Second quantization

Goals for the lecture:

I. introduce quantization of EM fields $\Rightarrow$ photon

II. "second quantization" – jargon from QFT

– different approach than "first quantization"

* we studied so far:
  observables $\leftrightarrow$ operators
  states $\leftrightarrow$ functions
* when dealing with (fermions) bosons:
  impose (anti)symmetry of states "by hand"

– second quantization formulation of Q.M.

* states themselves expressed in terms of "creation" and "annihilation" operators working on the "vacuum state"
* other operators (e.g. Hamiltonian) also described by creation/annihilation operators
* proper (anti)symmetry of states enforced via the algebra of these operators

Simple harmonic oscillator (revisited)

$$\hat{\mathcal{H}}|\psi\rangle = E|\psi\rangle \quad \text{with} \quad \hat{\mathcal{H}} = \frac{\hat{p}_x^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2$$

Rewrite:

$$\hat{\mathcal{H}} = \frac{m\omega^2}{2}\left(\frac{\hat{p}_x^2}{m^2\omega^2} + \hat{x}^2\right),$$

want to factor

E.g any 2 operators: $\hat{\alpha}, \hat{\beta}$

$$(\hat{\alpha}+i\hat{\beta})(\hat{\alpha}-i\hat{\beta}) = \hat{\alpha}^2 + i(\hat{\beta}\hat{\alpha}-\hat{\alpha}\hat{\beta}) + \hat{\beta}^2$$

$$\Rightarrow \hat{\alpha}^2 + \hat{\beta}^2 = (\hat{\alpha}+i\hat{\beta})(\hat{\alpha}-i\hat{\beta}) + i[\hat{\alpha},\hat{\beta}]$$

Define:

$$\hat{a} = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x} + \frac{i\hat{p}_x}{m\omega}\right) \quad \text{– "lowering operator"}$$

$$\hat{a}^+ = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x} - \frac{i\hat{p}_x}{m\omega}\right) \quad \text{– "raising operator"}$$

$(\hat{a})^+ = \hat{a}^+$ adjoint of one another (but not Hermitian, i.e. $\hat{a} \neq \hat{a}^+$)

self-adjoint

Can rewrite (Hermitian) operators in terms of $\hat{a}, \hat{a}^+$

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\hat{a}^+ + \hat{a}) \qquad \hat{p}_x = i\left(\frac{\hbar m\omega}{2}\right)^{1/2}(\hat{a}^+ - \hat{a})$$

Easy to show:

$$[\hat{a},\hat{a}^+] = \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1$$

$$[\hat{a}^+,\hat{a}] = -1$$

$$[\hat{a},\hat{a}] = 0$$

$$[\hat{a}^+,\hat{a}^+] = 0$$

E.g. $\hat{\mathcal{H}} = \frac{\hbar\omega}{2}\left(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\right)$

$$= \frac{\hbar\omega}{2}\left([\hat{a},\hat{a}^+] + 2\hat{a}^+\hat{a}\right)$$

$$= \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

Since we know $\hat{\mathcal{H}}|n\rangle = E_n|n\rangle$ with $E_n = \hbar\omega(n+\frac{1}{2})$

$\Rightarrow$ $\underbrace{\hat{a}^+\hat{a}}_{\hat{n}\text{ - number operator}}|n\rangle = n|n\rangle$

Raising & lowering operators' action

$$\hat{a}(\hat{a}^+\hat{a})|n\rangle = n\hat{a}|n\rangle$$

$\hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = [\hat{a},\hat{a}^+] = 1$, $\Rightarrow$ $\hat{a}\hat{a}^+ = 1 + \hat{a}^+\hat{a}$

$(1+\hat{a}^+\hat{a})\hat{a}|n\rangle = n\hat{a}|n\rangle$,

$$\underbrace{\hat{a}^+\hat{a}}_{\hat{n}}(\hat{a}|n\rangle) = \underbrace{(n-1)}_{\text{result of } \hat{n} \text{ operator}}(\hat{a}|n\rangle)$$

$\Rightarrow$ $\hat{a}|n\rangle = A_n|n-1\rangle$ – lowering operator

Similarly,

$\hat{a}^+|n\rangle = B_{n+1}|n+1\rangle$ – raising operator

Can find coefficients A_n, B_n (see Griffiths 2.3)

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

E.g. ground state $|0\rangle$:

$\hat{a}|0\rangle = 0$ (can use to find $\psi_0(x) = \langle x|0\rangle$)

$\hat{a}^+|0\rangle = |1\rangle$

Alternative interpretation: state $|n\rangle$ contains n identical quanta, each with energy $\hbar\omega$:

$$E_n = \underbrace{\frac{\hbar\omega}{2}}_{\text{zero fluctuations}} + \underbrace{n\hbar\omega}_{n \text{ bosons}}$$

These quanta behave like identical bosons

$\hat{a}^\dagger$ creates $\hbar\omega$ = creation operator

$\hat{a}$ destroys $\hbar\omega$ = annihilation operator

Quantization of EM fields

Hamiltonian equations:

$$\left.\begin{aligned} \frac{dp}{dt} &= -\frac{\partial H}{\partial q} \\ \frac{dq}{dt} &= \frac{\partial H}{\partial p} \end{aligned}\right\} \text{ describes time evolution of generalized momentum } p \text{ and position } q$$

E.g. classical $\mathcal{H} = \frac{p^2}{2m} + V(q)$

$$\left.\begin{aligned} \frac{dp}{dt} &= -\frac{\partial H}{\partial q} \\ \frac{dq}{dt} &= \frac{\partial H}{\partial p} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \frac{dp}{dt} &= -\frac{\partial V}{\partial q} \equiv F \\ \frac{dq}{dt} &= \frac{p}{m} \end{aligned}\right\} \text{Newton's } 2^{nd} \text{ law}$$

1) Find quantities similar to p and q for EM field, Hamiltonian eqs give classical time evolution of p and q

2) Proceed to quantization of $\hat{\mathcal{H}}$ $\left(\hat{q} \to q,\ \hat{p} \to -i\hbar\frac{d}{dq}\right)$

E.g. standing EM wave

$\vec{E}$ polarized along z-dir

	$x=0$	$x=L$
$E_z = p(t)\,D\sin kx$	node	node
$B_y = q(t)\,\frac{D}{c}\cos kx$	antinode	antinode

Check against Maxwell equ.

$$\left.\begin{aligned} \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\times\vec{B} &= \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \end{aligned}\right\} \quad \begin{aligned} &y\text{-comp}: \cancelto{0}{\frac{\partial E_x}{\partial z}} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \\ &z\text{-comp}: \frac{\partial B_y}{\partial x} - \cancelto{0}{\frac{\partial B_x}{\partial y}} = \varepsilon_0\mu_0\frac{\partial E_z}{\partial t} \end{aligned}$$

$$\left.\begin{aligned} kpD\cos kx &= \frac{D}{c}\frac{\partial q}{\partial t}\cos kx, & &\Rightarrow \frac{dq}{dt} = \omega p \\ -kq\frac{D}{c}\sin kx &= \varepsilon_0\mu_0\frac{\partial p}{\partial t}D\sin kx, & &\Rightarrow \frac{dp}{dt} = -\omega q \end{aligned}\right\}$$

$$\left(\omega = kc \text{ and } \varepsilon_0\mu_0 = \frac{1}{c^2}\right)$$

combining the two:

$$\frac{d^2q}{dt^2} + \omega^2 q = 0 \quad \text{same as SHO!}$$

Hamiltonian for the standing EM wave:

$$H = \frac{\omega}{2}\left(p^2+q^2\right) \qquad \text{if } D = \sqrt{\frac{2\omega}{L\varepsilon_0}}$$

Going quantum:

$$\hat{H} = \frac{\omega}{2}\left(\hat{p}^2+\hat{q}^2\right)$$

rewrite in terms of creation / annihilation operators

$$\left.\begin{aligned} \hat{a} &= \sqrt{\frac{1}{2\hbar}}\left(\hat{q}+i\hat{p}\right) \\ \hat{a}^+ &= \sqrt{\frac{1}{2\hbar}}\left(\hat{q}-i\hat{p}\right) \end{aligned}\right\} \Rightarrow \hat{H} = \hbar\omega\left(\hat{a}^+\hat{a}+\frac{1}{2}\right)$$

Extend to many modes $\hat{H} \to \hat{H}_\lambda$, $\hbar\omega \to \hbar\omega_\lambda$

$$\hat{H}_\lambda = \hbar\omega_\lambda\left(\underbrace{\hat{a}^+_\lambda\hat{a}_\lambda}_{\hat{n}_\lambda \text{ number operator}}+\frac{1}{2}\right)$$

photon energy

Same properties of $\hat{a}^+_\lambda, \hat{a}_\lambda$: $[\hat{a}_\lambda, \hat{a}^+_\lambda] = 1$

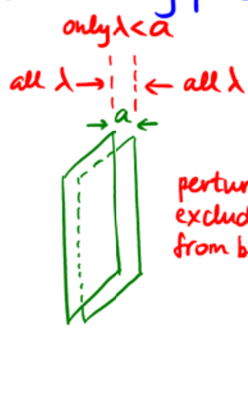
$\hat{a}^+_\lambda |n_\lambda\rangle = \sqrt{n_\lambda+1}\,|n_\lambda+1\rangle$ create $\hbar\omega_\lambda$ photon

$\hat{a}_\lambda |n_\lambda\rangle = \sqrt{n_\lambda}\,|n_\lambda-1\rangle$ destroy $\hbar\omega_\lambda$ photon

Physics of what's going on

* quantized EM field
  $\Rightarrow$ energy / single quantum, $\hbar\omega_\lambda \equiv$ "photon"
* can put as many photons into λ-mode as desired $\Rightarrow$ bosons
* zero-pt. fluctuations (E, B in $|0_\lambda\rangle$ fluctuates)
  $\Rightarrow$ energy of vacuum
* can quantize other waves: vibrations in solids $\to$ phonon
  plasma waves $\to$ plasmon, spin waves $\to$ magnon ...

Casimir effect: attractive force due to zero-point on two parallel conducting plates



perturbs zero-pt modes excluding $\lambda > a$ wavelengths from between the plates

$$\frac{\text{force}}{\text{area}} = -\frac{\hbar c \pi^2}{240\, a^4} \quad \text{(Casimir)}$$

E.g. 1 μm separation for $1\times1\,\text{m}^2$ plates $\Rightarrow$ force 1.3×10^{-3} N

Fermion creation & annihilation operators

recall: fermion wavefcn antisymmetric w.r.t. particle exchange

$$|2;a,b\rangle = \frac{1}{\sqrt{2}}\left(|1,a\rangle|2,b\rangle - |1,b\rangle|2,a\rangle\right)$$

state   particle

rewrite

$$|2;a,b\rangle = \frac{1}{\sqrt{2}}\begin{vmatrix} |1,a\rangle & |2,a\rangle \\ |1,b\rangle & |2,b\rangle \end{vmatrix}$$

Slater's determinant

Easy to extend to N identical fermions

$$|N;a,b,\dots,n\rangle = \frac{1}{\sqrt{N!}}\begin{vmatrix} |1,a\rangle & |2,a\rangle & \dots & |N,a\rangle \\ |1,b\rangle & |2,b\rangle & \cdots & |N,b\rangle \\ \vdots & & \ddots & \vdots \\ |1,n\rangle & |2,n\rangle & \dots & |N,n\rangle \end{vmatrix}$$

- exchange any two particles (columns)
  $\Rightarrow$ eigenfcn changes sign = antisymmetric
- if any two single-part. states the same (rows)
  $\Rightarrow$ eigenfcn goes to zero = Pauli principle

Can characterize multiparticle state:

$|\mu_a, \mu_b, \dots, \mu_n\rangle$ = Fock representation (same Fock as HF method)

occupation of state a, e.g. 0 or 1 for fermion

Creation operator for fermions : $\hat{b}_m^{\dagger}$ state m

Annihilation ——— " ——— : $\hat{b}_m$

Their action:

$$\hat{b}_m^{\dagger}|\mu_a \dots \mu_m = 0 \dots \mu_n\rangle = (-1)^{\sum_{m'>m} \mu_{m'}} |\mu_a \dots \mu_m = 1 \dots \mu_n\rangle$$

add row with m^{th} state in Slater determinant, interchange rows for standard form $\Rightarrow$ (−1) for each interchange

Ex.

$$\hat{b}_m^{\dagger}\begin{vmatrix} |1,\ell\rangle & |2,\ell\rangle \\ |1,n\rangle & |2,n\rangle \end{vmatrix} = \begin{vmatrix} |1,\ell\rangle & |2,\ell\rangle & |3,\ell\rangle \\ |1,n\rangle & |2,n\rangle & |3,n\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \end{vmatrix}$$

swap

$$= - \begin{vmatrix} |1,l\rangle & |2,l\rangle & |3,l\rangle \\ |1,m\rangle & |2,m\rangle & |3,m\rangle \\ |1,n\rangle & |2,n\rangle & |3,n\rangle \end{vmatrix}$$

$$\hat{b}_m^\dagger \,|\mu_a \ldots \mu_m = 1 \ldots \mu_n\rangle = 0$$

Similarly:

$$\hat{b}_m \,|\mu_a \ldots \mu_m = 1 \ldots \mu_n\rangle = (-1)^{\sum_{m'>m} \mu_{m'}} \,|\mu_a \ldots \mu_m = 0 \ldots \mu_n\rangle$$

$$\hat{b}_m \,|\mu_a \ldots \mu_m = 0 \ldots \mu_n\rangle = 0$$

Can prove that : $\underbrace{\hat{b}_m \hat{b}_m^\dagger + \hat{b}_m^\dagger \hat{b}_m = 1}$

$\equiv \{\hat{b}_m, \hat{b}_m^\dagger\} = 1$ anticommutator for fermions

What's next?

* can write operators $\hat{H}$, etc. (including $\hat{\psi}$) in form of creation / annihilation operators
* can mix fermions with bosons (i.e. photon emission by an electron, electron-positron pair production, $\Rightarrow$ QFT)
* math keeps track of proper state (anti) symmetry