Red, blue and green electrons

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Abstract: If the Fokker–Planck equation for the phase space density of electrons in a storage ring is given, the corresponding equation for the polarization density has a related, simple and elegant form which can be deduced by a simple illuminating argument.

1 Introduction

If the phase space density $W_{\rm orb}(\vec{u};s)$ of electrons in a storage ring evolves according to a Fokker–Planck equation:

$$\frac{\partial W_{\rm orb}}{\partial s} = \mathcal{L}_{\rm FP,orb}\; W_{\rm orb}\;, \tag{1}$$

where s is the distance around the ring, $\vec{u}$ is the vector of the six canonical phase space coordinates and $\mathcal{L}_{\rm FP,orb}$ is the Fokker–Planck operator for the orbital motion, how can we write a corresponding equation for the transport of spin?

The solution is to work with the polarization density $\vec{\mathcal{P}}(\vec{u};s)$ which is defined as $2/\hbar\ \vec{\mathcal{S}}$ where $\vec{\mathcal{S}}$ is the phase space density per particle of the spin angular momentum. It can then be shown [1] that $\vec{\mathcal{P}}(\vec{u};s)$ satisfies the equation:

$$\frac{\partial \vec{\mathcal{P}}}{\partial s} \;=\; \mathcal{L}_{\rm FP,orb}\vec{\mathcal{P}} + \vec{\Omega}(\vec{u};s)\times\vec{\mathcal{P}}\;, \tag{2}$$

where $\vec{\Omega}(\vec{u};s)$ is the spin precession vector in the Thomas–BMT spin precession equation, $d\vec{S}/ds = \vec{\Omega}\times\vec{S}$, describing the coupling of the spin expectation value $\vec{S}$ (“the spin”) of a particle to the electric and magnetic fields [2, 3].

A relationship as elegant and simple as that between Eqs. (1) and (2) begs a simple intuitive interpretation. We offer this in the next section.

2 A colourful analogy

Imagine that the particles carry three continuous "charges" which, for amusement, we can call redness, blueness and greeness and which are parametrized by the variables R, B, G which we group into a three component colour vector $\vec{C}$. Imagine also that these charges do not couple to the electric and magnetic fields so that the dynamics is blind to colour. The joint density in phase and colour space is $W(\vec{u}, \vec{C}; s)$ and $W_{\rm orb}(\vec{u}; s) = \int W(\vec{u}, \vec{C}; s)\ d^3C$.

Since the dynamics is blind to colour $W(\vec{u}, \vec{C}; s)$ obeys the usual transport equation:

$$\frac{\partial W(\vec{u}, \vec{C}; s)}{\partial s} = \mathcal{L}_{\rm FP,orb} W(\vec{u}, \vec{C}; s)\ . \tag{3}$$

Likewise the phase space density of the colour $\vec{\mathcal{C}}(\vec{u}; s) = \int \vec{C}\ W(\vec{u}, \vec{C}; s)\ d^3C$ also obeys this equation:

$$\frac{\partial \vec{\mathcal{C}}}{\partial s} = \mathcal{L}_{\rm FP,orb}\ \vec{\mathcal{C}}\ . \tag{4}$$

So the colour density vector obeys the same evolution equation as the phase space density! If there is a constraint such as $R^2 + B^2 + G^2 = A^2 = \text{constant}$, then the colour vectors lie on a spherical shell in colour space: $W = \tilde{W}\ \delta(\sqrt{R^2 + B^2 + G^2} - |A|)$.

We now make the replacements: $R \longrightarrow S_x,\ B \longrightarrow S_z,\ G \longrightarrow S_s,\ \vec{C} \longrightarrow \vec{S}$ where S_x, S_z and S_s are the spin components of the spin vector $\vec{S}$ of a particle and in analogy to Eq. (4) it is now clear that if spin were not to couple directly to electric and magnetic fields we would have

$$\frac{\partial \vec{\mathcal{P}}}{\partial s} = \mathcal{L}_{\rm FP,orb}\ \vec{\mathcal{P}}\ . \tag{5}$$

Equation (5) contains all contributions to $\partial \vec{\mathcal{P}}/\partial s$ from instantaneous orbital motion for the case where spin is just a spectator, i.e. in the absence of spin–to–field coupling. The alert reader will realise that this derivation just reflects the derivation of the first two terms of Eq. (11) in [1] using the first two terms in Eq. (10) in [1].

But spins precess in electric and magnetic fields according to the Thomas–BMT equation and $\vec{\mathcal{P}} = \vec{P}_{\rm loc}(\vec{u}; s)\ W_{\rm orb}$ where $W_{\rm orb}$ is now normalized to unity and $\vec{P}_{\rm loc}(\vec{u}; s)$ is the polarization at $(\vec{u}; s)$. Thus precession of $\vec{P}_{\rm loc}$ should be reflected in the precession of $\vec{\mathcal{P}}$ at the rate $\vec{\Omega}(\vec{u}; s) \times \vec{\mathcal{P}}$.

We now note [1] that the operator $\mathcal{L}_{\rm FP,orb}$ can be written as

$$\mathcal{L}_{\rm FP,orb} = \mathcal{L}_{\rm ham} + \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 \tag{6}$$

where $\mathcal{L}_{\rm ham}$ is the "Hamiltonian part" and accounts for noiseless and dissipationless motion. For "Hamiltonian motion", $dW_{\rm orb}/ds$ vanishes along trajectories so that $W_{\rm orb}$ is conserved. If there were no spin–to–field coupling $\vec{P}_{\rm loc}$ and $\vec{\mathcal{P}}$ would be conserved too. The terms $\mathcal{L}_0,\ \mathcal{L}_1,\ \mathcal{L}_2$ account for the deviation from "Hamiltonian motion" due to damping and noise and contain zeroth, first and second order derivatives w.r.t. the components of $\vec{u}$ respectively. The precession of $\vec{\mathcal{P}}$ is also a deviation from Hamiltonian motion. So to include spin–to–field coupling we just add a term $\vec{\Omega}(\vec{u}; s) \times \vec{\mathcal{P}}$ to the r.h.s. of Eq.(5). We have now arrived at Eq. (2) by heuristic arguments and by this means we have exposed the essence of the matter.

Note that the relationship between equations (1) and (2) survives if $\mathcal{L}_{\rm FP,orb}$ is replaced by any physically sensible transport operator $\mathcal{T}_{\rm orb}$ — which could even contain derivatives beyond second order.

3 Commentary

From Eqs.(1) and (2) and the relation $\vec{\mathcal{P}} = \vec{P}_{\rm loc}\, W_{\rm orb}$ it is simple to show that the corresponding equation for $\vec{P}_{\rm loc}$ is

$$\frac{\partial \vec{P}_{\rm loc}}{\partial s} \; = \; \mathcal{L}_{\rm ham} \vec{P}_{\rm loc} + \vec{\Omega} \times \vec{P}_{\rm loc} + \mathcal{L}_1 \; \vec{P}_{\rm loc} + \frac{\mathcal{L}_2 \; (\vec{P}_{\rm loc} \; W_{\rm orb}) - \vec{P}_{\rm loc}(\mathcal{L}_2 \; W_{\rm orb})}{W_{\rm orb}} \, . \tag{7}$$

This is considerably more complicated than Eq. (2) so that to obtain $\vec{P}_{\rm loc}$ it is better to solve Eqs. (1) and (2) separately and calculate $\vec{\mathcal{P}}/W_{\rm orb}$.

By appealing again to our heuristic picture, equation (2) can be extended to include other effects such as intrinsic spin flip due to the Sokolov—Ternov effect [4] so that we obtain

$$\frac{\partial \vec{\mathcal{P}}}{\partial s} \; = \; \underbrace{\underbrace{\mathcal{L}_{\rm ham} \vec{\mathcal{P}} + \vec{\Omega}(\vec{u}; s) \times \vec{\mathcal{P}}}_{\equiv \text{Noise and damping free part}}}_{\text{Large}} \; \underbrace{+ \mathcal{L}_0 \; \vec{\mathcal{P}} + \mathcal{L}_1 \; \vec{\mathcal{P}} + \mathcal{L}_2 \; \vec{\mathcal{P}} + \underbrace{\text{Intrinsic spin flip}}_{\text{e.g.Sokolov}-\text{Ternov}} + \underbrace{\text{Cross terms}}_{\text{e.g. Kinetic pol}}}_{\text{Small}} \tag{8}$$

and thereby arrive at an interpretation of the calculation in [6].

Equation (8) provides a clean starting point for perturbative estimates of the combined effects of damping, noise and spin flip close to equilbrium. In fact the part $\partial \vec{\mathcal{P}}/\partial s = \mathcal{L}_{\rm ham} \vec{\mathcal{P}} + \vec{\Omega}(\vec{u}; s) \times \vec{\mathcal{P}}$ in Eq. (8) is just the T–BMT equation for $\vec{\mathcal{P}}$ and the direction of the stationary (i.e. ring periodic) solution for $\vec{\mathcal{P}}$ is the invariant spin field $\vec{n}(\vec{u}; s)$ [5]. Since $\vec{\Omega}(\vec{u}; s)$ is usually large one can then evaluate the influence of the "Small" terms using perturbation theory to obtain expressions for the equilbrium polarization and the depolarization rate in terms of functionals of $\vec{n}$ and its derivatives [7]. Near to spin–orbit resonances [5], $\vec{n}$ is a sensitive function of $\vec{u}$ and this sensitivity feeds through to enhance the rate of depolarization near resonances. Further details will be given elsewhere.

References

[1] K. Heinemann and D. P. Barber, DESY report DESY 98-145 and at Los Alamos archive: physics/9901045. To be published in Nuovo Cimento A.

[2] L. Thomas, Philosophical Magazine, **3**, 1 (1927).

[3] V. Bargmann, L. Michel and V. L. Telegdi, Phys.Rev.Letts., **2**, 435 (1959).

[4] A. A. Sokolov and I. M. Ternov, Sov.Phys.Dokl., **8** $\mathrm{N}^0$12, 1203 (1964).

[5] Articles by D.P. Barber (some with co-authors), in proceedings of the ICFA workshop "Quantum Aspects of Beam Physics", Monterey, U.S.A., 1998, World Scientific (1999). Also as DESY Report 98-96 and at the Los Alamos archive: physics/9901038 – physics/9901044.

[6] Ya.S. Derbenev and A.M. Kondratenko, Sov.Phys.Dokl., **19**, 438 (1975).

[7] K. Heinemann, DESY report 97-166 (1997) and Los Alamos archive: physics/9709025.