Second quantization

Goals for the lecture:

I. Introduce quantization of EM fields \(\Rightarrow\) photon

II. "second quantization" - jargon from QFT
   - different approach than "first quantization"
     * we studied so far:
       observables \(\leftrightarrow\) operators
       states \(\leftrightarrow\) functions
     * when dealing with (fermions) bosons:
       impose (anti)symmetry of states "by hand"

   - second quantization formulation of Q.M.
     * states themselves expressed in terms of
       "creation" and "annihilation" operators
       working on the "vacuum state"
     * other operators (e.g. Hamiltonian) also
       described by creation/annihilation operators
     * proper (anti)symmetry of states enforced
       via the algebra of these operators

Simple harmonic oscillator (revisited)

\[ \hat{\mathcal{H}} |\psi\rangle = E |\psi\rangle \quad \text{with} \quad \hat{\mathcal{H}} = \frac{\hat{p}_x^2}{2m} + \frac{m\omega^2}{2} \hat{x}^2 \]

Rewrite:

\[ \hat{\mathcal{H}} = \frac{m\omega^2}{2} \left( \frac{\hat{p}_x^2}{m\omega^2} + \hat{x}^2 \right) \]

E.g. any 2 operators: \(\hat{\alpha}, \hat{\beta}\)

\[ (\hat{\alpha} + i\hat{\beta})(\hat{\alpha} - i\hat{\beta}) = \hat{\alpha}^2 + i(\hat{\beta}\hat{\alpha} - \hat{\alpha}\hat{\beta}) + \hat{\beta}^2 \]

\[ \Rightarrow \hat{\alpha}^2 + \hat{\beta}^2 = (\hat{\alpha} + i\hat{\beta})(\hat{\alpha} - i\hat{\beta}) + i[\hat{\alpha}, \hat{\beta}] \]

Define:

\[ \hat{\alpha} = \left( \frac{m\omega}{2\hbar} \right)^{1/2} \left( \hat{x} + \frac{i\hat{p}_x}{m\omega} \right) \quad \text{"lowering operator"} \]

\[ \hat{\alpha}^+ = \left( \frac{m\omega}{2\hbar} \right)^{1/2} \left( \hat{x} - \frac{i\hat{p}_x}{m\omega} \right) \quad \text{"raising operator"} \]

\[ (\hat{\alpha})^+ = \hat{\alpha}^+ \quad \text{adjoint of one another} \quad \text{self-adjoint} \]

(\text{but not Hermitian, i.e. } \hat{\alpha} \neq \hat{\alpha}^+) \]

Can rewrite (Hermitian) operators in terms of \(\hat{\alpha}, \hat{\alpha}^+\)

\[ \hat{x} = \left( \frac{\hbar}{2m\omega} \right)^{1/2} (\hat{\alpha}^+ + \hat{\alpha}) \]

\[ \hat{p}_x = i \left( \frac{\hbar m\omega}{2} \right)^{1/2} (\hat{\alpha}^+ - \hat{\alpha}) \]
Easy to show:
\[
[a, a^+] = a a^+ - a^+ a = 1 \\
[a, a] = -1 \\
[a, a] = 0 \\
[a^+, a^+] = 0
\]

E.g. \[
\hat{\mathcal{H}} = \frac{\hbar \omega}{2} (a a^+ + a^+ a)
\]
\[
= \frac{\hbar \omega}{2} ([a, a^+] + 2 a^+ a)
\]
\[
= \hbar \omega (a^+ a + \frac{1}{2})
\]

Since we know \[
\hat{\mathcal{H}} \ket{n} = E_n \ket{n} \] with \[
E_n = \hbar \omega (n + \frac{1}{2})
\]
\[\Rightarrow \quad a^+ a \ket{n} = n \ket{n}\]

\(\hat{n}\) - number operator

Raising & lowering operators' action

\[
\hat{a} (a^+ a) \ket{n} = n a \ket{n}
\]
\[
\hat{a} a^+ - a^+ a = [\hat{a}, a^+] = 1 \quad \Rightarrow \quad \hat{a} a^+ = 1 + a^+ a
\]
\[
(1 + a^+ a) \hat{a} \ket{n} = n a \hat{a} \ket{n}
\]
\[
\hat{a} \ket{n} = (n-1) \hat{a} \ket{n-1}
\]

\(\hat{n}\) - result of \(\hat{n}\) operator

\[\Rightarrow \quad \hat{a} \ket{n} = \sqrt{n} \ket{n-1}\] - lowering operator

Similarly,

\[\hat{a}^+ \ket{n} = B_n \ket{n+1}\] - raising operator

Can find coefficients \(A_n, B_n\) (see Griffiths 2.3)

\[\hat{a} \ket{n} = \sqrt{n} \ket{n-1}\]
\[\hat{a}^+ \ket{n} = \sqrt{n+1} \ket{n+1}\]

E.g. ground state \[\ket{0}\]:

\[\hat{a} \ket{0} = 0 \] (can use to find \(\Psi_0(x) = \langle x | 0 \rangle\))
\[\hat{a}^+ \ket{0} = 1 \ket{1}\]
Alternative interpretation: state \( |n\rangle \) contains \( n \) identical quanta, each with energy \( \hbar \omega \):
\[
E_n = \frac{n \hbar \omega}{2} + n \hbar \omega
\]
These quanta behave like identical bosons:
\( \hat{a}^\dagger \) creator \( \hbar \omega \) = creation operator
\( \hat{a} \) destroys \( \hbar \omega \) = annihilation operator

**Quantization of EM fields**

Hamiltonian equations:
\[
\begin{align*}
\frac{\partial p}{\partial t} &= -\frac{\partial H}{\partial q} \\
\frac{\partial q}{\partial t} &= \frac{\partial H}{\partial p}
\end{align*}
\]
describes time evolution of generalized momentum \( p \) and position \( q \)

E.g. classical
\[
H = \frac{p^2}{2m} + V(q)
\]
\[
\begin{align*}
\frac{\partial p}{\partial t} &= -\frac{\partial H}{\partial q} \\
\frac{\partial q}{\partial t} &= \frac{\partial H}{\partial p}
\end{align*}
\]
\[
\Rightarrow \begin{align*}
\frac{dp}{dt} &= -\frac{\partial V}{\partial q} = -F \\
\frac{dq}{dt} &= \frac{p}{m}
\end{align*}
\] Newton's 2nd law

1) Find quantities similar to \( p \) and \( q \) for EM field, Hamiltonian eqns give classical time evolution of \( p \) and \( q \)
2) Proceed to quantization of \( \hat{H} \) \( (\hat{q} \rightarrow \hat{q}, \hat{p} \rightarrow i\hbar \frac{\partial}{\partial q}) \)

E.g. standing EM wave
\( E \) polarized along \( z \)-dir

\[
\begin{align*}
E_z &= A(t) \sin kx \\
B_y &= A(t) \cos kx
\end{align*}
\]
node \hspace{1cm} antinode

Check against Maxwell eqns:
\[
\begin{align*}
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{B} &= \mu_0 \frac{\partial \vec{E}}{\partial t}
\end{align*}
\]
\[
\begin{align*}
y \text{-comp: } &\frac{\partial E_y}{\partial y} - \frac{\partial E_z}{\partial z} = -\frac{\partial B_y}{\partial t} \\
z \text{-comp: } &\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0 \frac{\partial E_z}{\partial t}
\end{align*}
\]
\[
\begin{align*}
\Rightarrow &\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0 \frac{\partial E_z}{\partial t} \\
kp \cos kx &= \frac{\partial^2}{\partial t^2} \cos kx \\
\Rightarrow &\frac{\partial^2}{\partial t^2} = \omega p \\
-kq \frac{\partial}{\partial t} \sin kx &= \frac{\partial^2}{\partial t^2} \sin kx \\
\Rightarrow &\frac{\partial^2}{\partial t^2} = -\omega q
\end{align*}
\]
\( (\omega = kc \text{ and } \epsilon_0 \mu_0 = \frac{1}{c^2}) \)
Combining the two:
\[
\frac{d^2q}{dt^2} + \omega^2 q = 0 \quad \text{\{same as SHO!\}}
\]

Hamiltonian for the standing EM wave:
\[
H = \frac{\omega}{2} (p^2 + q^2) \quad \text{if } D = \sqrt{\frac{2\mu}{\varepsilon_0}}
\]

Going quantum:
\[
\hat{H} = \frac{\omega}{2} (\hat{p}^2 + \hat{q}^2)
\]

Rewrite in terms of creation/annihilation operators
\[
\hat{a} = \frac{1}{\sqrt{2\hbar}} \left( \hat{q} + i\hat{p} \right) \quad \hat{a}^+ = \frac{1}{\sqrt{2\hbar}} \left( \hat{q} - i\hat{p} \right)
\]

\[
\Rightarrow \quad \hat{H} = \hbar \omega \left( \hat{a}^+ \hat{a} + \frac{1}{2} \right)
\]

Extend to many modes: \( \hat{a} \rightarrow \hat{a}_\lambda \), \( \hbar \omega \rightarrow \hbar \omega_\lambda \)

\[
\hat{H}_\lambda = \hbar \omega_\lambda \left( \hat{a}_\lambda^+ \hat{a}_\lambda + \frac{1}{2} \right)
\]

Same properties of \( \hat{a}_\lambda^+, \hat{a}_\lambda \): \([\hat{a}_\lambda, \hat{a}_\lambda^+] = 1\)

\[
\hat{a}_\lambda^+ |n_\lambda\rangle = \sqrt{n_\lambda + 1} |n_\lambda + 1\rangle \quad \text{create } \hbar \omega_\lambda \text{ photon}
\]

\[
\hat{a}_\lambda |n_\lambda\rangle = \sqrt{n_\lambda} |n_\lambda - 1\rangle \quad \text{destroy } \hbar \omega_\lambda \text{ photon}
\]

**Physics of what's going on**

* Quantized EM field
  * energy / single quantum, \( \hbar \omega_\lambda \) is "photon"
* Can put as many photons into \( \lambda \)-mode as desired ⇒ bosons
* Zero-point fluctuations (E, B \( \propto |\lambda\rangle \) fluctuates)
  * Energy of vacuum
* Can quantize other waves: vibrations in solids ⇒ phonon
  plasma waves ⇒ plasmon, \( \sigma \rightarrow \text{waves} \Rightarrow \text{magnon} \)...

**Casimir effect**:

Attractive force due to zero-point

\[
\text{force} = -\frac{\hbar c n_{\lambda=2}}{24 \alpha^4} \quad \text{(Casimir)}
\]

E.g. \( 1 \mu m \) separation for \( 1 \times 1 m^2 \)

plates ⇒ force \( 1.3 \times 10^{-3} N \)

Perturbs zero-point modes excluding \( \lambda > \alpha \) wavelengths from between the plates
Fermion creation & annihilation operators

recall: fermion wavefun antisymmetric w.r.t.
particle exchange

\[ |1_{2};a,b\rangle = \frac{1}{\sqrt{2}} \left( |1_{1},a\rangle |2_{b}\rangle - |1_{1},b\rangle |2_{a}\rangle \right) \]

rewrite

\[ |1_{2};a,b\rangle = \frac{1}{\sqrt{2}} \begin{vmatrix} 1_{1},a\rangle & 1_{2},a\rangle \\ 1_{1},b\rangle & 1_{2},b\rangle \end{vmatrix} \]

Slater's determinant

Easy to extend to \( N \) identical fermions

\[ |N_{;a,b,\ldots,n}\rangle = \frac{1}{\sqrt{N!}} \begin{vmatrix} 1_{1},a\rangle & 1_{2},a\rangle & \cdots & 1_{N},a\rangle \\ 1_{1},b\rangle & 1_{2},b\rangle & \cdots & 1_{N},b\rangle \\ \vdots & \vdots & \ddots & \vdots \\ 1_{1},n\rangle & 1_{2},n\rangle & \cdots & 1_{N},n\rangle \end{vmatrix} \]

- exchange any two particles (columns)
  \( \Rightarrow \) eigenfun changes sign =antisymmetric
- if any two signle-pair. states the same (rows)
  \( \Rightarrow \) eigenfun goes to zero = Pauli principle

Can characterize multiparticle state:

\[ |m_{1},m_{2},\ldots,m_{n}\rangle = \text{Fock representation} \]

occupation of state \( a \), e.g. 0 or 1 for fermion

Creation operator for fermions: \( \hat{b}_{m}^{\dagger} \)
Annihilation: \( \hat{b}_{m} \)

Their action:

\[ \hat{b}_{m} |m_{1},\ldots,m_{n-1},m_{n}\rangle = (-1)^{m_{n}} |m_{1},\ldots,m_{n-1},m_{n}-1\rangle \]

add row with \( m \)th state in Slater determinant, interchange rows
for standard form \( \Rightarrow (-1) \) for each interchange

Ex.

\[ \hat{b}_{m}^{\dagger} |1_{1},e\rangle |1_{2},e\rangle = \frac{1}{\sqrt{2}} \begin{vmatrix} 1_{1},e\rangle & 1_{2},e\rangle & 1_{3},e\rangle \\ 1_{1},h\rangle & 1_{2},h\rangle & 1_{3},h\rangle \\ 1_{1},m\rangle & 1_{2},m\rangle & 1_{3},m\rangle \end{vmatrix} \]

\( \Rightarrow \) swap
\[ \hat{b}_m^+ | \mu_a ... \mu_m=1 ... \mu_n > = 0 \]

Similarly:

\[ \hat{b}_m | \mu_a ... \mu_m=1 ... \mu_n > = (-1)^{\sum_{i=1}^{\mu_m} \mu_i} | \mu_a ... \mu_m=0 ... \mu_n > \]

\[ \hat{b}_m | \mu_a ... \mu_m=0 ... \mu_n > = 0 \]

Can prove that: \[ \{ \hat{b}_m, \hat{b}_n^+ \} = \{ \hat{b}_m^+, \hat{b}_n \} = 1 \quad \text{anticommutator for fermions} \]

What's next?

* can write operators \( \hat{A} \), etc. (including \( \hat{\psi} \)) in form of creation/annihilation operators

* can mix fermions with bosons (i.e. photon emission by an electron, electron-positron pair production, \( \Rightarrow \) QFT)

* math keeps track of proper state (anti)symmetry