Radiative Polarization, Computer Algorithms and Spin Matching in Electron Storage Rings [a]

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We present a set of notes, meant for quick reference, on radiative spin polarization, computer algorithms and spin matching in electron storage rings.

2.7.7. Radiative Polarization in Electron Storage Rings

The Sokolov-Ternov effect [1] Relativistic electrons in a storage ring emit synchrotron radiation (Sec.3.1 in [29]). A very small fraction of the radiated photons cause spin flip. For electron spins aligned along a uniform magnetic field, the $\uparrow\downarrow$ and $\downarrow\uparrow$ flip rates differ and this leads to a build-up of spin polarization antiparallel to the field. Positrons become polarized parallel to the field. The transition rates for electrons are

$$\begin{aligned} W_{\uparrow\downarrow} &= \frac{5\sqrt{3}}{16}\frac{r_{\rm e}\gamma^5\hbar}{m_{\rm e}|\rho|^3}\left(1+\frac{8}{5\sqrt{3}}\right) \\ W_{\downarrow\uparrow} &= \frac{5\sqrt{3}}{16}\frac{r_{\rm e}\gamma^5\hbar}{m_{\rm e}|\rho|^3}\left(1-\frac{8}{5\sqrt{3}}\right) \end{aligned} \tag{1}$$

For positrons, interchange plus and minus signs here and elsewhere [b].

The equilibrium polarization in a uniform magnetic field is independent of γ,

$$P_{\rm st} = \frac{W_{\uparrow\downarrow}-W_{\downarrow\uparrow}}{W_{\uparrow\downarrow}+W_{\downarrow\uparrow}} = \frac{8}{5\sqrt{3}} = 0.9238 \tag{2}$$

For a beam with zero initial polarization, the time dependence for build-up to equilibrium is

$$P(t) = P_{\rm st}\left[1-\exp\left(-t/\tau_0\right)\right] \tag{3}$$

where the build-up rate is

$$\tau_0^{-1} = \frac{5\sqrt{3}}{8}\frac{r_{\rm e}\gamma^5\hbar}{m_{\rm e}|\rho|^3} \tag{4}$$

[a] Contributions to the Handbook of Accelerator Physics and Engineering, Eds. A.W. Chao and M. Tigner, 2nd edition, World Scientific, 2002.

[b] ρ is the radius of curvature of the orbit, $r_{\rm e}$ is the classical electron radius and the other symbols have their usual meanings.

τ_0 depends strongly on γ and ρ but is typically minutes or hours. In a flat ring in which all bending magnets have the same ρ just average Eq.(4) over the circumference C:

$$\tau_0^{-1}[\mathrm{s}^{-1}] \;\approx\; \frac{2\pi}{99}\frac{E[\mathrm{GeV}]^5}{C[\mathrm{m}]\rho[\mathrm{m}]^2} \tag{5}$$

The Baier-Katkov flip rate For electron spins initially aligned along an arbitrary unit vector $\hat{\xi}$ the generalization of Eq.(1) is [2]

$$W \;=\; \frac{1}{2\tau_0}\left[1-\frac{2}{9}(\hat{\xi}\cdot\hat{s})^2+\frac{8}{5\sqrt{3}}\,\hat{\xi}\cdot\hat{b}\right] \tag{6}$$

where $\hat{s}$ = direction of motion and $\hat{b}=(\hat{s}\times\dot{\hat{s}})/|\dot{\hat{s}}|$. $\hat{b}$ is the magnetic field direction if the electric field vanishes and the motion is perpendicular to the magnetic field.

The corresponding instantaneous rate of build-up of polarization along $\hat{\xi}$ is

$$\tau_{\mathrm{bk}}^{-1} \;=\; \tau_0^{-1}\left[1-\frac{2}{9}(\hat{\xi}\cdot\hat{s})^2\right] \tag{7}$$

The T-BMT equation Neglecting radiative spin flip, the motion of the centre-of-mass spin expectation value $\vec{\xi}$ of a relativistic charged particle traveling in electric and magnetic fields is governed by the Thomas-BMT equation $d\vec{\xi}/dt=\vec{\Omega}\times\vec{\xi}$ (Sec.2.7.1 in [29]).

We write

$$\vec{\Omega}=\vec{\Omega}^{\mathrm{co}}+\vec{\omega}^{\mathrm{sb}} \tag{8}$$

where $\vec{\Omega}^{\mathrm{co}}$ is due to the fields on the closed orbit, whence $\vec{\Omega}^{\mathrm{co}}(s+C)=\vec{\Omega}^{\mathrm{co}}(s)$. $\vec{\Omega}^{\mathrm{co}}=\vec{\Omega}^{\mathrm{ref}}+\vec{\omega}^{\mathrm{imp}}$, where $\vec{\Omega}^{\mathrm{ref}}$ contains the design fields and $\vec{\omega}^{\mathrm{imp}}$ represents the effects of magnet misalignments, correction fields etc. $\vec{\omega}^{\mathrm{sb}}$ is due to synchrotron and/or betatron motion with respect to the closed orbit.

On the closed orbit the T-BMT equation

$$\frac{d}{dt}\vec{\xi} \;=\; \vec{\Omega}^{\mathrm{co}}\times\vec{\xi} \tag{9}$$

can be solved in the form

$$\vec{\xi}(s) \;=\; \mathbf{R}^{\mathrm{co}}_{3\times3}(s,s_0)\vec{\xi}(s_0) \tag{10}$$

where $\mathbf{R}^{\mathrm{co}}_{3\times3}$ is a rotation matrix. The real unit eigenvector (rot. axis) for the one turn matrix $\mathbf{R}^{\mathrm{co}}_{3\times3}(s+C,s)$, denoted by $\hat{n}_0(s)$, is the periodic spin solution on the closed orbit. For a perfectly aligned flat ring with no solenoids, $\hat{n}_0(s)=\pm\hat{y}$. The one turn matrix has a complex conjugate pair of eigenvalues $e^{\pm i2\pi\nu_{\mathrm{spin}}}$. Given $\hat{n}_0$, we introduce a pair of unit vectors $(\hat{m}_0,\hat{l}_0)$ such that $\hat{n}_0\times\hat{m}_0=\hat{l}_0$, $\hat{l}_0\times\hat{n}_0=\hat{m}_0$, fulfill Eq.(9), and obey

$$\hat{m}_0(s_0+C)+i\hat{l}_0(s_0+C) \;=\; e^{i2\pi\nu_{\mathrm{spin}}}\left[\hat{m}_0(s_0)+i\hat{l}_0(s_0)\right] \tag{11}$$

The $(\hat{m}_0, \hat{l}_0)$ are usually not periodic in s. But by applying a further rotation by an angle $\psi_{\rm spin}(s)$ around $\hat{n}_0$ we can construct the vectors $(\hat{m},\, \hat{l})$,

$$\hat{m}(s) + i\,\hat{l}(s) \;\;=\;\; e^{-i\psi_{\rm spin}(s)}\left[\hat{m}_0(s) + i\hat{l}_0(s)\right] \tag{12}$$

By choosing $\psi_{\rm spin}(s+C) - \psi_{\rm spin}(s) = 2\pi\nu_{\rm spin}$, the set $(\hat{n}_0, \hat{m}, \hat{l})$ is then periodic in s with period C. The vectors $(\hat{m}, \hat{l})$ are needed in Sec 2.7.8.

The closed orbit spin tune $\nu_{\rm spin}$ is the number of spin precessions per turn around $\hat{n}_0$. For a perfectly aligned flat ring without solenoids $\nu_{\rm spin} = a\gamma$, where $a = (g-2)/2$ (see Sec.2.7.1 in [29]). In this section and in Sec. 2.7.8 we use the symbol "a" instead of the symbol "G" used in the rest of the Handbook. Only the fractional part of the spin tune can be extracted from the numerical values of the eigenvalues $e^{\pm i2\pi\nu_{\rm spin}}$. For the definition of spin tune away from the closed orbit see [3].

The Baier-Katkov-Strakhovenko (BKS) equation Neglecting the effect of stochastic (synchrotron radiation) photon emission on the orbit and imagining that all particles remain on the closed orbit, the equation of motion for electron polarization is [4, 5]

$$\frac{d\vec{P}}{dt} \;\;=\;\; \vec{\Omega}^{\rm co} \times \vec{P} - \frac{1}{\tau_0(s)}\left[\vec{P} - \frac{2}{9}\hat{s}(\vec{P}\cdot\hat{s}) + \frac{8}{5\sqrt{3}}\hat{b}(s)\right] \tag{13}$$

In the case of horizontal motion in a vertical magnetic field, we have $\vec{\Omega} = (a\gamma c/\rho)\hat{y}$, and $\hat{b}(s) = \hat{y}$.

By integrating the BKS equation, one finds the generalized Sokolov-Ternov formula for the asymptotic electron polarization in arbitrary magnetic fields along the closed orbit,

$$\vec{P}_{\rm bks} \;\;=\;\; -\frac{8}{5\,\sqrt{3}}\;\hat{n}_0\;\frac{\oint ds \frac{\hat{n}_0(s)\cdot\hat{b}(s)}{|\rho(s)|^3}}{\oint ds \frac{\left[1-\frac{2}{9}(\hat{n}_0(s)\cdot\hat{s})^2\right]}{|\rho(s)|^3}} \tag{14}$$

See [6] for a compilation of time scales. Usually, in rings containing dipole spin rotators (Secs.2.7.3, 2.7.4 in [29]) the polarization $|\vec{P}_{\rm bks}|$ cannot reach 0.9238 [7].

The BKS polarization build-up rate is

$$\tau_{\rm bks}^{-1} \;\;=\;\; \frac{5\sqrt{3}}{8}\frac{r_{\rm e}\gamma^5\hbar}{m_{\rm e}}\frac{1}{C}\oint ds\,\frac{\left[1-\frac{2}{9}\,(\hat{n}_0\cdot\hat{s})^2\right]}{|\rho(s)|^3} \tag{15}$$

This is in accord with Eq.(7) by replacing $\hat{\xi} \to \hat{n}_0$ and averaging.

Radiative depolarization The stochastic element of photon emission together with damping determines the equilibrium phase space density distribution. The same photon emission also imparts a stochastic element to $\vec{\omega}^{\rm sb}$ and then, via the T-BMT equation, spin diffusion (and thus depolarization) can occur [8]. The polarization is the result of a balance between the Sokolov-Ternov effect and this radiative depolarization. In the approximation that the orbital motion is linear, the *value* of the polarization is essentially the same at each point in phase space and azimuth and the polarization is aligned along the Derbenev-Kondratenko vector $\hat{n}$ [9].

The unit vector field $\hat{n}$, which is also called the *"invariant spin field"* [3, 10, 11], depends on s and $\vec{u} \equiv (x, p_x, y, p_y, z, \delta)$. $\hat{n}(\vec{u}; s)$ satisfies the T-BMT equation at $(\vec{u}; s)$ and is periodic: $\hat{n}(\vec{u}; s) = \hat{n}(\vec{u}; s + C)$. On the closed orbit $\hat{n}(\vec{u}; s)$ reduces to $\hat{n}_0(s)$.

The Derbenev–Kondratenko–Mane formula Taking into account radiative depolarization due to photon-induced longitudinal recoils, the equilibrium electron polarization along the $\hat{n}$ field is [9, 12, 3]

$$P_{\rm dk} \;=\; -\frac{8}{5\sqrt{3}} \frac{\oint ds \left\langle \frac{1}{|\rho(s)|^3} \hat{b} \cdot (\hat{n} - \frac{\partial \hat{n}}{\partial \delta}) \right\rangle_s}{\oint ds \left\langle \frac{1}{|\rho(s)|^3} (1 - \frac{2}{9} (\hat{n} \cdot \hat{s})^2 + \frac{11}{18} \left(\frac{\partial \hat{n}}{\partial \delta}\right)^2) \right\rangle_s} \tag{16}$$

where $< >_s$ denotes an average over phase space at azimuth s. This formula differs from Eq.(14) by the inclusion of the terms with $\frac{\partial \hat{n}}{\partial \delta}$ and use of $\hat{n}$ instead of $\hat{n}_0$. The ensemble average of the polarization is

$$\vec{P}_{\rm ens,dk}(s) \;=\; P_{\rm dk} \, \langle \hat{n} \rangle_s \tag{17}$$

and $\langle \hat{n} \rangle_s$ is very nearly aligned along $\hat{n}_0(s)$ (see the angle estimate below). The *value* of the ensemble average, $P_{\rm ens,dk}(s)$, is essentially independent of s.

The effect of transverse recoil can also be included but contributes derivative terms analogous to $\frac{\partial \hat{n}}{\partial \delta}$ which are typically a factor γ smaller than $\frac{\partial \hat{n}}{\partial \delta}$ and can be neglected unless $\frac{\partial \hat{n}}{\partial \delta}$ is very small [13, 14]. If $\frac{\partial \hat{n}}{\partial \delta}$ were to vanish, a $P_{\rm dk}$ of 99.2 % could be reached [13, 14, 3].

In the presence of radiative depolarization Eq.(15) becomes

$$\tau_{\rm dk}^{-1} \;=\; \frac{5\sqrt{3}}{8} \frac{r_{\rm e} \gamma^5 \hbar}{m_{\rm e}} \frac{1}{C} \oint ds \left\langle \frac{1 - \frac{2}{9} (\hat{n} \cdot \hat{s})^2 + \frac{11}{18} \left(\frac{\partial \hat{n}}{\partial \delta}\right)^2}{|\rho(s)|^3} \right\rangle_s \tag{18}$$

This can be written in the form:

$$\frac{1}{\tau_{\rm dk}} \;=\; \frac{1}{\tau_{\rm st}} + \frac{1}{\tau_{\rm dep}} \, , \tag{19}$$

where $\tau_{\rm st}^{-1}$ can be (very well) approximated by $\tau_{\rm bks}^{-1}$ in (15) and

$$\tau_{\rm dep}^{-1} \;=\; \frac{5\sqrt{3}}{8} \frac{r_{\rm e} \gamma^5 \hbar}{m_{\rm e}} \frac{1}{C} \oint ds \left\langle \frac{\frac{11}{18} \left(\frac{\partial \hat{n}}{\partial \delta}\right)^2}{|\rho(s)|^3} \right\rangle_s \tag{20}$$

The time dependence for build-up from an initial polarization P_0 to equilibrium is

$$P(t) \;=\; P_{\rm ens,dk} \left[1 - e^{-t/\tau_{\rm dk}}\right] + P_0 e^{-t/\tau_{\rm dk}} \tag{21}$$

This formula can be used to calibrate polarimeters (see Eqs.(21) and (22), Sec.2.7.8) [15]. However, the calibration will be imprecise if $\frac{\partial \hat{n}}{\partial \delta}$ in the numerator of Eq.(16) is not well enough known. For examples of build-up curves see [7].

Resonances Away from the spin–orbit resonances (see also Eq.(11), Sec.2.7.8)

$$\nu_{\rm spin} \;=\; k + k_x\nu_x + k_y\nu_y + k_z\nu_z \qquad (22)$$

$\hat{n}(\vec{u};s) \approx \hat{n}_0(s)$. But near resonances $\hat{n}(\vec{u};s)$ deviates from $\hat{n}_0(s)$ by typically tens of milliradians at a few tens of GeV and the deviation increases with distance in phase space from the closed orbit. The *"spin–orbit coupling function"* $\frac{\partial \hat{n}}{\partial \delta}$, which quantifies the depolarization, can then be large and the equilibrium polarization can then be small. Note that even very close to resonances, $|\langle \hat{n} \rangle_s| \approx 1$: the ensemble average polarization is mainly influenced by the value of $P_{\rm dk}$ in Eq.(16).

To get high polarization, one must have $(\partial \hat{n}/\partial \delta)^2 \ll 1$ in dipole magnets. The machine optimization required to make $\frac{\partial \hat{n}}{\partial \delta}$ small is called *"spin matching"* (Sec.2.7.8).

Asymmetric wigglers If $\tau_{\rm bks}^{-1}$ is very low because the energy is low and/or the average curvature is small the polarization rate can be enhanced (see Eq.(15)) by installing an *"asymmetric wiggler"*, i.e. a string of dipoles in which short dipoles with high fields are interleaved with long dipoles with low fields of opposite polarity while ensuring that the field integral of the string vanishes. For more details, and discussion of advantages and disadvantages see [6]. A particular potential disadvantage is that the enhanced radiation loss can require that extra rf power be installed and that the energy spread increases so that the depolarization rate increases owing to stronger synchrotron sideband resonances (Sec.2.7.8).

Kinetic polarization The (numerator) term linear in $\frac{\partial \hat{n}}{\partial \delta}$ in Eq.(16) is due to a correlation between the spin orientation and the radiation power [6]. In rings where $\hat{n}_0$ is horizontal due, say, to the presence of a solenoid Siberian Snake (Secs.2.7.3, 2.7.4 in [29]) [16], $\frac{\partial \hat{n}}{\partial \delta}$ has a vertical component in the dipole fields. This can lead to a build-up of polarization (*"kinetic polarization"*) even though the pure Sokolov–Ternov effect vanishes. The rate is $\tau_{\rm dk}^{-1}$.

Phase space and polarization evolution equations If the orbital phase space density ψ obeys an equation of the Fokker–Planck type (Sec.2.5.4 in [29])

$$\frac{\partial \psi}{\partial s} \;=\; \mathcal{L}_{\rm FP}\ \psi \qquad (23)$$

where $\mathcal{L}_{\rm FP}$ is the orbital Fokker–Planck operator, then the spin diffusion is described by the "Bloch" equation

$$\frac{\partial \vec{\mathcal{P}}}{\partial s} \;=\; \mathcal{L}_{\rm FP}\,\vec{\mathcal{P}} + \vec{\bar{\Omega}} \times \vec{\mathcal{P}} \tag{24}$$

where $\vec{\bar{\Omega}} = \vec{\Omega}/(ds/dt)$ and $\vec{\mathcal{P}}$ is the *"polarization density"* $\equiv 2/\hbar\times$(density in phase space per particle of spin angular momentum) [17, 18]. To include the Sokolov–Ternov effect see [19].

Beam energy calibration A polarized electron beam can be depolarized by applying a weak oscillating magnetic field perpendicular to $\hat{n}_0$ with a frequency $f_{\rm rf}$ related to the fractional part of the spin tune $\tilde{\nu}_{\rm spin}$ by

$$f_{\rm rf} \;=\; f_{\rm c}\tilde{\nu}_{\rm spin} \quad \text{or} \quad f_{\rm rf} \;=\; f_{\rm c}(1-\tilde{\nu}_{\rm spin}) \tag{25}$$

where $f_{\rm c}$ is the circulation frequency of the beam [20]. Thus the required $f_{\rm rf}$ gives an accurate measurement of $\tilde{\nu}_{\rm spin}$ and this gives high relative precision knowledge of $\nu_{\rm spin}$. By relating $\nu_{\rm spin}$ to the average energy of each beam, high precision measurements of the centre–of–mass energy of colliding e^{+}–e^{-} beams and of the masses of vector mesons such as the Υ's and the Z can be obtained [21, 22, 23, 24, 25]. Other beam parameters can also be measured [26]. The polarization need not be large for these measurements so that by Eq.(21) the depolarization can be repeated at intervals of about $\tau_{\rm dk}$.

Concluding remarks For an overview of measurements see [27, 7, 28]. For an overview of the theoretical background see [3].

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2.7.8. Computer Algorithms and Spin Matching

There are two classes of computer algorithm for estimating the equilibrium polarization in real rings:

(i) Methods based on evaluating $\frac{\partial \hat{n}}{\partial \delta}$ in the Derbenev–Kondratenko–Mane (DKM) formula (Eq.(16) of Sec.2.7.7) given the ring layout and magnet strengths; and

(ii) The SITROS [1] algorithm which estimates τ_{dep} (Sec.2.7.7) using Monte–Carlo tracking.

The class (i) algorithms are further divided according to the degree of linearization of the spin and orbital motion:

(ia) The SLIM family (SLIM [2, 3], SLICK [4], SITF [1]) and SOM [5] and ASPIRRIN [6]. The latter two utilize the *"betatron–dispersion"* formalism outlined below and all are based on a linearization of the orbital and spin motion.

(ib) SMILE [7]: Linearized orbital motion but nonlinear spin motion;

(ic) SODOM [8]: Linearized orbital motion but nonlinear spin motion;

(id) SpinLie: Nonlinear orbital motion and nonlinear spin motion (Sec.2.7.9 in [48]); and

(ie) SPRINT [9]: Linearized orbital motion but nonlinear spin motion.

The linear approximation – SLIM We now present expressions for $\frac{\partial \hat{n}}{\partial \delta}$ in an approximation in which the orbit and spin motion are linearized and in which $\vec{\omega}^{\text{sb}}$ (Sec.2.7.7) is linearized as in Eq.(2) below (the SLIM formalism). In linear approximation we write (see Sec.2.7.7)

$$\hat{n}(\vec{u};s) \;=\; \hat{n}_0(s) + \alpha(\vec{u};s)\hat{m}(s) + \beta(\vec{u};s)\hat{l}(s) \qquad (1)$$

valid for $\sqrt{\alpha^2+\beta^2} \ll 1$ and we write the components $\omega_z^{\text{sb}},\ \omega_x^{\text{sb}},\ \omega_y^{\text{sb}}$ in the form [10, 11]

$$\begin{pmatrix} \omega_z^{\text{sb}} \\ \omega_x^{\text{sb}} \\ \omega_y^{\text{sb}} \end{pmatrix} \;=\; \mathbf{F}_{3\times 6} \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix} \qquad (2)$$

where $\vec{u} \equiv (x, p_x, y, p_y, z, \delta)$ describes motion with respect to the closed orbit. In particular $p_x = x'$ and $p_y = y'$ (except in solenoids).

The detailed forms of the matrix $\mathbf{F}_{3\times 6}$ for bending magnets, quadrupoles, skew quadrupoles, solenoids and rf cavities can be found in [11, 12]. The orbit motion in sextupoles is linearized. For example for a quadrupole, defining $\tilde{g} = -(1 + a\gamma)\,g$ where $g = \frac{e}{p_0}\frac{\partial B_y}{\partial x}$ one has

$$\mathbf{F}(s) \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{g} & 0 & 0 & 0 \\ \tilde{g} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

In linear approximation the combined orbit and spin motion is described by 8×8 transport matrices of the form

$$\hat{\mathbf{M}} \;=\; \begin{pmatrix} \mathbf{M_{6\times 6}} & \mathbf{0_{6\times 2}} \\ \mathbf{G_{2\times 6}} & \mathbf{D_{2\times 2}} \end{pmatrix} \tag{4}$$

acting on the vector $(\vec{u},\, \alpha,\, \beta)$, where $\mathbf{M_{6\times 6}}$ is a symplectic matrix describing orbital motion and $\mathbf{G_{2\times 6}}$ describes the coupling of the spin variables $(\alpha,\, \beta)$ to the orbit and depends on $\hat{m}(s)$ and $\hat{l}(s)$ (see e.g. Eq.(14)). $\mathbf{D_{2\times 2}}$ is a rotation matrix associated with the spin basis rotation of Eq.(12) in Sec.2.7.7 [10, 11].

The eigenvectors for one turn defined by $\hat{\mathbf{M}}(s_0 + C, s_0) \cdot \vec{q}_\mu = \hat{\lambda}_\mu \cdot \vec{q}_\mu$ are written in the form

$$\vec{q}_k(s_0) \;=\; \begin{pmatrix} \vec{v}_k(s_0) \\ \vec{w}_k(s_0) \end{pmatrix}, \quad \vec{q}_{-k}(s_0) = [\vec{q}_k(s_0)]^*$$

$$\text{for } k = I,\ II,\ III\ ;$$

$$\vec{q}_k(s_0) \;=\; \begin{pmatrix} \vec{0}_6(s_0) \\ \vec{w}_k(s_0) \end{pmatrix}\ , \quad \vec{q}_{-k}(s_0) = [\vec{q}_k(s_0)]^*$$

$$\text{for } k = IV \tag{5}$$

for arbitrary s_0. The $\vec{v}_k$ are the eigenvectors for orbital motion with eigenvalues $\lambda_k = e^{-i2\pi\nu_k}$ and with $\nu_{-k} = -\nu_k$ $(k = I,\ II,\ III)$. These eigenvectors obey the orthogonality relations, and have the normalization of [2]. The corresponding eigenvalues of $\hat{\mathbf{M}}(s_0 + C, s_0)$ are $\hat{\lambda}_k = \lambda_k$ $(k = I,\ II,\ III)$ and $\hat{\lambda}_{IV} = e^{-i2\pi\nu_{IV}}$ with $\nu_{IV} = \nu_{\text{spin}}$ and with $\nu_{-IV} = -\nu_{IV}$.

The spin parts of the eigenvectors $\vec{w}_k(s_0)$ $(k = I,\ II,\ III)$ and $\vec{w}_{IV}(s_0)$ can be written as

$$\begin{aligned} \vec{w}_k(s_0) \;&=\; -\left[\mathbf{D}(s_0 + C, s_0) - \hat{\lambda}_k\right]^{-1} \mathbf{G}(s_0 + C, s_0)\vec{v}_k(s_0) \\ &\text{for } k = I,\ II,\ III\ ; \\ \vec{w}_{IV}(s_0) \;&=\; \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i\,\psi_{\text{spin}}(s_0)} \\ &\text{for } k = IV \end{aligned} \tag{6}$$

and

$$\vec{w}_{-k}(s_0) \;=\; [\vec{w}_k(s_0)]^*\,, (k = I, II, III, IV)$$

In this linear approximation $\hat{n}(\vec{u}; s)$ can be obtained via [13, 14, 11]

$$\hat{n}(\vec{u}; s) - \hat{n}_0(s) \;\equiv\; \begin{pmatrix} \alpha(\vec{u}; s) \\ \beta(\vec{u}; s) \end{pmatrix} = \sum_{k=I,II,III} \{A_k \vec{w}_k(s) + A_{-k}\vec{w}_{-k}(s)\} \qquad (7)$$

where the amplitudes A_k are determined by the orbit via

$$\vec{u}\,(s) \;=\; \sum_{k=I,II,III} \{A_k \vec{v}_k(s) + A_{-k}\vec{v}_{-k}(s)\} \qquad (8)$$

Then with respect to the $(\hat{n}_0, \hat{m}, \hat{l})$ frame,

$$\begin{aligned} \frac{\partial \hat{n}}{\partial \delta} \;&\equiv\; i \sum_{k=I,II,III} \{v_{k5}^* \vec{w}_k - v_{k5}\vec{w}_k^*\} \\ &=\; -2\ \mathrm{Im} \sum_{k=I,II,III} v_{k5}^* \vec{w}_k \end{aligned} \qquad (9)$$

Note that this is independent of the phase space vector $\vec{u}$ and that $\frac{\partial \hat{n}}{\partial \delta}$ is periodic in azimuth in the machine coordinate system. In this approximation the depolarization time is then (Eq.(20), Sec.2.7.7)

$$\tau_{\mathrm{dep,lin}}^{-1} \;=\; \frac{55\sqrt{3}}{36}\frac{r_{\mathrm{e}}\gamma^5\hbar}{m_{\mathrm{e}}}\frac{1}{C}\int_{s_0}^{s_0+C} d\tilde{s}\,\frac{1}{|\rho(\tilde{s})|^3}\sum_{\mu=1}^{2}\left(\mathrm{Im}\sum_{k=I-III}[v_{k5}^*(\tilde{s})w_{k\mu}(\tilde{s})]\right)^2 \qquad (10)$$

This is the formula used in SLIM to calculate the depolarization rate. SLIM is based on thin lens optics. SLIM–like programs for thick lens optics are SLICK and SITF. Each term in Eq. (9) is basically the product of the sensitivity of an orbit amplitude to a change of δ and the sensitivity of $\hat{n}$ to a change of that orbit amplitude.

Using the 6×6 symplectic unit matrix $\mathbf{S}$ defined in [2] and the relation $A_k = -i\vec{v}_k^\dagger \mathbf{S}\vec{u}$, Eq. (7) can be written to display the explicit dependence of $\hat{n}$ on $\vec{u}$ as

$$\begin{pmatrix} \alpha(\vec{u}; s) \\ \beta(\vec{u}; s) \end{pmatrix} = 2\ \mathrm{Im}\left\{\sum_{k=I,II,III} \vec{w}_k(s)\cdot\vec{v}_k^\dagger(s)\mathbf{S}\right\}\vec{u}(s) = \mathbf{H_{2\times 6}}\ \vec{u}(s)$$

In this linearized theory the vectors $\hat{n}(\vec{u}; s)$ and $\frac{\partial \hat{n}}{\partial \delta}$ display only first order resonance behaviour, namely the resonances

$$\nu_{\mathrm{spin}} \;=\; k + k_I\nu_I + k_{II}\nu_{II} + k_{III}\nu_{III} \qquad (11)$$

with $|k_I| + |k_{II}| + |k_{III}| = 1$. They arise from the denominator matrix in Eq.(6). The theory is not valid beyond the limit $\sqrt{\alpha^2 + \beta^2} \ll 1$. In a fully uncoupled optic $I \to x$, $II \to y$ and $III \to z$.

This formalism forms the natural language for the method of maximizing the polarization called *"spin matching"*. Thus comments on the other programs will be postponed until later.

Spin matching in the SLIM formalism In practice the spin matching of real rings takes place in stages as follows.

Stage 1: *Strong synchrobeta spin matching of the perfectly aligned ring*

From Sec.2.7.7 it is clear that to maximize the polarization we must minimize $\tau_{\rm dep}^{-1}$. Then by Eq.(10) we need to minimize v_{k5}^* $(k = I,\ II,\ III)$ or the components of $\vec{w}_k$ at azimuths where $1/|\rho(s)|^3$ is large. The v_{k5}^* determine the orbit excitation due to synchrotron radiation (Sec.2.1.4 in [48]) [15]. In particular, for rings without $x-y$ coupling, $v_{_{II5}}^*$ usually vanishes in the arcs since the vertical dispersion η_y vanishes. However, $v_{_{II5}}^*$ does not vanish inside spin rotators (Secs.2.7.3, 2.7.4 in [48]) containing vertical bends. On the other hand $v_{_{I5}}^*$ tends not to vanish in the arcs since the horizontal dispersion $\eta_x \neq 0$. Finally, $v_{_{III5}}^*$ essentially never vanishes. Each case must be evaluated individually but the minimal recipe is to try to minimize $\vec{w}_k$ for $(k = I,\ II,\ III)$ only at azimuths where $|v_{_{k5}}(s)|^2/|\rho(s)|^3$ is sufficiently large. This in turn requires (Eq.(6)) that $\mathbf{G}(s+C,\ s)\cdot\vec{v}_k(s)$ for $(k = I,\ II,\ III)$ be minimized. This must be achieved by designing the ring layout with this in mind and then providing sufficient flexibility in the optics by providing enough independently powered quadrupoles. Subsequent calculations with SLIM will indicate whether the match criteria for the adopted design suffice.

Consider, for example, a specific mode, k. Label those bending magnets at which $|v_{_{k5}}(s)|^2/|\rho(s)|^3$ is large by ${\mu_1}^{(k)}$, ${\mu_2}^{(k)}$, $\cdots$, ${\mu_{n_k}}^{(k)}$. Then the suppression of depolarization associated with the kth mode requires that $\vec{w}_k(s_{\mu_i}) = 0$ for all $(i = 1$ to $n_k)$. In general (see Eq.(6)) this in turn requires [16]

$$
\begin{aligned}
\mathbf{G}(s_{\mu_2},\ s_{\mu_1})\,\vec{v}_k(s_{\mu_1}) &= 0 \\
\mathbf{G}(s_{\mu_3},\ s_{\mu_2})\,\vec{v}_k(s_{\mu_2}) &= 0 \\
&\vdots \\
\mathbf{G}(s_{\mu_1}+C,\ s_{\mu_n})\,\vec{v}_k(s_{\mu_n}) &= 0
\end{aligned}
\qquad (12)
$$

where we suppressed the superscript label "k". To fulfill Eq.(12) we then require the $G_{ij}(s_{\mu_{l+1}},\ s_{\mu_l})$ to vanish when the jth component of $\vec{v}_k$ does not vanish. The matrix $\mathbf{G}$ can be written in the form

$$
\mathbf{G}(s_2,\ s_1) \;=\; \int_{s_1}^{s_2} d\tilde{s}\ \mathbf{D}(s_2,\ \tilde{s})\,\mathbf{G}_0(\tilde{s})\,\mathbf{M}(\tilde{s},\ s_1)
$$

where

$$
\mathbf{G}_0 \;=\; \begin{pmatrix} l_s & l_x & l_y \\ -m_s & -m_x & -m_y \end{pmatrix}\cdot\mathbf{F} \qquad (13)
$$

Thus $G_{ij}(s_{\mu_{l+1}},\ s_{\mu_l})$ depends on the orientation of the $(\hat{m},\ \hat{l})$ vectors so that in some cases some elements of $G_{ij}(s_{\mu_{l+1}},\ s_{\mu_l})$ vanish automatically. But in general these conditions can only be fulfilled by adjusting quadrupole strengths — while maintaining other necessary features of the orbital optics. We call this **strong synchrobeta spin**

matching. A section of the ring satisfying a condition in Eq.(12) is *"spin transparent"* for mode k. The interpretation is immediate: the overall spin–orbit coupling for the section vanishes for mode k. Clearly, the exact spin matching conditions are very dependent on the layout of a machine and each case must be handled individually. In thin lens approximation the $\mathbf{G}$ matrix for a quadrupole of length l_{q} is

$$\mathbf{G} \;=\; \begin{pmatrix} -\tilde{q} l_y & 0 & -\tilde{q} l_x & 0 & 0 & 0 \\ +\tilde{q} m_y & 0 & +\tilde{q} m_x & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

where $\tilde{q} = (1 + a\gamma)\, g\, l_{\mathrm{q}}$. The thin and thick lens forms of $\mathbf{G}$ for other magnet types are given in [2, 3, 17].

If the $G_{ij}(s_{\mu_{l+1}},\, s_{\mu_l})$ cannot be brought to zero while maintaining an acceptable optic, then the $\mathbf{G}(s_{\mu_l} + C,\, s_{\mu_l}) \cdot \vec{v}_k(s_{\mu_l})$ themselves should be minimized. This essentially means that the effects of elements of the $\mathbf{G}$ matrices of sections of the ring are made to partially cancel one another. The spin matching of a ring with a solenoid Siberian Snake (Secs.2.7.3, 2.7.4 in [48]) has provided an example of this [6]. By Eq.(7) reduction of $\mathbf{G}(s + C,\, s) \cdot \vec{v}_k(s)$ for $(k = I,\ II,\ III)$ also reduces the angle between $\hat{n}$ and $\hat{n}_0$ at azimuth s.

Alternative Stage 1: *Harmonic synchrobeta spin matching of the perfectly aligned ring*

If the strong spin matching methods just described are impractical for some reason, another approach aimed at minimizing the strengths of depolarizing resonances can be adopted.

Rewrite Eq.(6) as

$$[\,w_{k1}(s_0) \mp i\, w_{k2}(s_0)\,] \;=\; -\frac{e^{\pm i\psi_{\mathrm{spin}}(s_0+C)}}{\left[e^{\pm i2\pi\nu_{\mathrm{spin}}} - e^{-i2\pi\nu_k}\right]} \int_{s_0}^{s_0+C} d\tilde{s}\, j_k^{(\mp)}(\tilde{s})\, e^{-i2\pi\left[\nu_k \pm \nu_{\mathrm{spin}}\right]\tilde{s}/C}$$

with

$$\begin{aligned} j_k^{(\mp)}(\tilde{s}) \;&=\; e^{\pm i[2\pi\nu_{\mathrm{spin}}\tilde{s}/C - \psi_{\mathrm{spin}}(\tilde{s})]} \begin{pmatrix} l_s \pm i m_s & l_x \pm i m_x & l_y \pm i m_y \end{pmatrix} \mathbf{F} \vec{v}_k(\tilde{s}) e^{+i2\pi\nu_k \tilde{s}/C} \\ &=\; j_k^{(\mp)}(\tilde{s} + C) = \sum_{p=-\infty}^{+\infty} c_{kp}^{(\mp)} e^{+i2\pi p\tilde{s}/C} \end{aligned}$$

$\Longrightarrow$

$$c_{kp}^{(\mp)} \;=\; \frac{1}{C}\int_0^C d\tilde{s} e^{i2\pi\left[\,\nu_k \pm \nu_{\mathrm{spin}} - p\,\right]\tilde{s}/C} e^{\mp i\psi_{\mathrm{spin}}(\tilde{s})} \begin{pmatrix} l_s \pm i m_s & l_x \pm i m_x & l_y \pm i m_y \end{pmatrix} \mathbf{F} \vec{v}_k(\tilde{s})$$

so that

$$[\,w_{k1}(s_0) \mp i\, w_{k2}(s_0)\,] \;=\; e^{\pm i\psi_{\mathrm{spin}}(s_0)}\, i\frac{C}{2\pi} \sum_{p=-\infty}^{+\infty} c_{kp}^{(\mp)} \frac{e^{-i2\pi\left[\,\nu_k \pm \nu_{\mathrm{spin}} - p\,\right] s_0/C}}{[\,\nu_k \pm \nu_{\mathrm{spin}} - p\,]}$$

The condition that $\vec{w}_k(s_{\mu_i}) = 0$ for all ($i = 1$ to n_k) is now be replaced by

$$[\, w_{k1}(s_{\mu_i}) \mp i\, w_{k2}(s_{\mu_i})\,] \;=\; e^{\pm i\psi_{\rm spin}(s_{\mu_i})}\, i\frac{C}{2\pi} \sum_{p=-\infty}^{+\infty} c_{kp}^{(\mp)} \frac{e^{-i2\pi[\,\nu_k \pm \nu_{\rm spin} - p\,] s_{\mu_i}/C}}{[\,\nu_k \pm \nu_{\rm spin} - p\,]} = 0$$

Near to the resonance $\nu_k \pm \nu_{\rm spin} - \tilde{p} \;=\; 0$ the sum over p is dominated by the term containing $c_{k\tilde{p}}^{(\mp)}$. This corresponds to the spins' seeing a stationary field in the $(\hat{n}_0,\, \hat{m}_0,\, \hat{l}_0)$ frame, proportional to $c_{k\tilde{p}}^{(\mp)}$, which rotates spins away from $\hat{n}_0$. Note that $c_{k\tilde{p}}^{(\mp)}$ is independent of s_{μ_i}.

Approximate spin matching can be achieved for all s_{μ_i} by adjusting the optics so that an appropriate set of the $c_{k\tilde{p}}^{(\mp)}$ are small. This is called **harmonic synchrobeta spin matching**. See also [16, 18]. For mode k and orbit amplitudes A_k and A_{-k} the strengths of the resonances are proportional to $A_k c_{k\tilde{p}}^{+}$ and $A_{-k} c_{k\tilde{p}}^{-}$. The concept of resonance strength is important for the acceleration of polarized protons (Sec.2.7.5 in [48]). Normally only the case of flat rings is considered so that $\hat{n}_0$ is vertical. The formalism presented here shows how to define resonance strength for arbitrary orientations of $\hat{n}_0$. See also [19, 20]. On resonance with $e^{i2\pi[\,\nu_k \pm \nu_{\rm spin} - \tilde{p}\,]\tilde{s}/C} = 1$, the coefficients $c_{k\tilde{p}}^{(\mp)}$ take the form

$$\begin{aligned}
c_{k\tilde{p}}^{(-)} \;&=\; \frac{1}{C}\int_0^C d\tilde{s}\, e^{-i\psi_{\rm spin}(\tilde{s})} \left(\begin{array}{ccc} l_s + i m_s & l_x + i m_x & l_y + i m_y \end{array}\right) \cdot \mathbf{F}\vec{v}_k(\tilde{s}) \\
&\quad \text{for} \;\; \nu_k + \nu_{\rm spin} \;=\; \tilde{p}\;; \\
c_{k\tilde{p}}^{(+)} \;&=\; \frac{1}{C}\int_0^C d\tilde{s}\, e^{+i\psi_{\rm spin}(\tilde{s})} \left(\begin{array}{ccc} l_s - i m_s & l_x - i m_x & l_y - i m_y \end{array}\right) \cdot \mathbf{F}\vec{v}_k(\tilde{s}) \\
&\quad \text{for} \;\; \nu_k - \nu_{\rm spin} \;=\; \tilde{p}
\end{aligned} \tag{15}$$

These can be obtained from the SLIM algorithm by calculating the matrix $\mathbf{G}$ at the resonance for one turn but without the backward spin basis rotation (Eq.(12), Sec.2.7.7) that, in SLIM, is applied at the end of one turn [12].

Reformulation in terms of beta functions and dispersion[21]

We can reformulate Stage 1 by making a transformation of the particle coordinates from $\vec{u} \equiv (x, p_x, y, p_y, z, \delta)$ to $\vec{\tilde{u}} \equiv (\tilde{x}, \tilde{p}_x, \tilde{y}, \tilde{p}_y, \tilde{z}, \delta)$ via the transformation

$$\vec{\tilde{u}} \;=\; \mathbf{K} \cdot \vec{u}$$

where

$$\mathbf{K}(s) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\eta_1 \\ 0 & 1 & 0 & 0 & 0 & -\eta_2 \\ 0 & 0 & 1 & 0 & 0 & -\eta_3 \\ 0 & 0 & 0 & 1 & 0 & -\eta_4 \\ \eta_2 & -\eta_1 & \eta_4 & -\eta_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

whereby the dispersion vector $\vec{\eta} \equiv (\eta_1, \eta_2, \eta_3, \eta_4)$ is the periodic solution of the linearized equations of motion for (x, p_x, y, p_y) with $\delta = 1$ and without the rf cavities. Then with $\eta_x \equiv \eta_1, \eta_y \equiv \eta_3$

$$\tilde{x} = x - \delta\eta_x, \quad \tilde{y} = y - \delta\eta_y \, .$$

The matrix $\mathbf{K}$ is symplectic so that the formalism remains canonical. In particular, the new transfer matrices $\tilde{\mathbf{M}}$ and eigenvectors $\tilde{\vec{v}}_\mu$ are obtained via

$$\tilde{\mathbf{M}}(s_2, s_1) = \mathbf{K}(s_2) \cdot \mathbf{M}(s_2, s_1) \cdot \mathbf{K^{-1}}(s_1)$$

and

$$\tilde{\mathbf{M}}(s + C, s) = \mathbf{K}(s) \cdot \mathbf{M}(s + C, s) \cdot \mathbf{K^{-1}}(s)$$

$$\Longrightarrow \quad \tilde{\vec{v}}_\mu(s) = \mathbf{K}(s) \, \vec{v}_\mu(s)$$

so that the eigenvalues and orthogonality conditions are unchanged [12]. Furthermore the new matrices $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ are

$$\tilde{\mathbf{F}}(s) = \mathbf{F}(s) \cdot \mathbf{K^{-1}}(s)$$

and

$$\tilde{\mathbf{G}}(s_2, s_1) = \mathbf{G}(s_2, s_1) \cdot \mathbf{K^{-1}}(s_1)$$

The depolarization rate then takes the form

$$\tau^{-1}_{\text{dep,lin}} = \frac{55\sqrt{3}}{36} \frac{r_e \gamma^5 \hbar}{m_e} \frac{1}{C} \int_{s_0}^{s_0 + C} d\tilde{s} \frac{1}{|\rho(\tilde{s})|^3} \sum_{\mu=1}^{2} \left(\mathrm{Im} \sum_{k=I-III} [\, f_k(\tilde{s}) \tilde{w}_{k\mu}(\tilde{s}) \,] \right)^2$$

with $f_k = \sum_{n=1}^{6} [K^{-1}]_{5n} \cdot \tilde{v}^*_{kn} = v^*_{k5}$ and $\tilde{\vec{w}}_k = \vec{w}_k$. This formulation has the advantage that in the special case, or the approximation, of no orbital coupling, the 6×6 orbit matrices just consist of three 2×2 matrices on the diagonal.

The eigenvectors $\vec{\tilde{v}}_k(s)$ of the revolution matrix are then written in the form

$$\vec{\tilde{v}}_I = \begin{pmatrix} \vec{t}_x \\ \vec{0}_2 \\ \vec{0}_2 \end{pmatrix}, \; \vec{\tilde{v}}_{II} = \begin{pmatrix} \vec{0}_2 \\ \vec{t}_y \\ \vec{0}_2 \end{pmatrix}, \; \vec{\tilde{v}}_{III} = \begin{pmatrix} \vec{0}_2 \\ \vec{0}_2 \\ \vec{t}_z \end{pmatrix} ;$$

$$\vec{t}_r = \frac{1}{\sqrt{2\beta_r(s)}} \begin{pmatrix} \beta_r(s) \\ -[\alpha_r(s) + i] \end{pmatrix} e^{-i\psi_r(s)}$$

$(r \equiv x, y, z)$ and the f_k are given by $f_I \equiv f_x = -(\tilde{v}_{I1}\eta_2 - \tilde{v}_{I2}\eta_1)$; $f_{II} \equiv f_y = -(\tilde{v}_{II3}\eta_4 - \tilde{v}_{II4}\eta_3)$ and $f_{III}(s) \equiv f_z = \sqrt{\frac{\beta_z}{2}} e^{-i\psi_z(s)}$. The functions f_I and f_{II} are closely related to the factor

$$\frac{{\eta_r}^2 + (\alpha_r\eta_r + \beta_r\eta_r')^2}{\beta_r} \quad (r = x, y)$$

in [22]. f_{III} is in practice almost independent of s (see below). Note that these α's and β's are Courant–Snyder parameters and should not be confused with the quantities in Eq.(1). With these coordinates the $\tilde{\mathbf{F}}$ matrix for a quadrupole takes the form

$$\tilde{\mathbf{F}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{g} & 0 & 0 & \tilde{g}\eta_3 \\ \tilde{g} & 0 & 0 & 0 & 0 & \tilde{g}\eta_1 \end{pmatrix}$$

We can write

$$\vec{\tilde{w}}_k(s_0) = -\left[\mathbf{D}(s_0 + C,\, s_0) - \hat{\lambda}_k\right]^{-1} \tilde{\mathbf{G}}(s_0 + C,\, s_0) \cdot \vec{\tilde{v}}_k(s_0)$$

for $(k = I,\; II,\; III)$ and we use a representation of the $\tilde{\mathbf{G}}$ matrix in the form

$$\tilde{\mathbf{G}}(s_2,\, s_1) = \int_{s_1}^{s_2} d\tilde{s}\; \mathbf{D}(s_2,\, \tilde{s})\, \tilde{\mathbf{G}}_0(\tilde{s})\, \tilde{\mathbf{M}}(\tilde{s},\, s_1)$$

with

$$\tilde{\mathbf{G}}_0 = \begin{pmatrix} l_s & l_x & l_y \\ -m_s & -m_x & -m_y \end{pmatrix} \cdot \tilde{\mathbf{F}}$$

In thin lens approximation the $\tilde{\mathbf{G}}$ matrix for a quadrupole is

$$\tilde{\mathbf{G}} = \begin{pmatrix} -\tilde{q}l_y & 0 & -\tilde{q}l_x & 0 & 0 & \kappa_1 \\ +\tilde{q}m_y & 0 & +\tilde{q}m_x & 0 & 0 & \kappa_2 \end{pmatrix}$$

where $\kappa_1 = -\tilde{q}l_y\eta_1 - \tilde{q}l_x\eta_3$ and $\kappa_2 = +\tilde{q}m_y\eta_1 + \tilde{q}m_x\eta_3$. We see that as a result of separating the transverse coordinates into betatron and dispersion contributions, columns six of $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ contain terms depending on dispersions.

The strong spin matching condition $\vec{\tilde{w}}_k = 0$ for suppressing depolarization now amounts to setting the $\tilde{\mathbf{G}}(s_{\mu_{l+1}}, s_{\mu_l})\,\vec{\tilde{v}}_k(s_{\mu_l})$ to zero in analogy with Eq.(12). Then in the special case, or approximation, of a fully uncoupled optic and by taking into account only the depolarizing influence of quadrupoles this is equivalent to requiring [23, 24]:

For horizontal motion:

$$-\frac{(1+a\gamma)}{\sqrt{2}}\frac{1}{C}\int_{s_{\mu_l}}^{s_{\mu_{l+1}}} d\tilde{s}\sqrt{\beta_x(\tilde{s})}g(\tilde{s})e^{-i\,\psi_x(\tilde{s})}\,[\,l_y(\tilde{s}) \pm i\,m_y(\tilde{s})\,]\,e^{\mp i\psi_{\rm spin}(\tilde{s})} \;=\; 0 \tag{16}$$

For vertical motion:

$$-\frac{(1+a\gamma)}{\sqrt{2}}\frac{1}{C}\int_{s_{\mu_l}}^{s_{\mu_{l+1}}} d\tilde{s}\sqrt{\beta_y(\tilde{s})}g(\tilde{s})e^{-i\,\psi_y(\tilde{s})}\,[\,l_x(\tilde{s}) \pm i\,m_x(\tilde{s})\,]\,e^{\mp i\psi_{\rm spin}(\tilde{s})} \;=\; 0 \tag{17}$$

For longitudinal motion:

$$\begin{aligned}&-\frac{(1+a\gamma)}{\sqrt{2}}\frac{1}{C}\int_{s_{\mu_l}}^{s_{\mu_{l+1}}} d\tilde{s}\frac{[\alpha_z(\tilde{s})+i]}{\sqrt{\beta_z(\tilde{s})}}g(\tilde{s})e^{-i\psi_z(\tilde{s})}\\ &\times\{\eta_y\,[l_x(\tilde{s}) \pm i m_x(\tilde{s})] + \eta_x\,[l_y(\tilde{s}) \pm i m_y(\tilde{s})]\}e^{\mp i\psi_{\rm spin}(\tilde{s})} = 0\end{aligned} \tag{18}$$

Since in practice synchrotron motion is well approximated by simple harmonic motion [26], $\beta_z(s)$ is almost independent of s and $\alpha_z(s) \approx 0$. Then Eq.(18) may be approximated by

$$\begin{aligned}&-\frac{(1+a\gamma)}{\sqrt{2}}\frac{i}{\sqrt{\beta_z}}\frac{1}{C}\int_{s_{\mu_l}}^{s_{\mu_{l+1}}} d\tilde{s}g(\tilde{s})e^{-i\psi_z(\tilde{s})}\\ \times\quad &e^{\mp i\psi_{\rm spin}(\tilde{s})}\,\{\eta_y\,[l_x(\tilde{s}) \pm i\,m_x(\tilde{s})] + \eta_x\,[l_y(\tilde{s}) \pm i\,m_y(\tilde{s})]\} \;=\; 0\end{aligned} \tag{19}$$

Harmonic synchrobeta spin matching in terms of beta functions and dispersion follows the path detailed earlier under "Alternative Stage 1" but with the eigenvectors $\vec{\tilde{v}}_k$ and the matrices $\tilde{\mathbf{F}}$. Typical expressions can be found in [24, 25].

Commentary

Spin matching should be carried out using thick lenses so that the optic is correct. Strong spin matching by minimizing the integrals in Eqs.(16–18) requires explicit integration. Furthermore Eqs.(16–18) must be modified if there is significant orbital coupling. Thus in practice the numerical fitting involved in strong spin matching can be carried out most simply by minimizing the $G_{ij}(s_{\mu_{l+1}}, s_{\mu_l})$ since these already represent integrals and do not need knowledge of the dispersion. Moreover these matrices are precisely those contained in the SLIM program so that cross checks between programs are simplified. Another advantage of working with the $\mathbf{G}$ matrix is that it allows sections of the ring to be studied and made transparent in isolation since no knowledge of Courant–Snyder parameters is needed; use of **G** emphasizes the local nature of spin transparency. On the other hand Eqs.(16–18) and the split–up versions depend on Courant–Snyder parameters and these in turn depend on the structure of the whole ring so that the *"locality"* is masked. When studying the spin transparency of a ring, it is often useful for diagnostic purposes to set elements of the **G** or the $\tilde{\mathbf{G}}$ matrices to zero artificially and thereby obtain an impression of which sections of the ring are most dangerous. For example by switching off column six of $\tilde{\mathbf{G}}$ in quadrupoles, the effect of dispersion can be cleanly separated from the effect of betatron motion. One can also investigate the system by using the matrix handling facilities in symbolic algebra programs and the fact that the **G** and $\tilde{\mathbf{G}}$ of magnets or strings of magnets often depend in a simple way on the elements of the corresponding **M** and $\tilde{\mathbf{M}}$ [17]. Finally, the **G** and $\tilde{\mathbf{G}}$ matrices are in general energy dependent. But a spin match made at the design energy is usually still effective for a few tens of MeV above and below, except near resonances.

Some examples

In a perfectly aligned flat ring (no vertical bends) with no solenoids and no $x-y$ coupling, the depolarization rate $\tau^{-1}_{\mathrm{dep,lin}}$ vanishes (see below under *Harmonic closed orbit spin matching*) so that no spin matching is needed.

A spin rotator (Secs.2.7.3, 2.7.4 in [48]) based on dipoles and containing no quadrupoles is automatically almost spin transparent since the elements of **G** are usually much smaller in dipoles than in quadrupoles [17]. Dipole rotators containing quadrupoles need explicit spin matching [27].

Spin rotators based on a combination of solenoids (which rotate $\hat{n}_0$ from the vertical into the horizontal) and dipoles (to make the polarization longitudinal at an IP) [17] are not automatically transparent. They also cause $x-y$ coupling. However, by sandwiching quadrupoles and skew quadrupoles among sections of solenoid the coupling can be eliminated and by careful choice of the sandwich structure some terms in columns 1 to 4 of **G** for the rotator can be made small at the same time [17]. Column 6 remains troublesome but for antisymmetric solenoid schemes [17] the columns 6 of the rotators cancel each other. For further discussion on solenoids see [28, 29].

For a straight section (e.g. surrounding an IP) where the polarization is longitudinal and which only contains quadrupoles and drifts, the spin precession angle is a linear combination of the overall orbit deflections Δp_x and Δp_y in the quadrupole fields [17]. Thus spin transparency implies making Δp_x and Δp_y vanish for all orbits. This can also be deduced from Eqs. (16) and (17). If the straight section is geometrically and optically left–right symmetric, this can be achieved with an optic for which $\tan \Delta\psi_x = -\alpha_x$ and $\tan \Delta\psi_y = -\alpha_y$ where the $\Delta\psi$'s are the phase advances between the IP and the outer end of the straight section and the α's appertain to the outer end. So the eight conditions that columns 1 to 4 of the **G** matrix vanish have been reduced to two conditions by the symmetry. Furthermore, this is an example where the spin matching conditions reduce to purely optical conditions.

These conditions can also be formulated directly in terms of $\mathbf{G}$. By choosing $\hat{l} = \hat{y}$ and $\hat{m} = \hat{x}$ and requiring that the elements G_{11} and G_{23} vanish for the stretch from the IP to the outer end, $\mathbf{G}$ vanishes for the whole straight section for arbitrary orientation of $\hat{m}$, $\hat{l}$ around the longitudinal $\hat{n}_0$.

For a straight section modified to contain horizontally bending dipoles with $\hat{n}_0$ in the horizontal plane, Eq. (19) is equivalent to requiring that the total change of η_2 due to the quadrupoles vanishes over the section.

If the straight section contains rf cavities, their influence on the spin transparency can often be neglected.

Other examples of the use of symmetry to simplify the spin match can be found in [24] where spin matching using variants of Eqs.(16–18) for a ring with dipole rotators is discussed. The results of a calculation with SLICK before and after a spin match can be found in [30]. Experimental observations resulting from successful spin matching involving spin rotators are described in [31].

Computer programs for strong spin matching

Strong spin matching facilities based on evaluation of spin–orbit integrals (e.g. Eqs.(16–18)) are built into the programs ASPIRRIN and SOM. To do spin matching in terms of $\mathbf{G}$ the code SPINOR [32] can be used.

Stage 2: *Harmonic closed orbit spin matching*

Once the perfectly aligned ring has been spin matched, the effects of misalignment must be addressed. In a perfectly aligned flat ring with no solenoids, $\hat{n}_0$ is vertical so that l_y and m_y are zero. Then by inspection of the $\mathbf{G}$ matrix elements for horizontal bends, quadrupoles and rf cavities it is clear that for no $x-y$ coupling, columns 1, 2, 5 and 6 of $\mathbf{G}(s+C,\, s)$ vanish. Moreover with no $x-y$ coupling the one turn orbital matrix $\mathbf{M_{6\times 6}}$ and its eigenvectors have the structures [33]

$$\mathbf{M_{6\times 6}} = \begin{pmatrix} \star & \star & 0 & 0 & \star & \star \\ \star & \star & 0 & 0 & \star & \star \\ 0 & 0 & \star & \star & 0 & 0 \\ 0 & 0 & \star & \star & 0 & 0 \\ \star & \star & 0 & 0 & \star & \star \\ \star & \star & 0 & 0 & \star & \star \end{pmatrix} \; ; \; \vec{v}_x = \begin{pmatrix} \star \\ \star \\ 0 \\ 0 \\ \star \\ \star \end{pmatrix} \; ; \; \vec{v}_y = \begin{pmatrix} 0 \\ 0 \\ \star \\ \star \\ 0 \\ 0 \end{pmatrix} \; ; \; \vec{v}_z = \begin{pmatrix} \star \\ \star \\ 0 \\ 0 \\ \star \\ \star \end{pmatrix}$$

where the $\star$'s denote nonzero elements. Therefore by Eq.(6) $\vec{w}_I(s)$ and $\vec{w}_{III}(s)$ are zero. In particular columns 1 and 2 of $\mathbf{G}$ and columns 1, 2 and 6 of $\tilde{\mathbf{G}}$ vanish. Note that for no $x-y$ coupling $v^*_{II5} \equiv v^*_{y5}$ vanishes. Then by Eq.(10) $\tau^{-1}_{\mathrm{dep,lin}}$ is automatically zero. In rings with vertical bends (e.g. in spin rotators) $\hat{n}_0$ is made vertical in the arcs by design.

In real misaligned rings there is a vertical closed orbit distortion and $\hat{n}_0$ is tilted from the vertical in the arcs (see below) so that the above mentioned columns of $\mathbf{G}$ and $\tilde{\mathbf{G}}$ for the arc quadrupoles do not vanish. In practice the tilts can be tens of milliradians and they increase with energy (they are roughly proportional to $a\gamma$) but even these small angles can lead to strong depolarization so that it is essential that the ring be very well aligned from the beginning. Note that *vertical* closed orbit distortion leads primarily to depolarization due to *horizontal* synchrobetatron motion in the arcs. Note also that tilts of tens of milliradians cause a negligible decrease of the underlying ST polarization (Eq.(14), Sec.2.7.7).

If there is a vertical correction coil and a BPM near each quadrupole, one can try to minimize the combined vertical kick (*"kick minimization"*) [34] applied to the orbit by each quadrupole and its correction coil and thereby reduce the tilt of $\hat{n}_0$ due to the distorted orbit's being off centre in the (misaligned) quadrupoles. This also reduces the generation of spurious vertical dispersion so that the driving of ν_y and ν_z (Eq.(11)) resonances is avoided. This presupposes that the positions with respect to the quadrupoles of the BPM's are well known. These relative positions can be estimated using beam–based calibration (Sec.4.5.5 in [48])[34]. However, kick minimization will not be effective if, say, the dipoles have significant tilt misalignments.

If these measures are insufficient, a further method for bringing $\hat{n}_0$ closer to the vertical is needed. $\hat{n}_0$, and thus its tilt, for the distorted ring can be obtained as described in Sec.2.7.7 but one gains more insight by using a perturbation theory based on SLIM concepts [35]. Viewed from the ($\hat{n}_0$, $\hat{m}$, $\hat{l}$) frame calculated for the design orbit, the first order deviation of $\hat{n}_0$ from the design orientation can be written as

$$[\delta n_{01}(s) - i\delta n_{02}(s)] \quad = \quad -i\frac{C}{2\pi}\sum_k h_k \frac{e^{i2\pi ks/C}}{k-\nu_{\text{spin}}}$$

where the h_k are Fourier coefficients given by

$$h_k \quad = \quad \frac{1}{C}\int_{s_0}^{s_0+C} d\tilde{s}\,[d_1(\tilde{s}) - id_2(\tilde{s})]\,e^{-ik2\pi\tilde{s}/C}$$

Here

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} l_s & l_x & l_y \\ -m_s & -m_x & -m_y \end{pmatrix} \left\{ \mathbf{F}\cdot\vec{u}_{\text{co}} - \tfrac{e}{p_0} \begin{pmatrix} \Delta B_s \frac{1+a\gamma_0}{1+\gamma_0} \\ \Delta B_x(1+a\gamma_0) \\ \Delta B_y(1+a\gamma_0) \end{pmatrix} \right\}$$

where the $\Delta B_{x,y,s}$ are field errors and $\vec{u}_{\text{co}}$ is the deviation of the 6–D closed orbit from the design orbit. $\delta\hat{n}_0$ can be minimized by using correction coils to adjust the closed orbit (e.g. by generating closed bumps so that the luminosity is not affected) in such a way that the real and imaginary parts of h_k, with k near ν_{spin}, are small. This technique is called **harmonic closed orbit spin matching** and is embodied in the program FIDO [36, 37]. See [18] also. If the machine distortions are not well known and if the closed orbit cannot be measured well enough, the closed orbit correction must be carried out empirically by observing the polarization. If the distortions and the orbit are well enough known the correction coil strength can be calculated *ab initio* (deterministic harmonic closed orbit spin matching) [38]. The correction scheme should be chosen so that it achieves the maximum effect on $\delta\hat{n}_0$ with the smallest possible additional orbit distortion.

Harmonic closed orbit spin matching can in principle be used to minimize the $\delta\hat{n}_0$ due to an uncompensated solenoid placed at the position of a nominally vertical $\hat{n}_0$. However, this is achieved more efficiently by generating relatively antisymmetric vertical orbit bumps (spanning horizontal bend magnets) on each side of the solenoid [39, 40].

It might also be useful to weight $\delta\hat{n}_0(s)$ by a periodic function $p(s)$ [41]. In that case one tries to minimize $p(s)\delta\hat{n}_0(s)$. This is worth trying, for example, if the main source of depolarization due to misalignments is the coupling of non-zero l_y and m_y to the horizontal dispersion in the arcs (see Eq.(19)). This is often the case, as can be seen by examining the numerical values of the contributions of each mode (I, II, III) in Eq.(10). Then $p(s)$ is taken to be $\eta_x(s)g(s)$.

To minimize $p(s)\delta\hat{n}_0(s)$ one must minimize the harmonics $\tilde{h}_k$ of

$$\begin{aligned} \tilde{h}(s) &= p(s)(d_1 - id_2) + p'(s)\left[\delta n_{01}(s) - i\delta n_{02}(s)\right] \\ &= \tilde{h}(s + C) \end{aligned}$$

whereby

$$p(s)\left[\delta n_{01}(s) - i\delta n_{02}(s)\right] = -i\frac{C}{2\pi}\sum_k \tilde{h}_k \frac{e^{i2\pi ks/C}}{k - \nu_{\text{spin}}}$$

Stage 3: *Further tuning*

Harmonic closed orbit spin matching can generate spurious vertical dispersion and this in turn generates vertical emittance (nonzero v^*_{II5} (Sec.2.1.4 in [48])) and also ensures that column 6 of $\tilde{\mathbf{G}}$ for the quadrupoles does not vanish. Thus extra depolarization can occur. It might then be useful to overlay a harmonic vertical betatron match ($k = II$ in Eq.(15)) on any existing Stage 1 match, assuming that is possible. Likewise, to overcome the effect of spurious vertical dispersion in column 6 of $\tilde{\mathbf{G}}$ one could use extra vertical correction coils to overlay a harmonic vertical dispersion match ($k = III$ in Eq.(15)). Usually both of these two extra matches would be empirical. One could also try to combine the harmonic closed orbit match and the harmonic vertical dispersion match into one procedure.

Stage 4: *Beam–beam spin matching*

The beam–beam interaction is equivalent to a nonlinear lens and can spoil a spin match. The effect of the beam–beam interaction on the polarization is not fully understood but it has been suggested that the beam–beam depolarization can be reduced by balancing the beam–beam deflection of spins against subsequent deflections taking place in the ring quadrupoles. The condition for minimizing the effect of vertical kicks is independent of the current and charge distribution in the opposing beam and takes the form [42]

$$\frac{m_x - il_x}{\sqrt{\beta_y^*}} + \sum_{\pm} \pm \frac{e^{-\frac{i}{2}(\nu_{\rm spin} \pm \nu_y)}}{4 \sin \frac{\nu_{\rm spin} \pm \nu_y}{2}} \int_0^C ds \; g \; \sqrt{\beta_y} e^{\pm i\psi_y} \; (m_x + il_x) \;=\; 0$$

An equivalent prescription in SLIM formalism allows an arbitrarily coupled optic to be treated [43].

Higher order resonances To go beyond the linearization of spin contained in Eq.(1) one writes

$$\hat{n}(\vec{\tilde{u}}; s) \;=\; (1 - \alpha^2 - \beta^2)^{1/2} \hat{n}_0(s) + \alpha \hat{m}(s) + \beta \hat{l}(s) \tag{20}$$

(for $\alpha^2 + \beta^2 \leq 1$) and does not linearize the T–BMT equation. Then spin–orbit resonances of arbitrarily high order can appear in $\frac{\partial \hat{n}}{\partial \delta}$ [7]. The strength decreases with the order ($\equiv |k_I| + |k_{II}| + |k_{III}|$). In practice the most intrusive higher order resonances are those for which $\nu_{\rm spin} = k \pm \nu_k + k_{III}\nu_{III}$. These *"synchrotron sideband resonances"* of the first order parent resonances are due to modulation by energy oscillations of the instantaneous rate of spin precession around $\hat{n}_0$. They originate in the part due to synchrotron motion in the term $\vec{\omega}^{\rm sb} \cdot \hat{n}_0$ appearing in the full equations of spin motion (i.e. beyond the SLIM level) [44]. The depolarization rate associated with sidebands of isolated parent resonances ($\nu_{\rm spin} = k \pm \nu_k$) is approximately proportional to the depolarization rate for the parent resonances. Thus the effects of synchrotron sideband resonances can be reduced by doing the spin matches described above. Explicit formulae for the proportionality constants (*"enhancement factors"*) can be found in [45, 46]. The underlying strength parameter (the *"modulation index"*) of synchrotron sideband resonances is $(a\gamma\sigma_\delta/\nu_z)^2$ which increases strongly with the energy and energy spread.

Other computer codes [47] The SMILE algorithm is restricted to linearized orbital motion and calculates $\frac{\partial \hat{n}}{\partial \delta}$ by an extension of the first order perturbation theory of SLIM to high order using Eq.(20) and full 3–D spin motion. The algorithm involves multi–turn spin–orbit tracking. High order resonance effects are manifested by resonance denominators but the formalism ensures that the vector $\hat{n}$ is of unit length. The highest required absolute values of the k_I, k_{II}, k_{III} are specified as input parameters.

SODOM represents $\hat{n}$ by a spinor notation. The periodicity condition $\hat{n}(\vec{u}; s) = \hat{n}(\vec{u}; s + C)$ (Sec.2.7.7) is equivalent to periodicity in the three phases of linearized orbital motion and the one turn 2×2 spinor transfer matrix on a synchrobeta orbit is also periodic in the initial orbital phases. The spinor transfer matrix and $\hat{n}(\vec{u}; s)$ are then represented by Fourier series. The Fourier coefficients are obtained numerically and $\hat{n}(\vec{u}; s)$ can then be reconstructed. By constructing $\hat{n}$ at many points in phase space $\frac{\partial \hat{n}}{\partial \delta}$ can be obtained by numerical differentiation.The highest required absolute values of the k_I, k_{II}, k_{III} are specified as input parameters.

The algorithm SpinLie utilizes Lie algebraic methods (Sec.2.7.9 in [48]) and can handle non-linear orbit motion and 3–D spin motion.

The vector $\hat{n}(\vec{u}; s)$ can also be obtained by *"stroboscopic averaging"* using the code SPRINT. $\frac{\partial \hat{n}}{\partial \delta}$ can then be calculated by numerical differentiation. This algorithm automatically includes all orders of resonance.

The above algorithms all exploit the DKM formula (Eq.(16),Sec.2.7.7) but the SITROS program simulates the depolarization process directly using a Monte–Carlo tracking simulation of the effects on the orbit and then on the spin of stochastic photon emission and damping and delivers an estimate of $\tau_{\rm dep}$. The equilibrium polarization is then obtained from the approximation (Sec.2.7.7)

$$P_{\rm eq} = P_{\rm bks} \frac{\tau_{\rm tot}}{\tau_{\rm bks}} \tag{21}$$

where

$$\frac{1}{\tau_{\rm tot}} = \frac{1}{\tau_{\rm bks}} + \frac{1}{\tau_{\rm dep}} \ . \tag{22}$$

This ignores the (normally small) term $\frac{\partial \hat{n}}{\partial \delta}$ in the numerator of the DKM formula.

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