}$ and $\vec{\xi}$ both obey the T–BMT equation they are very different objects; $\hat{n}$ is a function of the dynamical phase space variables but $\vec{\xi}$ is a dynamical spin variable and by Eq. (14) the Poisson bracket $\{\hat{n}, \vec{\xi}\}$ vanishes. Now that we have a classical integral of motion for the spin, namely $\check{I}_{spin}$, we recognize $\hat{n}$ as a phase space dependent semiclassical quantisation axis corresponding to the quantum observable $(\hbar/2)\vec{\sigma}_{op} \cdot \hat{n}$. We also see that the quantisation axis coincides with the direction of the equilibrium spin field. As we will see later $(\hbar/2)\vec{\sigma}_{op} \cdot \hat{n}$ is a key object in the analytical theory of equilibrium electron polarisation and indeed it was originally introduced as an aid to calculating the electron polarisation [30, 7]. The analysis becomes more complicated if the orbital motion is nonlinear but in practice one tries to use an optic for which the nonlinear effects have been minimised and tries to restrict the beam to a phase space volume such that the motion is almost integrable.

One last point on the virtues of $\hat{n}$: a calculation of electron polarisation with the computer program SODOM [19] which exploits $\hat{n}$ agrees well with a calculation using the Monte–Carlo spin tracking program SITROS [39] which contains no notion of $\hat{n}$.

The material on spin distributions presented in this section is applicable both to electrons and protons. The application to protons is the topic of Article III so that for the remainder of this article I will focus on electrons and in particular on the modifications by synchrotron radiation to the concepts already presented.

5 The effects of synchrotron radiation

Synchrotron radiation (SR) emitted by stored electrons has three main effects: it determines the phase space distribution and it brings about spin polarisation owing to spin flip associated with synchrotron radiation (the ST effect) but the stochastic element of SR also causes depolarisation. Thus SR brings polarisation but it also takes it away! As we have seen already and as we will see below spin motion is irrevocably intertwined with the orbital effects. I will begin by summarising the orbital dynamics and then discuss the polarisation and depolarisation effects in detail.

5.1 Orbital phase space

Although SR spectra can be estimated by classical means [6] SR is fundamentally a quantum phenomenon; it consists of single photons so that one can only make reliable predictions by using quantised radiation theory. One then finds corrections to the classical spectrum [40]. The work of Huang and Ruth [41] presented at this meeting is a good example of recent quantum calculations.

Most of the SR in conventional storage rings is generated in the fields of the dipole magnets defining the design orbit. A quantum treatment for this case of the effects of SR on the orbital phase space distribution was carried out in 1975 [42] using the Hamiltonian of Eq. (10). I will return to this later but here I will follow another route which has the advantage of exhibiting the transparency needed for this article.

Photon emission in the dipole fields is largely incoherent and detailed calculations show that one can consider the photons to be emitted over short distances of the order of ρ/γ where ρ is the orbit radius [q]. Furthermore in practical storage rings the energy loss per turn of a single particle is small compared to the nominal energy. Thus the dissipative effect is weak and for example in HERA (Article II) an electron at 27.5 GeV loses about 80 MeV per turn. Then for many purposes it suffices to describe the radiation reaction power $\mathfrak{p}(s)$ from SR using a classical model in which smooth classical radiation reaction power $\mathfrak{p}_{cl}(s)$ is overlayed with a 'delta correlated' ('white') stochastic component $\delta\mathfrak{p}(s)$:

$$\mathfrak{p}(s) = \mathfrak{p}_{cl}(s) + \delta\mathfrak{p}(s)\ , \ \langle\delta\mathfrak{p}(s)\delta\mathfrak{p}(s')\rangle \ = R(E_0, K)\delta(s - s') \tag{23}$$

where the parameter R quantifies the intensity of the noise and depends on the design energy E_0 and the curvature K [43].

The equations for $\vec{u}$ of deterministic orbital motion derived from a Hamiltonian are then modified by inclusion of damping and stochastic terms and in the (very good) approximation that the photons are emitted parallel to the particle trajectory and neglecting interparticle interactions the resultant linearised *stochastic differential equation* describing motion with respect to the closed orbit can be used to construct the Fokker–Planck equation for the evolution of the phase space density [43, 44, 45, 46] [r]. I write this as

$$\frac{\partial w}{\partial s} = \mathcal{L}_{_{FP,orb}}\ w\ , \tag{24}$$

where the orbital Fokker–Planck operator can be decomposed into the form:

$$\mathcal{L}_{_{FP,orb}} = \mathcal{L}_{ham} + \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2\,.$$

The term $\mathcal{L}_{ham}w$ is associated with the original symplectic (i.e. phase space density preserving) motion and it contains just first order derivatives with respect to the components u_i $(i = 1...6)$. The operators $\mathcal{L}_0$ and $\mathcal{L}_1$ contain zeroth and first order derivatives and account for damping effects. The operator $\mathcal{L}_2$ contains second order derivatives originating in diffusion effects.

[q] At this point I recommend the reader to consult the chart of time scales for electron dynamics in [17]. We will need this on several occasions. Indeed, an appreciation of these time scales is indispensible for understanding the physics of electron storage rings.

[r] Restriction to linearised motion enables me to describe the chief qualitative features to be discerned without undue complication.

A central property of Eq. (24) is that $w(\vec{u};s)$ reaches equilibrium with $w(\vec{u};s) = w(\vec{u};s+C)$ within a few damping times. At HERA at 27.5 GeV the longitudinal damping time is about 7 milliseconds $\approx$ 350 turns $\approx$ (design energy)/(energy loss per turn) [47]. Furthermore $w(\vec{u};s)$ is gaussian and since the radiation effects are weak, $w(\vec{u};s)$ is very close to being a solution of the radiationless transport equation Eq. (17) but with the radiation effects determining the beam size and causing a tiny ripple in the emittances [s] as functions of s.

Now that we understand the effects of SR on orbital phase space we can move on to spin.

5.2 The Sokolov-Ternov effect

Only a very small fraction of the radiated photons cause spin flip but for electron spins aligned along a uniform magnetic field, the $\uparrow\downarrow$ and $\downarrow\uparrow$ flip rates differ and this leads to a build-up of spin polarisation antiparallel to the field. Positrons become polarised parallel to the field. The transition rates for electrons are [2]:

$$\begin{aligned} W_{\uparrow\downarrow} &= \frac{5\sqrt{3}}{16}\frac{e^2\gamma^5\hbar}{m_e^2c^2|\rho|^3}\left(1+\frac{8}{5\sqrt{3}}\right) \\ W_{\downarrow\uparrow} &= \frac{5\sqrt{3}}{16}\frac{e^2\gamma^5\hbar}{m_e^2c^2|\rho|^3}\left(1-\frac{8}{5\sqrt{3}}\right). \end{aligned} \qquad (25)$$

For positrons, interchange plus and minus signs here and elsewhere.

The equilibrium polarisation in a uniform magnetic field is independent of γ,

$$P_{st} = \frac{W_{\uparrow\downarrow} - W_{\downarrow\uparrow}}{W_{\uparrow\downarrow} + W_{\downarrow\uparrow}} = \frac{8}{5\sqrt{3}} = 92.38\%\,. \qquad (26)$$

For a beam with zero initial polarisation, the time dependence for build-up to equilibrium is

$$P(t) = P_{st}\left[1 - \exp\left(-t/\tau_0\right)\right] \qquad (27)$$

where the build-up rate is

$$\tau_0^{-1} = \frac{5\sqrt{3}}{8}\frac{e^2\gamma^5\hbar}{m_e^2c^2|\rho|^3}\,. \qquad (28)$$

The time τ_0 depends strongly on γ and ρ but is typically minutes or hours.

However, the fields in storage rings are far from uniform but since the system is semiclassical, Eq. (25), which was originally obtained from solutions of the Dirac equation, can be generalised and according to Baier and Katkov[48] for electron spins initially aligned along an arbitrary unit vector $\hat{\xi}$ the transition rate is

$$W = \frac{1}{2\tau_0}\left[1 - \frac{2}{9}(\hat{\xi}\cdot\hat{s})^2 + \frac{8}{5\sqrt{3}}\,\hat{\xi}\cdot\hat{b}\right] \qquad (29)$$

[s] For electrons I define the emittance of a mode to be the r.m.s. action of the mode.

where $\hat{s}$ = direction of motion and $\hat{b} = (\hat{s} \wedge \dot{\hat{s}})/|\dot{\hat{s}}|$. This is the magnetic field direction if the electric field vanishes and the motion is perpendicular to the magnetic field.

The corresponding instantaneous rate of build-up of polarisation along $\hat{\xi}$ is

$$\tau_{bk}^{-1} = {\tau_0}^{-1} \left[1 - \frac{2}{9} (\hat{\xi} \cdot \hat{s})^2 \right] . \qquad (30)$$

But instead of spin flip rates we really need an EOM for the polarisation itself and if we neglect the effect of stochastic (synchrotron radiation) photon emission on the orbit and imagine that all particles remain on the closed orbit (CO), the equation of motion for electron polarisation as given by Baier, Katkov and Strakhovenko (BKS) is [49, 50]

$$\frac{d\vec{P}}{dt} = \vec{\Omega}_{co} \wedge \vec{P} - \frac{1}{\tau_0(s)} \left[\vec{P} - \frac{2}{9} \hat{s} (\vec{P} \cdot \hat{s}) + \frac{8}{5\sqrt{3}} \hat{b}(s) \right] . \qquad (31)$$

Note that the T–BMT term $\vec{\Omega}_{co} \wedge \vec{P}$ appears here as the output of the radiation calculation itself.

By noting that the characteristic time for polarisation build up is much larger than the circulation time [t], and integrating the BKS equation (Eq. (31)) one finds the generalised Sokolov–Ternov formula for the asymptotic electron polarisation in arbitrary magnetic fields along the closed orbit [17]:

$$\vec{P}_{bks} = -\frac{8}{5\sqrt{3}} \; \hat{n}_0 \; \frac{\oint ds \; (\hat{n}_0(s) \cdot \hat{b}(s))/|\rho(s)|^3}{\oint ds \; \left[1 - \frac{2}{9} (\hat{n}_0(s) \cdot \hat{s})^2 \right] / |\rho(s)|^3} . \qquad (32)$$

So the polarisation settles down *aligned with* $\hat{n}_0(s)$, the periodic solution to the T-BMT equation on the closed orbit. In rings containing dipole spin rotators (Article II) the polarization can usually not reach 92.38% since $\hat{n}_0(s)$ is then not parallel to the field everywhere. The corresponding polarisation build-up rate is

$$\tau_{bks}^{-1} = \frac{5\sqrt{3}}{8} \frac{e^2 \gamma^5 \hbar}{m_e^2 c^2} \frac{1}{C} \oint ds \, \frac{\left[1 - \frac{2}{9} (\hat{n}_0 \cdot \hat{s})^2 \right]}{|\rho(s)|^3} . \qquad (33)$$

The above formulae show that in the absence of stochastic motion the maximum attainable polarisation is 92.38% instead of 100%. Why should this be?

At the simplest level the reason is clear: the probability for reverse spin flip is nonzero (Eq. (25)). Nevertheless, 'lay observers' often imagine that spin flip has something to do with spin's trying to reach the lowest energy level of the two levels of a magnetic dipole in a magnetic field and that once the spin is in its lowest level it will stay there. Then 100% polarisation would be achieved. Also, reverse flip by radiation emission would defy energy conservation.

However, we are not dealing with spins at rest but with spins 'sitting' on relativistic electrons which already have quantised orbital energy levels so that the prohibition of

[t] Again, see [17] for a compilation of time scales.

reverse flip by energy conservation no longer applies. From Eq. (5) applied to electrons in a uniform vertical magnetic field it is clear that the energy change associated with spin reversal from up to down is $(1+a\gamma = \gamma(1/\gamma - 1 + g/2))$ larger than the separation of orbital energy levels $\hbar\omega_c$ where ω_c is the angular frequency of the orbit. So one could naively imagine spin flip occuring without radiation but simply by a change of orbital energy level. A related phenomenon involving exchange of orbital and spin energy has been proposed by Derbenev [51, 13] while commenting on the possible use of transverse SG forces in storage rings. See Article III.

Note also that the splitting of spin energy levels is not simply proportional to $g/2$ but contains a Thomas precession term, which indicates that the spin motion is coupled to the orbital motion. Furthermore, the average energy of a synchrotron radiation photon is tens of KeV. This is many orders of magnitude greater than the separation of spin levels. Moreover, photons emitted during spin flip tend to have higher energies than those emitted without spin flip. In addition, the polarisation does not reverse its sign with respect to the magnetic field at $g = 0$ but at $g \approx 1.2$ [7, 40, 48]. This results from the fact that $(1/\gamma - 1 + g/2)$ appears in the Hamiltonian h^{dk}_{int} (Eq. (10))instead of just $g/2$.

Finally, it is interesting to note that the synchrotron radiation spectrum and the polarisation effects just depend on the curvature (i.e. the geometry) of the orbit [48]. So the same effects could be obtained by using electric fields to bend the trajectory instead of magnetic fields.

These comments should convince the reader that in the laboratory frame we are not dealing with a simple two level spin system. For further discussions relevant to this topic the reader is directed to the articles by W. G. Unruh and J.D. Jackson in these proceedings and elsewhere [40].

5.3 Radiative depolarisation

The stochastic element of photon emission together with damping determines the equilibrium phase space density distribution. The same photon emission also imparts a stochastic element to the $\vec{u}$ in $\vec{\Omega}(\vec{u}; s)$ and then, via the T-BMT equation applied to spin motion in the (inhomogeneous) focusing fields and in a simple classical picture, spin diffusion (and thus depolarisation) can occur [52]. The polarisation level reached is the result of a balance between the Sokolov–Ternov effect and this radiative depolarisation. In practice, depolarisation can be strong and it is therefore essential that it is well understood.

But how can we calculate the equilibrium polarisation? After all, the polarisation at a point in phase space is the average of the unit spins $2\vec{\xi}/\hbar$ contained in a small packet of phase space at that point. Now, for protons, the phase space density is conserved along a trajectory so that no particles are lost from such a packet but for electrons their stochastic motion means that spins are continually diffusing from packet to packet. For the orbital motion one then employs a Fokker–Planck equation for the particle density. But polarisation is not a density so that it is not immediately clear how to proceed. Moreover the ST effect must be included so that an analogue of the BKS expression for stochastic orbits is needed. I will mention the best solution to this puzzle later but in the meantime I will follow a path which roughly reflects the way that estimates have been made in practice.

A clue to the next step is contained in the above comments about the equilibrium phase space distribution resulting from weak dissipation. There, the phase space distribution settles down to a distribution close to an invariant solution for the dissipationless problem but with the widths of the distribution determined by the radiation. Assuming that one has significant asymptotic polarisation the characteristic depolarisation time must be similar to the polarisation time, namely minutes or hours. Both are orders of magnitude larger than the orbital damping times [u]. Thus the analogue for the polarisation would be that the direction of the equilibrium polarisation at each point in phase space would settle down close to the equilibrium solution of the radiationless problem, namely $\hat{n}(\vec{u}; s)$. Furthermore, the 'spin emittance' i.e. the average of $\check{I}_{spin} = \vec{\xi} \cdot \hat{n}$ at each point in phase space, would be independent of $\vec{u}$ and s.

As has been customary I will now adopt these plausible notions as working *assumptions* that at equilibrium a) the polarisation is parallel to $\hat{n}(\vec{u}; s)$ and b) the *value* of the polarisation is independent of $\vec{u}$ and s. In particular, it is assumed that the spin tune hardly varies across phase space so that there are no 'local' spin–orbit resonances and therefore no polarisation 'absorbers'. I will offer support for the first assumption at the end of this article but in the meantime some support for these assumptions comes from noting that by integrating the BKS equation along a deterministic synchrobetatron orbit the polarization settles down very nearly parallel to $\hat{n}$ [53] in analogy with the solution on the closed orbit (Eq. 32) [v]. Furthermore, a study of a special but exactly solvable model of spin diffusion [15] shows that far from resonance the polarization settles down asymptotically very nearly parallel to $\hat{n}$.

This picture was first proposed by Derbenev and Kondratenko [7]. In the absence of radiation $s_n = (\hbar/2)\vec{\sigma}_{op} \cdot \hat{n}$ is conserved. But in the presence of radiation one has

$$\frac{ds_n}{dt} = \frac{i}{\hbar}[h^{dk}_{rad} + h^{dk}_{int}, s_n]\,. \tag{34}$$

This is evaluated in the equations following Eq. (4.2) in [7] and by writing $\vec{\mathfrak{s}} = (\hbar/2)\vec{\sigma}_{op}$ the essence of the physics can be stated (very) symbolically in the form:

$$\frac{ds_n}{dt} = \frac{d\vec{\mathfrak{s}}}{dt} \cdot \vec{n} + \vec{\mathfrak{s}} \cdot \frac{d\vec{n}}{dt}\,. \tag{35}$$

The first term describes the rate of change of s_n due to pure spin flip at a point in phase space (pure ST effect). The consequent build–up of polarisation is a 'spin damping' analogous to orbital damping. The second term describes the change in s_n due to the fact that when a photon is emitted, the particle jumps without a change of spin to a new position in phase space where it finds a new $\hat{n}$ which will in general not be parallel to the $\hat{n}$ at the initial point. The projection of the spin on the $\hat{n}$–axis has thus decreased stochastically so that s_n diffuses in analogy with the diffusion of

[u] Time scales [17] again!

[v] 'very nearly' means that the angle between the polarization and $\hat{n}$ is much less than the angle between $\hat{n}_0$ and $\hat{n}$.

the orbital actions. This is where the depolarisation comes in. Thus the effect on the polarisation of the stochastic journey of a particle though phase space is accounted for by defining an appropriate quantisation axis at each point in phase space. Photon emission imparts both transverse and longitudinal recoils to the electron but since a photon is emitted typically within an angle $1/\gamma$ with respect to the direction of the electron, the effect of the longitudinal recoil (i.e. the energy jump) dominates: the electron remains at almost the same point in x and z but can suffer a significant change in energy. Then by neglecting the effect of transverse recoil Derbenev and Kondratenko arrive at the following expression for the equilibrium polarisation along the axes $\hat{n}$:

$$P_{dk} = -\frac{8}{5\sqrt{3}} \; \frac{\oint ds \left\langle \hat{b} \cdot (\hat{n} - \frac{\partial \hat{n}}{\partial \delta})/|\rho(s)|^3 \right\rangle_s}{\oint ds \left\langle (1 - \frac{2}{9}(\hat{n} \cdot \hat{s})^2 + \frac{11}{18} \left(\frac{\partial \hat{n}}{\partial \delta}\right)^2)/|\rho(s)|^3 \right\rangle_s} \qquad (36)$$

where $\langle\ \rangle_s$ denotes an average over phase space at azimuth s and $\delta = \Delta E/E_0$ is the fractional energy deviation from the design energy [w]. This formula differs from Eq. (32) by the inclusion of the terms with $\partial\hat{n}/\partial\delta$ and use of $\hat{n}$ instead of $\hat{n}_0$. The derivative $\partial\hat{n}/\partial\delta$ is a measure of the change of $\hat{n}$ caused by fractional energy jumps δ and its presence corresponds to the fact that the main consequence of a photon emission is a change in particle energy. The phase space average of the polarisation is

$$\vec{P}_{av,dk}(s) = P_{dk} \; \langle \hat{n} \rangle_s \qquad (37)$$

and $\langle \hat{n} \rangle_s$ is very nearly aligned along $\hat{n}_0(s)$ (see the angle estimate below). The *value* of the phase space average, $P_{av,dk}(s)$, is essentially independent of s.

The effect of transverse recoil can also be included but contributes derivative terms (see Article IV, Eq. (2)) analogous to $\partial\hat{n}/\partial\delta$ which are typically a factor γ smaller than $\partial\hat{n}/\partial\delta$ and can usually be neglected [54, 55]. This point will be dealt with again in Article IV .

In the presence of radiative depolarisation Eq. (33) becomes

$$\tau_{dk}^{-1} = \frac{5\sqrt{3}}{8} \frac{e^2\gamma^5\hbar}{m_e^2c^2} \frac{1}{C} \; \oint ds \left\langle \frac{1 - \frac{2}{9}(\hat{n} \cdot \hat{s})^2 + \frac{11}{18} \left(\frac{\partial \hat{n}}{\partial \delta}\right)^2}{|\rho(s)|^3} \right\rangle_s . \qquad (38)$$

Away from the spin–orbit resonances of Eqs. (21a, b) $\hat{n}(\vec{u}; s) \approx \hat{n}_0(s)$. But near resonances $\hat{n}(\vec{u}; s)$ deviates from $\hat{n}_0(s)$ by typically several tens of milliradians at a few tens of GeV and the deviation increases with distance in phase space from the closed orbit. The *spin orbit coupling function* $\partial\hat{n}/\partial\delta$, whose square $(\partial\hat{n}/\partial\delta)^2$ in Eq. (36) quantifies the depolarisation, can then be large and the equilibrium polarisation can then be small. For example if $|\partial\hat{n}/\partial\delta|$ is 1 the polarisation will not exceed about 57%.

Note that even very close to resonances, $|\langle \hat{n} \rangle_s| \approx 1$: the phase space average polarisation measured by a polarimeter is mainly influenced by the value of P_{dk} in Eq. (37).

[w] This is sometimes written as $\delta\gamma/\gamma$ [7]. See also footnote j.

The nice thing about this formulation is that a very complicated calculation of the effects of radiation has been distilled into a formula involving a few strange coefficients (emanating from the radiation theory) and a classical solution to the T–BMT equation, namely $\hat{n}$ whose behaviour encapsulates all of the depolarisation effects.

To get high polarisation, one must have $(\partial\hat{n}/\partial\delta)^2 \ll 1$ in dipole magnets. If $\hat{n}$ is independent of the position in phase space, the derivative is zero: all points in phase space have the same quantisation axis and there is no depolarisation. But storage ring fields are inhomogeneous so that $\hat{n}$ varies across phase space. Thus the vector $\frac{\partial\hat{n}}{\partial\delta}$ depends on the optic of the machine. The optimisation of the optic required to make $\partial\hat{n}/\partial\delta$ small is called *spin matching* [23]. This will be mentioned again in Article II. An example can be found in [56].

The term linear in $\partial\hat{n}/\partial\delta$ in Eq. (36) is due to a correlation between the spin orientation and the radiation power [17, 57]. Alternatively, it can be considered to result from the interference between the two terms in Eq. (35). In rings where $\hat{n}_0$ is horizontal due, say, to the presence of a solenoid Siberian Snake [58], $\partial\hat{n}/\partial\delta$ has a vertical component in the dipole fields. This can lead to a build-up of polarisation (*'kinetic polarisation'*) even though the pure Sokolov–Ternov effect vanishes. The rate is τ_{dk}^{-1}.

The expression for τ_{dk}^{-1} in Eq. (38) can be found from a purely classical calculation of spin diffusion by evaluating the effects of the second term in Eq. (35) or by other means [30, 28, 43, 23] and indeed this was the first use for $\hat{n}$ [30]. Then we have a mixed calculation: the spin flip is described by quantum mechanics and the depolarisation is described by classical diffusion. But obviously kinetic polarisation will not be found by that route and the exotic resonance structure examined in Article IV would be missed. So it is clear that a quantum mechanical approach is needed to get the full picture. An observation of kinetic polarisation [58] would be a nice vindication of this viewpoint.

The Derbenev-Kondratenko formula (Eq. (36)) has been rederived in a very elegant way by Mane [27]. He introduces the concept of generalised spin flip whereby he calculates the transition rates due to photon emission from 'spin up along $\hat{n}(x, p_x, z, p_z, \Delta t, \Delta E; s)$' to 'spin down along $\hat{n}(x, p_x, z, p_z, \Delta t, \Delta E - \Delta_{ph}; s)$' where Δ_{ph} is the energy of the emitted photon. One also needs the corresponding 'spin down' to 'spin up' rates. Then by requiring that the polarisation has the same value over all of phase space and imposing the constraint that the total generalised up–to–down rate equals the total down–to–up rate, and solving for the polarisation one arrives at Eq. (36)! In this formulation, the concept of depolarisation never arises! Instead one just has a statistical spin equilibrium. In a perturbative calculation of $\langle(\partial\hat{n}/\partial\delta)^2\rangle$ [27] one finds a series expansion in powers of emittances, i.e. in powers of $\hbar$, in which each term contains a product of resonance denominators which impart the resonant structure to $\hat{n}$. As the order of $\hbar$ increases along the series so does the order of the resonances contained in the successive terms.

Apart from the calculation of Bell and Leinaas (Article IV), there are two further quantum calculations which should be mentioned, namely those of Hand and Skuja [55, 59]. They choose $\hat{n}_0$ as the spin quantisation axis. When a photon is emitted, the electron jumps to a new orbit. By writing the phase space coordinates as functions of the radiation fields and including damping phenomenologically, they calculate the rates of spin flip along $\hat{n}_0$ [55] and obtain an expression for the equilibrium polarisation which is equivalent to the Derbenev–Kondratenko–Mane (DKM) expression in the limit in which the derivatives of $\hat{n}$ are evaluated in the linear approximation as in the SLIM formalism [23, 21, 22]. Thus they only find the first order spin–orbit resonances [23]. Indeed, since their (quantum) representation of the orbit has a form similar to the classical representation in [60] their final expression contains terms which are equivalent and identical in form to terms obtained in [60] in a model of classical spin diffusion. The fact that they only find first order resonances can then be traced to their use of just a first order perturbative calculation and the choice of $\hat{n}_0$ as quantisation axis. To find the higher order resonances one should use $\hat{n}$. However, in a second calculation [59], again using $\hat{n}_0$, they calculated to higher order and although the outcome is not very transparent, the terms beyond the leading order in $\hbar$ contain high order resonant behaviour. But in the end the moral seems to be that it is more efficient to choose a quantisation axis (i.e. an unperturbed eigenstate), namely $\hat{n}$, which reflects the physics and invest the numerical effort in working with this. In this respect the DKM formalism provides a clean practical framework in which to calculate higher order effects. Radiation fields can cause spin flip. Now we can see how, by treating the external fields experienced by the spin as functions of the radiation fields, spin precession can be regarded as spin flip as suggested at the beginning.

We have seen that there are several ways to approach the estimation of the equilibrium polarisation. In practice analytical calculations are carried out using the DKM formalism and this has been a successful and essential tool for predicting the main qualitative features of polarisation in electron storage rings. However, I now present an approach which is in many ways more satisfying.

5.4 Phase space and polarisation evolution equations

Earlier, while discussing the difficulty of finding a Fokker–Planck treatment for the polarisation I promised further insights and they follow now.

We have seen that the orbital phase space density w obeys the Fokker–Planck equation, Eq. (24). Then if the ST effect is ignored and it is recognised that spin is a passenger subject to the T–BMT equation it may be demonstrated using a classical picture [15, 61] that the spin diffusion is described by the 'Bloch' equation:

$$\frac{\partial \vec{\mathcal{P}}}{\partial s} = \mathcal{L}_{_{FP,orb}}\, \vec{\mathcal{P}} + \vec{\tilde{\Omega}} \wedge \vec{\mathcal{P}}\ , \tag{39}$$

where $\vec{\tilde{\Omega}}$ was defined by Eq. (16) and $\vec{\mathcal{P}}$ is the *'polarisation density'* which is defined as $2/\hbar\times$(density in phase space per particle of spin angular momentum). The Bloch equation for the polarisation density is linear and it is universal in the sense that it does not contain the phase space density [15]. It is also valid far from spin–orbit equilibrium. So the trick is to work with the polarisation density instead of the local polarisation $\vec{P}(\vec{u};s)$. It is simple to show that $\vec{\mathcal{P}}(\vec{u};s) = \vec{P}(\vec{u};s)w(\vec{u};s)$ and then by combining Eqs. (17) and (19) it is already clear that in the absence of radiation, $\vec{\mathcal{P}}$ obeys the radiationless limit of Eq. (39), namely Eq. (40) below. The evolution equation for the local polarisation in the presence of radiation is obtained by combining Eq. (24) with Eq. (39). One finds that it has a more complicated form than Eq. (39) owing to the presence of the second derivatives in the $\mathcal{L}_2$ in $\mathcal{L}_{_{FP,orb}}$. It also contains the phase space density so that it is not universal. But the local polarisation can always be obtained instead as $\vec{\mathcal{P}}(\vec{u};s)/w(\vec{u};s)$.

The ST effect can be included by adding in terms from the BKS equation multiplied by the phase space density, together with some terms to represent the interference between ST effect and diffusion[15] and in fact the full Bloch equation for the polarisation density and the Fokker–Planck equation for the phase space density can be obtained from quantum radiation theory [42].

In the absence of radiation we obtain:

$$\frac{\partial \vec{\mathcal{P}}}{\partial s} \;=\; \{\tilde{h}_{orb}, \vec{\mathcal{P}}\} + \tilde{\vec{\Omega}} \wedge \vec{\mathcal{P}} \; . \tag{40}$$

From here it is easy to see, in analogy with the case of the phase space, that since the orbital damping, orbital diffusion and ST terms are very small compared to the remaining symplectic and T–BMT terms, the equilibrium (i.e. periodic) $\vec{P}(\vec{u};s)$ will indeed be almost parallel to $\hat{n}(\vec{u};s)$, at least away from resonances [x].

The Bloch equation for the polarisation density is free from assumptions of the kind we needed earlier and in principle it allows us to calculate everything we need from scratch by looking at the beam as a whole instead of focusing on individual particles to begin with. It is clearly the best starting point for discussing radiative polarisation. Furthermore, the spin diffusion part (Eq. (39)) can be set up for any source of noise in the orbital motion — we just need the appropriate $\mathcal{L}_{_{FP,orb}}$. For example it could be applied to scattering of protons by gas molecules.

Conclusion

Spin polarisation in high energy storage rings is an exciting and exotic topic. I hope that the reader now has a solid overview of the status of our understanding and will pass on to Articles II, III, IV and V. An overview of the experimental aspects of electron polarization and plans for the future can be found in [63, 64].

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[x]The commentary in [62] on the calculations described in Article IV [54], contains the opinion that it is unsafe to employ $\hat{n}$ as the appropriate quantisation axis. The above discussion should be an adequate response.

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